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Calculus of variations. — *Necessary optimality conditions for geodesics in weighted Wasserstein spaces*, by LUIGI AMBROSIO and FILIPPO SANTAMBROGIO, communicated on 23 June 2006.

ABSTRACT. — The geodesic problem in Wasserstein spaces with a metric perturbed by a conformal factor is considered, and necessary optimality conditions are established in a case where this conformal factor favours the spreading of the probability measure along the curve. These conditions have the form of a system of PDEs of the kind of the compressible Euler equations. Moreover, self-similar solutions to this system are discussed.

KEY WORDS: Optimal transport theory; Wasserstein distance; compressible Euler equation.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35K55, 47J35, 49K99.

1. INTRODUCTION

Let us consider a closed convex set $\Omega \subset \mathbb{R}^d$ and the set $\mathcal{P}(\Omega)$ of probability measures in Ω. Given $p \in (1, \infty)$, we denote by $W_p(\Omega)$ the subspace of measures with finite p-th moment, i.e.

$$W_p(\Omega) := \left\{ \mu \in \mathcal{P}(\Omega) \,:\, \int_\Omega |x|^p \, d\mu < \infty \right\}.$$

We endow $W_p(\Omega)$ with the canonical Wasserstein distance $W_p(\mu, \nu)$ of order p (see [1], [10] for the basic facts about W_p).

It is well known that $W_p(\Omega)$ is a length space, and that (constant speed) geodesics of $W_p(\Omega)$ are in one-to-one correspondence with optimal transport plans, via McCann's linear interpolation procedure (see for instance Proposition 7.2.2 of [1]). Here we consider, instead, the case when the Wasserstein metric is perturbed by a conformal factor $L(\mu)$: by minimizing

$$\int_0^1 L^2(\mu_t) |\mu'|^2(t) \, dt \tag{1.1}$$

among all curves μ connecting $\mu_0 = \mu$ to $\mu_1 = \nu$, one obtains a new squared distance depending on p and L, and we are interested in computing the geodesics relative to this distance. In (1.1), $|\mu'|(t)$ is the rate of change of W_p, also called the metric derivative, along the curve μ (see (2.1)).

This problem has been introduced in [2], where the main goal was to choose a factor L favouring atomic measures in order to give a time-dependent approach to some branched transport problems which may be applied to the study of river networks, pipe systems, blood vessels, tree structures etc. In fact, by setting $L(\mu) = \sum_i a_i^r$ (for $0 < r < 1$) if $\mu = \sum_i a_i \delta_{x_i}$, and $L = +\infty$ on measures which are not purely atomic, there is a strong

link between this variational problem and those which were first presented in [11] and [9] (the latter uses in fact a time-dependent approach, but by means of measures on the space of paths instead of paths on the space of measures). This choice of L is in fact a local functional on measures which, among probability measures, favours the most concentrated ones. In [2], as a natural counterpart, the case of local functionals L which prefer spread measures is considered as well and the two problems sound somehow specular. The aim of the present paper is in fact to consider this second problem and to find optimality conditions in the form of PDEs.

In particular, we study in detail the case when $L(\mu)$ is the γ-th power of the L^q norm of the density of μ with respect to Lebesgue measure $\mathcal{L}^d$, with $q > 1$ and $\gamma > 0$ given, and $L(\mu) = +\infty$ if μ is a singular measure. So, geodesics with respect to the new metric tend to spread the density as much as possible. Denoting by u_t the density of μ_t, we find that a necessary optimality condition for geodesics is (for $p = 2$; see (2.5) for general p)

$$\frac{d}{dt}(K(t)\boldsymbol{v}u) + K(t)\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{v}u) + H(t)\nabla u^q = 0, \tag{1.2}$$

where $\boldsymbol{v}_t$ is the tangent velocity field of μ_t, linked to u_t via the continuity equation $\frac{d}{dt}u_t + \nabla\cdot(\boldsymbol{v}_t u_t) = 0$. Here $H(t) < 0$ and $K(t) > 0$ are suitable functions depending only on the metric derivative of μ_t and on $L(\mu_t)$. As Brenier pointed out to us, this equation is very similar to the compressible Euler equation, but with a negative pressure field $p = H(t)u^q$; a similar equation, with H constant and $q = 3$, recently appeared also in [7], in the one-dimensional case. In fact the main difference appears in the relationship between the L part and the speed part: here it is multiplicative, while in [7] it is additive, as we will explain in a while.

The appearance of the Euler equation as an optimality condition is not very surprising, taking into account the approach developed, in the incompressible case, by Brenier (first in a purely Lagrangian framework in [3], [4], and then in a mixed Eulerian-Lagrangian one in [5], [6]). In this connection, we mention that our derivation of the optimality condition differs from [4], [6], where duality is used to perform first variations, and uses instead a perturbation argument directly at the level of the primal problem.

Due to the non-convex nature of this problem, we do not know of any sufficient minimality condition for the geodesics. In this connection, one may notice that, in the case $\gamma = q/2$ and $p = 2$, we have

$$\inf_{\delta>0}\ \delta\int_\Omega u^q\,dx + \frac{1}{\delta}\int_\Omega |\boldsymbol{v}|^2 u\,dx = 2L(u\mathcal{L}^d)\left(\int_\Omega |\boldsymbol{v}|^2 u\,dx\right)^{1/2}$$

and the minimal $L^2(\mu)$ norm of $\boldsymbol{v}$ is strictly linked to the metric derivative. This suggests a connection between the "multiplicative" model studied here and in [2], and the "additive" model

$$\min\left\{\int_0^1\int_\Omega (u^q + |\boldsymbol{v}|^2 u)\,dx\,dt : \frac{d}{dt}u + \nabla\cdot(\boldsymbol{v}u) = 0\right\}$$

subject to Dirichlet conditions at $t = 0$ and $t = 1$. This additive model, in the case $q = 3$, is exactly the one studied in [7] (in this connection, see also [8]). Notice that this problem is *convex* in the pair $(u, \boldsymbol{v}u)$. It turns out, indeed, that the (necessary and sufficient, by the

convex nature of the problem) optimality conditions for the additive model are very similar to (1.2), the only difference being that H and K do not depend on time.

In the last part of the paper we compute and characterize particular self-similar solutions of (1.2).

2. Optimality conditions for weighted Wasserstein geodesics

2.1. *A new velocity vector field*

Definition 1. If we are given a Lipschitz curve $\mu : [0,1] \to W_p(\Omega)$, we define a *velocity field* of the curve to be any vector field $\boldsymbol{v} : [0,1] \times \Omega \to \mathbb{R}^d$ such that for a.e. $t \in [0,1]$ the vector field $\boldsymbol{v}_t = \boldsymbol{v}(t,\cdot)$ belongs to $[L^p(\mu_t)]^d$ and the continuity equation

$$\frac{d}{dt}\mu_t + \nabla \cdot (\boldsymbol{v}\mu_t) = 0$$

is satisfied in the sense of distributions: this means that for all $\phi \in C_c^1(\Omega)$ and any $t_1 < t_2 \in [0,1]$,

$$\int \phi \, d\mu_{t_2} - \int \phi \, d\mu_{t_1} = \int_{t_1}^{t_2} ds \int_\Omega \nabla\phi \cdot \boldsymbol{v}_s \, d\mu_s,$$

or, equivalently, in differential form,

$$\frac{d}{dt}\int \phi \, d\mu_t = \int_\Omega \nabla\phi \cdot \boldsymbol{v}_t \, d\mu_t \quad \text{for a.e. } t \in [0,1].$$

We say that $\boldsymbol{v}$ is the *tangent* field to the curve μ_t if, for a.e. t, $\boldsymbol{v}_t$ has minimal $[L^p(\mu_t)]^d$ norm for any t among all the velocity fields.

It is now well known (see for instance Theorem 8.3.1 and Proposition 8.4.5 in [1]) that for any Lipschitz or absolutely continuous curve μ_t with values in $W_p(\Omega)$ there exists a unique tangent field and moreover it is characterized by

$$\|\boldsymbol{v}_t\|_{L^p(\mu_t)} = |\mu'|(t) = \lim_{h\to 0} \frac{W_p(\mu_{t+h}, \mu_t)}{|h|} \quad \text{for a.e. } t \in [0,1]. \tag{2.1}$$

The right hand side in the equality above is the rate of change of W_p along the curve μ_t, also called the *metric derivative* of μ_t.

We now want to investigate how velocity fields change if we modify the curve μ_t.

Theorem 2.1. *Let a Lipschitz function $\mu_t : [0,1] \to W_p(\Omega)$ and a smooth function $T : [0,1] \times \Omega \to \Omega$ be such that for any t the function $T_t := T(t,\cdot)$ is a diffeomorphism. Consider the new curve μ'_t given by $\mu'_t = (T_t)_\sharp \mu_t$. If $\boldsymbol{v}_t$ is a velocity field for μ_t, then the vector field $\boldsymbol{v}'$ defined by*

$$\boldsymbol{v}'_t \cdot \mu'_t = (T_t)_\sharp \left[\left(\nabla T_t \cdot \boldsymbol{v}_t + \frac{\partial T}{\partial t} \right) \mu_t \right]$$

is a velocity field for μ'_t.

PROOF. We have

$$\begin{aligned}\int_\Omega \phi\, d\mu'_{t+h} - \int_\Omega \phi\, d\mu'_t &= \int_\Omega \phi\circ T_{t+h}\, d\mu_{t+h} - \int_\Omega \phi\circ T_t\, d\mu_t\\ &= \int_\Omega (\phi\circ T_{t+h} - \phi\circ T_t)\, d\mu_{t+h} + \int_\Omega \phi\circ T_t\, d(\mu_{t+h}-\mu_t)\\ &= \int_\Omega \left(\int_t^{t+h} (\nabla\phi)\circ T_s\cdot \frac{\partial T}{\partial t}\Big|_s\, ds\right) d\mu_{t+h} + \int_t^{t+h} ds\int_\Omega (\nabla\phi)\circ T_t\cdot \nabla T_t\cdot \boldsymbol{v}_s\, d\mu_s,\end{aligned}$$

where in the last equality we have used the fact that $\boldsymbol{v}_t$ is a velocity field for μ, with test function $\phi\circ T_t$. It is now convenient to divide by h, rewrite and pass to the limit as $h\to 0$:

$$\begin{aligned}\frac{\int_\Omega \phi\, d\mu'_{t+h} - \int_\Omega \phi\, d\mu'_t}{h} &= \int_\Omega d\mu_{t+h}\,\frac{1}{h}\int_t^{t+h} (\nabla\phi)\circ T_s\cdot \frac{\partial T}{\partial t}\Big|_s\, ds\\ &\quad + \int_\Omega (\nabla\phi)\circ T_t\cdot\nabla T_t\cdot \boldsymbol{v}_t\, d\mu_t + \frac{1}{h}\int_t^{t+h} ds\int_\Omega \nabla\psi_t\cdot(\boldsymbol{v}_s\, d\mu_s - \boldsymbol{v}_t\, d\mu_t),\end{aligned}\tag{2.2}$$

where $\psi_t = \phi\circ T_t$. In the first term on the right hand side the measures μ_{t+h} weakly converge to μ_t, since $t\mapsto \mu_t$ is Lipschitz continuous, while the integrand uniformly converges as a function of the space variable x to $(\nabla\phi)\circ T_t\cdot\frac{\partial T}{\partial t}$ as $h\to 0$. Hence we get convergence of the integral. If we prove that the last term tends to zero at least for a.e. $t\in[0,1]$ we get the assertion, since then we would have

$$\begin{aligned}&\lim_{h\to 0}\frac{\int_\Omega \phi\, d\mu'_{t+h} - \int_\Omega \phi\, d\mu'_t}{h}\\ &\qquad = \int_\Omega \left((\nabla\phi)\circ T_t\cdot\frac{\partial T}{\partial t} + (\nabla\phi)\circ T_t\cdot\nabla T_t\cdot\boldsymbol{v}_t\right) d\mu_t = \int_\Omega \nabla\phi\cdot\boldsymbol{v}'_t\, d\mu'_t,\end{aligned}$$

and this is nothing but the differential version of the continuity equation for $\boldsymbol{v}'$ and μ' (it remains to prove $\boldsymbol{v}'_t\in L^p(\mu'_t)$ but this is straightforward since T_t is a diffeomorphism and this allows us to write down the densities and estimate them). To prove that the last term vanishes at the limit we see that, for fixed $\psi\in \mathrm{Lip}(\Omega)$, the function

$$s\mapsto g_\psi(s) := \int_\Omega \nabla\psi\cdot\boldsymbol{v}_s\, d\mu_s = \frac{d}{ds}\int_\Omega \psi\, d\mu_s$$

is L^∞ since μ_t is a Lipschitz curve in $W_p(\Omega)$ and hence almost any $s\in[0,1]$ is a Lebesgue point. This allows us to fix a negligible set $N\subset[0,1]$ such that any point $t\in[0,1]\setminus N$ is a Lebesgue point for all the functions $g_{\psi_{t_1}}$ for $t_1\in\mathbb{Q}$. We now fix $t\in[0,1]\setminus N$ and try to prove that the last integral in (2.2) tends to zero. For $t_1\in\mathbb{Q}$ we have

$$\begin{aligned}&\left|\frac{1}{h}\int_t^{t+h} ds\int_\Omega \nabla\psi_t\cdot(\boldsymbol{v}_s\, d\mu_s - \boldsymbol{v}_t\, d\mu_t)\right|\\ &\qquad\le \frac{1}{h}\int_t^{t+h} ds\left|\int_\Omega \nabla(\psi_t-\psi_{t_1})\cdot\boldsymbol{v}_s\, d\mu_s\right| + \left|\int_\Omega \nabla(\psi_t-\psi_{t_1})\cdot\boldsymbol{v}_t\, d\mu_t\right|\\ &\qquad\quad + \left|\frac{1}{h}\int_t^{t+h} ds\int_\Omega \nabla\psi_{t_1}\cdot(\boldsymbol{v}_s\, d\mu_s - \boldsymbol{v}_t\, d\mu_t)\right|\end{aligned}$$

$$\leq \operatorname{Lip}(\psi_t - \psi_{t_1}) \operatorname{Lip}_{W_p}(\mu) + \left| \frac{1}{h} \int_t^{t+h} ds \int_\Omega \nabla \psi_{t_1} \cdot (\boldsymbol{v}_s \, d\mu_s - \boldsymbol{v}_t \, d\mu_t) \right|.$$

In the last sum the second term tends to zero since t is a Lebesgue point for $g_{\psi_{t_1}}$, and the first term may be made as small as we want by choosing t_1 close to t, since $\psi_s = \phi \circ T_s$ and both ϕ and T are regular. □

2.2. *Derivation of the optimality conditions*

We consider the minimization problem presented in [2], i.e. finding a curve of measures in $W_p(\Omega)$ of minimal length according to a metric which, roughly speaking, is the Wasserstein (infinitesimal) metric multiplied by a conformal factor. Precisely, if we define for $q > 1$ the functional

$$L_q(\nu) = \begin{cases} \int_\Omega u^q \, d\mathcal{L}^d & \text{if } \nu = u \cdot \mathcal{L}^d, \\ +\infty & \text{otherwise,} \end{cases}$$

we want to minimize

$$\int_0^1 L_q(\mu_t) |\mu'|(t) \, dt,$$

where $|\mu'|(t)$ is the metric derivative of the curve μ and the minimization occurs among all the Wasserstein–Lipschitz curves $t \mapsto \mu_t$ with given initial and final points, i.e. μ_0 and μ_1 are given probability measures in $W_p(\Omega)$. We will always consider only the non-trivial case $\mu_0 \neq \mu_1$. If we define $V(\mu, t) = \int_\Omega |\boldsymbol{v}_t|^p \, d\mu_t$, where $\boldsymbol{v}$ is the tangent field to the curve μ_t, we know that $|\mu'|(t) = V(\mu, t)^{1/p}$. We may generalize the functional we want to minimize by considering

$$\mathfrak{F}(\mu) := \int_0^1 L_q(\mu_t)^\alpha \, V(\mu, t)^\beta \, dt$$

which reduces to the case studied in [2] if $\alpha = 1$ and $\beta = 1/p$. Notice that in this case the functional does not change under reparametrization of curves, while if $\beta > 1/p$ the minimization selects a particular parametrization. For $\beta \leq 1/p$ the existence of a minimum is not ensured. Anyway we do not deal here with existence results (see [2]), but we only look for necessary optimality conditions. We will consider variations of μ of the form

$$\mu_t^\varepsilon = (T_t^\varepsilon)_\sharp \mu_t \quad \text{with} \quad T^\varepsilon(t, x) = x + \varepsilon \xi(t, x), \quad T^\varepsilon = \mathrm{id} + \varepsilon \xi(t, \cdot),$$

for arbitrary regular functions $\xi \in C_c^\infty([0, 1] \times \Omega; \mathbb{R}^d)$. In the end optimality conditions will be expressed through a system of PDEs: we will obtain the result after collecting some lemmas. What we want to do now is exploiting the fact that for a minimizing curve μ the following quantity must be minimal for $\varepsilon = 0$:

$$\mathfrak{F}(\mu_t^\varepsilon) = \left(\int_0^1 F_\varepsilon(t)^\alpha V_\varepsilon(t)^\beta \, dt \right),$$

provided we define $F_\varepsilon(t) = L_q(\mu_t^\varepsilon)$ and $V_\varepsilon(t) = V(\mu^\varepsilon, t)$. Since it is not easy to deal with the term $V_\varepsilon(t)$, we will replace it by $\tilde{V}_\varepsilon(t)$ given by

$$\tilde{V}_\varepsilon(t) = \int_\Omega |(\boldsymbol{v}^\varepsilon)_t|^p \, d\mu_t^\varepsilon.$$

Here the vector field v^ε is the one we get by Theorem 2.1 when the map T is T^ε and the initial field v_t is the tangent field to μ_t. In this way we have $\tilde V_\varepsilon(t) \geq V_\varepsilon(t)$ (since v_t^ε is not necessarily of minimal L^p norm) but $\tilde V_\varepsilon(0) = V_\varepsilon(0)$. Thus we may switch to $\tilde V_\varepsilon(t)$ instead of $V_\varepsilon(t)$, getting

$$\tilde{\mathfrak{F}}(\mu_t^\varepsilon) = \left(\int_0^1 F_\varepsilon(t)^\alpha \tilde V_\varepsilon(t)^\beta \, dt\right).$$

We will compute the derivative of $\tilde{\mathfrak{F}}(\mu_t^\varepsilon)$ with respect to ε and get the conditions we are looking for.

LEMMA 2.2. *If μ is a curve given by $\mu_t = u_t\mathcal{L}^d$ and such that $\mathfrak{F}(\mu) < \infty$, then for almost any $t \in [0,1]$,*

$$\frac{d}{d\varepsilon} F_\varepsilon(t) = (1-q)\int_\Omega (JT_t^\varepsilon)'\left(\frac{u_t}{JT_t^\varepsilon}\right)^q d\mathcal{L}^d.$$

In particular,

$$\frac{d}{d\varepsilon} F_\varepsilon(t)|_{\varepsilon=0} = (1-q)\int_\Omega (\nabla\cdot\xi) u_t^q \, d\mathcal{L}^d.$$

Moreover, for ε sufficiently small (depending on T, but not on t),

$$\frac{d}{d\varepsilon} F_\varepsilon(t) \leq C L_q(\mu_t).$$

PROOF. We look at the integrand in the definition of F_ε: to do this it is necessary to look at the density of the measure μ_t^ε. Thanks to the change of variables formula, this density can be easily seen to be given by

$$u_t^\varepsilon = \frac{u_t}{JT_t^\varepsilon} \circ (T_t^\varepsilon)^{-1},$$

where J stands for the Jacobian (this formula is a consequence of T_t^ε being a diffeomorphism at least for small ε). Thus, after changing variables, we have

$$F_\varepsilon(t) = L_q(\mu_t^\varepsilon) = \int_\Omega \left(\frac{u_t}{JT_t^\varepsilon}\right)^q JT_t^\varepsilon \, d\mathcal{L}^d.$$

The derivative of the integral is

$$(1-q)(JT_t^\varepsilon)'\left(\frac{u_t}{JT_t^\varepsilon}\right)^q,$$

where $(JT_t^\varepsilon)'$ stands for the derivative of JT_t^ε with respect to ε. This quantity may be easily estimated by Cu_t^q, since $1-a \leq JT_t^\varepsilon \leq 1+a$ and $(JT_t^\varepsilon)' \leq B$ for suitable constants a and B. Since for almost any t the function u_t must belong to L^q (because the functional we are minimizing is finite) we can apply the dominated convergence theorem and get the assertion. To obtain the derivative at $\varepsilon = 0$ it is sufficient to notice that $(JT_t^\varepsilon)'|_{\varepsilon=0} = \nabla\cdot\xi$, which is well known. The same estimate we used to get dominated convergence may be used to get the last inequality. □

In the next lemma we consider the term $\tilde V_\varepsilon$.

LEMMA 2.3. *If μ is a curve such that $\mathfrak{F}(\mu) < \infty$, then for almost any $t \in [0,1]$,*

$$\frac{d}{d\varepsilon}\tilde{V}_\varepsilon(t) = p\int_\Omega \left|\nabla T_t^\varepsilon \cdot \boldsymbol{v}_t + \frac{\partial T^\varepsilon}{\partial t}\right|^{p-2}\left(\nabla T_t^\varepsilon \cdot \boldsymbol{v}_t + \frac{\partial T^\varepsilon}{\partial t}\right)\cdot\left(\nabla\xi\cdot\boldsymbol{v}_t + \frac{\partial\xi}{\partial t}\right)d\mu_t. \tag{2.3}$$

In particular,

$$\frac{d}{d\varepsilon}\tilde{V}_\varepsilon(t)|_{\varepsilon=0} = p\int_\Omega |\boldsymbol{v}_t|^{p-2}\boldsymbol{v}_t\cdot\left(\nabla\xi\cdot\boldsymbol{v}_t + \frac{\partial\xi}{\partial t}\right)d\mu_t.$$

Moreover, for ε sufficiently small (depending on T, but not on t),

$$\frac{d}{d\varepsilon}\tilde{V}_\varepsilon(t) \le C(V(\mu,t)+1).$$

PROOF. If we compute the densities of μ_t^ε and the expression of the new velocity field and we change variable in the integral by means of $y = T_t^\varepsilon(x)$, as we did in the previous lemma, we get

$$\tilde{V}_\varepsilon(t) = \int_\Omega \left|\nabla T_t^\varepsilon\cdot\boldsymbol{v}_t + \frac{\partial T^\varepsilon}{\partial t}\right|^p d\mu_t. \tag{2.4}$$

When we differentiate the integrand we get exactly the integrand in (2.3), and we only need to show that this expression is uniformly dominated, at least for small ε and almost every t, to get the result. By boundedness of the derivatives of T^ε it is not difficult to see that the norm of the first vector in the scalar product in the integrand may be estimated by

$$\left|\nabla T_t^\varepsilon\cdot\boldsymbol{v}_t + \frac{\partial T^\varepsilon}{\partial t}\right|^{p-1} \le (C|v_t| + C)^{p-1},$$

while for the second we have

$$\left|\nabla\xi\cdot\boldsymbol{v}_t + \frac{\partial\xi}{\partial t}\right| \le C|\boldsymbol{v}_t| + C$$

for a suitable constant C. Hence, since $\boldsymbol{v}_t \in [L^p(\mu_t)]^d$ for almost every t the integrability is proved and the differentiation under the integral sign can be performed. □

To conclude, we put together the previous two results in order to compute the derivative of the integral in t.

THEOREM 2.4. *If μ is a curve with $\mathfrak{F}(\mu) < \infty$ and $V(\mu,t) \ge V_0 > 0$ for almost every t, then*

$$\begin{aligned}\frac{d}{d\varepsilon}\tilde{\mathfrak{F}}(\mu^\varepsilon)|_{\varepsilon=0} &= \alpha(1-q)\int_0^1 F^{\alpha-1}V^\beta\int_\Omega(\nabla\cdot\xi)u_t^q\,d\mathcal{L}^d\,dt\\ &\quad + p\beta\int_0^1 F^\alpha V^{\beta-1}\int_\Omega|\boldsymbol{v}_t|^{p-2}\left(\nabla\xi\cdot\boldsymbol{v}_t+\frac{\partial\xi}{\partial t}\right)\cdot\boldsymbol{v}_t\,d\mu_t\,dt,\end{aligned}$$

where $F(t) = L_q(\mu_t)$ and $V(t)$ has the usual meaning.

PROOF. By the definition of $\tilde{\mathfrak{F}}(\mu^\varepsilon)$ we see that the pointwise derivative of the integrand is

$$\alpha F_\varepsilon(t)^{\alpha-1}\frac{dF}{d\varepsilon}\tilde{V}_\varepsilon(t)^\beta + \beta F_\varepsilon(t)^\alpha \tilde{V}_\varepsilon(t)^{\beta-1}\frac{d\tilde{V}}{d\varepsilon}.$$

By the regularity of T^ε the term $F_\varepsilon(t)$ may be estimated both from above and below by $F(t)$, up to multiplicative constants. As far as $\tilde{V}^\varepsilon(t)$ is concerned, the argument is a little more tricky. Indeed, we must write $\tilde{V}^\varepsilon(t)$ according to (2.4), then estimate

$$A^-|\boldsymbol{v}_t| - B \le \left|\nabla T_t^\varepsilon \cdot \boldsymbol{v}_t + \frac{\partial T^\varepsilon}{\partial t}\right| \le A^+|\boldsymbol{v}_t| + B,$$

for ε small enough, where the constants $A^\pm$ are as close to 1 as we want and the constant B is as small as we want (this comes from $\nabla T_t^\varepsilon = \mathrm{id} + O(\varepsilon)$ and $\partial T^\varepsilon/\partial t = O(\varepsilon)$), and get

$$A^-\tilde{V}^0 - B \le \tilde{V}^\varepsilon \le A^+\tilde{V}^0 + B.$$

The assumption $V \ge V_0 > 0$ allows us to infer from these inequalities that also $\tilde{V}^\varepsilon$ may be estimated both from above and below by V up to multiplicative constants. Finally, by the estimates in Lemmas 2.2 and 2.3, we bound the whole pointwise derivative by $CF^\alpha V^\beta$ since we have

$$\frac{dF}{d\varepsilon} \le CF, \qquad \frac{d\tilde{V}}{d\varepsilon} \le C(V+1) \le C\left(1+\frac{1}{V_0}\right)V,$$

where the last inequality too follows from $V \ge V_0$. Since $L^\alpha_q V^\beta$ is integrable on $[0,1]$, we may differentiate under the integral sign and get

$$\frac{d}{d\varepsilon}\mathfrak{F}(\mu^\varepsilon)|_{\varepsilon=0} = \int_0^1 \left(\alpha F(t)^{\alpha-1}\frac{dF}{d\varepsilon}\bigg|_{\varepsilon=0}\tilde{V}(t)^\beta + \beta F(t)^\alpha \tilde{V}(t)^{\beta-1}\frac{d\tilde{V}}{d\varepsilon}\bigg|_{\varepsilon=0}\right)dt.$$

The result follows when we replace the derivatives in ε by the explicit expressions we computed in Lemmas 2.2 and 2.3. □

REMARK 1. If $\beta = 1/p$ and μ is a minimizer, it is always possible to get the lower bound $V \ge V_0$ by reparametrizing in time, for instance by choosing the constant speed parametrization.

COROLLARY 2.5. *If μ minimizes $\mathfrak{F}$ with given boundary conditions μ_0 and μ_1, then its density u and its tangent field $\boldsymbol{v}$ satisfy*

$$\alpha(1-q)\int_0^1 F(t)^{\alpha-1}V(t)^\beta \int_\Omega (\nabla\cdot\xi)u_t^q\,d\mathcal{L}^d\,dt + p\beta\int_0^1 F(t)^\alpha V(t)^{\beta-1}\int_\Omega u_t|\boldsymbol{v}_t|^{p-2}\left(\nabla\xi\cdot\boldsymbol{v}_t + \frac{\partial\xi}{\partial t}\right)\cdot\boldsymbol{v}_t\,d\mathcal{L}^d\,dt = 0$$

for any vector field $\xi \in C_c^\infty(]0,1[\times\Omega;\mathbb{R}^d)$.

PROOF. It is sufficient to notice that when we create the modified curve μ^ε starting from the vector field ξ we do not change the initial and final points of the curve, so that the minimality implies that the derivative of $\tilde{\mathfrak{F}}(\mu^\varepsilon)$ at $\varepsilon = 0$ vanishes. □

2.3. *A system of PDEs*

The following theorem follows directly from the previous section.

THEOREM 2.6. *Let $\mu_0,\ \mu_1 \in W_p(\Omega)$ and let μ be a curve minimizing $\mathfrak{F}$ on $\Gamma(\mu_0, \mu_1)$, with a finite minimum value. Then, denoting by $u(t,\cdot)$ the density of μ_t and by $\boldsymbol{v}(t,\cdot)$ the tangent field to the curve μ, $(u, \boldsymbol{v})$ provide a weak (distributional) solution of the system*

$$
(2.5)\qquad \begin{cases} H(t)\nabla u^q + K(t)\nabla\cdot(u|\boldsymbol{v}|^{p-2}\boldsymbol{v}\otimes\boldsymbol{v}) + \dfrac{d}{dt}(K(t)u|\boldsymbol{v}|^{p-2}\boldsymbol{v}) = 0 & \text{in } \Omega,\\ \dfrac{d}{dt}u + \nabla\cdot(\boldsymbol{v}u) = 0 & \text{in } \Omega,\\ u\boldsymbol{v}\cdot n = 0 & \text{on } \partial\Omega,\\ \lim\limits_{t\downarrow 0} u(t,\cdot)\mathcal{L}^d = \mu_0, \quad \lim\limits_{t\uparrow 1} u(t,\cdot)\mathcal{L}^d = \mu_1, \end{cases}
$$

where $H(t) = \alpha(1-q)F(t)^{\alpha-1}V(t)^{\beta}$ and $K(t) = p\beta F(t)^{\alpha}V(t)^{\beta-1}$.

Given (μ_0, μ_1), existence of minimizers is ensured whenever $q < 1 + 1/d$ or, for general q, under the assumption that $\mu_0 = u_0\mathcal{L}^d$ and $\mu_1 = u_1\mathcal{L}^d$ with $u_0, u_1 \in L^q(\Omega)$ (see [2]*), hence under these conditions existence of solutions to this system is ensured.*

It is interesting to rewrite the equations, make some formal simplification and look at some particular cases.

First we expand all the terms in the first equation of system (2.5), obtaining

$$
\begin{aligned}
(2.6)\qquad & H(t)\nabla u^q + K(t)(u|\boldsymbol{v}|^{p-2}v\cdot\nabla\boldsymbol{v} + \boldsymbol{v}|\boldsymbol{v}|^{p-2}\nabla\cdot(u\boldsymbol{v}) + u(\boldsymbol{v}\cdot\nabla|\boldsymbol{v}|^{p-2})\boldsymbol{v})\\
&\qquad + K(t)\left(\boldsymbol{v}|\boldsymbol{v}|^{p-2}\frac{d}{dt}u + u\frac{d}{dt}(\boldsymbol{v}|\boldsymbol{v}|^{p-2})\right) + \frac{d}{dt}K(t)u|\boldsymbol{v}|^{p-2}\boldsymbol{v} = 0.
\end{aligned}
$$

Notice that this is always a vector equation, i.e. a system itself, consisting of d equations with $d+1$ unknown functions (the components of $\boldsymbol{v}$ and the density u). This system is then completed by the continuity equation. As usual, by $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ we mean the vector whose i-th component is $\sum_j \boldsymbol{v}_j\partial\boldsymbol{v}_i/\partial x_j$.

A formal simplification in (2.6) may be done: in fact there is a term $(K(t)\boldsymbol{v}|\boldsymbol{v}|^{p-2})(du/dt + \nabla\cdot(u\boldsymbol{v}))$ that might be removed by using the continuity equation. This is actually possible only under extra regularity assumptions on K and $\boldsymbol{v}$ (it consists in testing the continuity equation against the product $K(t)\boldsymbol{v}|\boldsymbol{v}|^{p-2}$ which is not in general C^1 or regular enough). Anyway, after this formal simplification, (2.6) becomes

$$
\begin{aligned}
(2.7)\qquad & H(t)\nabla u^q + K(t)(u|\boldsymbol{v}|^{p-2}\boldsymbol{v}\cdot\nabla\boldsymbol{v} + u(\boldsymbol{v}\cdot\nabla|\boldsymbol{v}|^{p-2})\boldsymbol{v})\\
&\qquad + K(t)u\frac{d}{dt}(\boldsymbol{v}|\boldsymbol{v}|^{p-2}) + \frac{d}{dt}K(t)u|\boldsymbol{v}|^{p-2}v = 0.
\end{aligned}
$$

Notice that in the case $\beta = 1/p$ we can reparametrize in time the solution and there are several possible parametrization choices that present some advantages. For instance, we could choose a parametrization so that $K(t)$ is constant, to get rid of the final derivative in time. This choice implies

$$
V(t) = \left(\frac{F^{\alpha}}{K}\right)^{p/(p-1)},
$$

and this, in the case of a bounded $|\Omega| < \infty$, is sufficient to have the lower bound $V \geq V_0$, since in this case F is bounded from below by a positive constant.

Another important fact to be noticed is that in (2.7) there is a common u factor. It is still formal, but in this way we should get, on $\{u > 0\}$,

$$H(t)u^{q-2}\nabla u + K(t)(|\boldsymbol{v}|^{p-2}\boldsymbol{v}\cdot\nabla\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla|\boldsymbol{v}|^{p-2})\boldsymbol{v}) + K(t)\frac{d}{dt}(\boldsymbol{v}|\boldsymbol{v}|^{p-2}) + \frac{d}{dt}K(t)|\boldsymbol{v}|^{p-2}\boldsymbol{v} = 0.$$

REMARK 2. One might wonder whether the solutions u are automatically positive a.e. in Ω for $t \in]0,1[$. This could be suggested by the fact that in the minimization problem spreadness of the density is favoured. In the next section we will see with explicit examples that this is not necessarily the case.

We finish this overview of simplifications of the system by looking at the simplest case, i.e. $p = q = 2$, $\alpha = 1$, $\beta = 1/2$, in the parametrization regime where K is constant. In this case we get

$$\begin{cases} -2V(t)^{1/2}\nabla u + K\left(\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \dfrac{d}{dt}\boldsymbol{v}\right) = 0 & \text{in } \{u > 0\}, \\ \dfrac{d}{dt}u + \nabla\cdot(\boldsymbol{v}u) = 0 & \text{in } \Omega, \\ u\boldsymbol{v}\cdot n = 0 & \text{on } \partial\Omega, \\ \lim\limits_{t\downarrow 0} u(t,\cdot)\mathcal{L}^d = \mu_0, \quad \lim\limits_{t\uparrow 1} u(t,\cdot)\mathcal{L}^d = \mu_1. \end{cases} \tag{2.8}$$

Under no constraint on the parametrization we have, instead,

$$\begin{cases} -2V(t)^{1/2}\nabla u + K(t)\left(\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \dfrac{d}{dt}\boldsymbol{v}\right) + \boldsymbol{v}\dfrac{dK}{dt} = 0 & \text{in } \{u > 0\}, \\ \dfrac{d}{dt}u + \nabla\cdot(\boldsymbol{v}u) = 0 & \text{in } \Omega, \\ u\boldsymbol{v}\cdot n = 0 & \text{on } \partial\Omega, \\ \lim\limits_{t\downarrow 0} u(t,\cdot)\mathcal{L}^d = \mu_0; \quad \lim\limits_{t\uparrow 1} u(t,\cdot)\mathcal{L}^d = \mu_1. \end{cases} \tag{2.9}$$

3. SELF-SIMILAR SOLUTIONS

3.1. *Homothetic solutions with fixed center*

In this section we look for particular solutions of system (2.5) which are self-similar in the sense that, for any t, the measure μ_t is the image under a homothety of a fixed measure. For simplicity we will consider only the case of system (2.9), i.e. with $p = q = 2$, and we assume that $0 \in \Omega$. The regularity of the candidate solutions we will propose will be enough to ensure that we can use this simplified system, instead of system (2.5). To start this analysis it is necessary to establish the following lemma.

LEMMA 3.1. *If μ is a curve in $W_2(\Omega)$ of the form $\mu_t = (T_{R(t)})_\sharp\bar{\mu}$ for a certain regular function $R : [0,1] \to]0,1]$ (where $T_R(x) = Rx$ is the multiplication by a factor R, hence a homothety), then its tangent field is given by $\boldsymbol{v}_t(x) = xR'(t)/R(t)$.*

PROOF. It is not difficult to prove that the field we have just defined solves the continuity equation and hence is a velocity field. Indeed, if $\phi \in C^1_c(\Omega)$, then

$$\frac{d}{dt}\int_\Omega \phi\, d\mu_t = \frac{d}{dt}\int_\Omega \phi(R(t)x)\, d\mu(x) = \int_\Omega \nabla\phi(R(t)x)\cdot R'(t)x\, d\mu(x)$$
$$= \int_\Omega \nabla\phi(R(t)x)\cdot \frac{R'(t)}{R(t)}R(t)x\, d\mu(x) = \int_\Omega \nabla\phi\cdot \boldsymbol{v}_t\, d\mu_t.$$

It remains to prove that $\boldsymbol{v}$ is actually the tangent velocity field, i.e. that its L^2 norm is minimal for a.e. t. This is achieved if we are able to prove that $\|\boldsymbol{v}_t\|_{L^2(\mu_t)} = |\mu|'(t)$ for a.e. $t \in [0,1]$. To do this, fix two times $t < t+h$ and see that the map $T(x) = xR(t+h)/R(t)$ is a transport between μ_t and μ_{t+h}. Since it is the gradient of the convex function $x \mapsto x^2R(t+h)/2R(t)$, it is actually the optimal transport according to the quadratic cost. Hence

$$\frac{W_2^2(\mu_t,\mu_{t+h})}{h^2} = \frac{1}{h^2}\int_\Omega \left(\frac{R(t+h)}{R(t)}-1\right)^2 x^2\, d\mu_t(x) \to \int_\Omega \left(\frac{R'(t)}{R(t)}\right)^2 x^2\, d\mu_t(x).$$

Since this last quantity is exactly the norm of $\boldsymbol{v}_t$ in $L^2(\mu_t)$, this proves that $\boldsymbol{v}$ is the tangent field to the curve μ. □

REMARK 3. In the case $p \neq 2$ the same result is true, but one has to use the characterization of tangent velocity fields in terms of limits of gradients of smooth maps (see Proposition 8.4.5 of [1]).

A first result we prove is the following:

THEOREM 3.2. *If $(u,\boldsymbol{v})$ is a self-similar solution of system* (2.5) *with u Lipschitz continuous, then necessarily u is of the form*

$$u(t,x) = (A_t - B_t|x|^2)\vee 0 \quad \textit{for suitable coefficients } A_t,\, B_t > 0.$$

PROOF. We look at the equation (2.7) with $p = q = 2$, which is valid on $\{u > 0\}$, and we freeze time, i.e. we look at the resulting space equation for fixed t. We use the fact that $\boldsymbol{v}$ is of the form $\boldsymbol{v}_t(x) = c_t x$, which implies that all the terms $\boldsymbol{v}$, $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ and $d\boldsymbol{v}/dt$ are of the same form. This easily implies that also ∇u is of the same form. Hence, at time t, on $\{u > 0\}$, we have $u(x) = A_t - B_t x^2$, where a priori B_t could also be negative. However, we can prove that B_t cannot be negative. In fact, if it were, and if Ω were a convex unbounded domain, then u could not be the density of a probability measure. On the other hand, one can easily see that on bounded convex domains Ω self-similar solutions must vanish on $\partial\Omega$, otherwise we should get a jump of the density at the boundary of $\{u > 0\}$ when rescaling, but u was supposed to be Lipschitz (except in the case that the solution is constant in time). This implies that also in the case of Ω bounded the coefficient B_t must be positive. For the same continuity reason we see that the region $\{u > 0\}$ must agree with $\Omega \cap \{A_t - B_t x^2 > 0\}$ in order to have continuity of u, and this proves the formula. □

REMARK 4. A similar result could be obtained for generic Wasserstein spaces with exponent $p > 1$ any self-similar solution should be of the form $u(t,x) = (A_t - B_t|x|^p)\vee 0$.

THEOREM 3.3. *If $\bar\mu$ is a probability measure on Ω with density*

$$u(x) = A[(R^2 - |x|^2) \vee 0],$$

then for any regular and monotone function $R : [0,1] \to [0,1]$ the curve $\mu_t = (T_{R(t)})_\sharp \bar\mu$ is a solution to system (2.5) *together with its tangent field $\boldsymbol{v}$.*

PROOF. It is sufficient to check the first vector equation in (2.9). First we compute the correct constant A: we must have

$$1 = A \int_0^R (R^2 - r^2) d\omega_d r^{d-1}\, dr = A R^{d+2} \omega_d \frac{2}{d+2},$$

and hence $A = R^{-d-2}(d+2)/(2\omega_d)$. This allows us to compute the term $F(t)$:

$$F = A^2 \int_0^R (R^2 - r^2)^2\, d\omega_d r^{d-1}\, dr = R^{-d} \frac{2(d+2)}{(d+4)\omega_d}.$$

Then we compute V by recalling that $\boldsymbol{v}_t(x) = xR'(t)/R(t)$:

$$V = \left(\frac{R'}{R}\right)^2 A \int_0^R r^2 (R^2 - r^2)\, d\omega_d r^{d-1}\, dr = \frac{d}{d+4}(R')^2.$$

We must also compute $d\boldsymbol{v}/dt$ and $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$:

$$\frac{\partial \boldsymbol{v}}{\partial t} = x \frac{R''R - (R')^2}{R^2}, \qquad \nabla \boldsymbol{v} = \left(\frac{R'}{R}\right) I, \qquad \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \left(\frac{R'}{R}\right)^2 x.$$

We now compute

$$K(t) = F(t) V(t)^{-1/2} = R^{-d} |R'|^{-1} \frac{2(d+2)}{\sqrt{d(d+4)}\omega_d},$$

$$K'(t) = \operatorname{sign}(R')(-dR^{-d-1} - R^{-d}(R')^{-2} R'') \frac{2(d+2)}{\sqrt{d(d+4)}\omega_d}.$$

If we set $c = \operatorname{sign}(R') \frac{2(d+2)}{\sqrt{d(d+4)}\omega_d}$ then $K = cR^{-d}(R')^{-1}$ and $K' = c(-dR^{-d-1} - R^{-d}(R')^{-2}R'')$, but also $-2V^{1/2}\nabla u(x) = cdR'R^{-d-2}x$. Inserting everything in the equation we must check that

$$dR'xR^{-d-2} + R^{-d}(R')^{-1} x \frac{R''}{R} - (dR^{-d-1} + R^{-d}(R')^{-2}R'') x \frac{R'}{R} = 0.$$

The proof is complete as this last equation is (miraculously enough) always satisfied. □

REMARK 5. By a similar proof we can show that, for $p \neq 2$, if $\bar\mu$ has a density of the form $u(x) = A[(R^p - |x|^p) \vee 0]$, then μ gives rise to a self-similar solution.

REMARK 6. This kind of self-similar solutions can join two different probability measures which are homothetic, and in particular arrive at the Dirac mass δ_0. Moreover, it is not in general possible to link a measure to δ_0 by a curve with finite energy: in [2], conditions to ensure this possibility are provided, but in general they are not satisfied in the case $q = 2$.

3.2. *Moving self-similar solutions*

We have characterized all the self-similar solutions which link two homothetic probability measures. It is however interesting to look also at the moving self-similar solutions, i.e. at solutions obtained by homotheties and translations together.

In this case we consider a reference measure $\bar\mu$ and we look for solutions of the form $(T^t)_\sharp\bar\mu$, where $T^t(x) = R(t)x + \bar x(t)$. It is not difficult to replace Lemma 3.1 with the following:

LEMMA 3.4. *If μ is a curve of the form $\mu_t = (T^t)_\sharp\bar\mu$, then its tangent field is given by*

$$\boldsymbol{v}_t(x) = \frac{R'(t)}{R(t)}(x - \bar x(t)) + \bar x'(t).$$

PROOF. The result may be proved very similarly to Lemma 3.1: it is sufficient to check the continuity equation

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi(R(t)x + \bar x(t))\,d\mu(x) &= \int_\Omega \nabla\phi(R(t)x + \bar x(t))\cdot(R'(t)x + \bar x'(t))\,d\mu(x)\\ &= \int_\Omega \nabla\phi(R(t)x + \bar x(t))\cdot\frac{R'(t)}{R(t)}(R(t)x + \bar x'(t))\,d\mu(x) = \int_\Omega \nabla\phi\cdot\boldsymbol{v}_t\,d\mu_t,\end{aligned}$$

and then to check the optimality of the norm by the fact that the map

$$x \mapsto \frac{R(t+h)}{R(t)}(x - \bar x(t)) + \bar x(t+h)$$

transports μ_t to μ_{t+h} and is optimal, and that

$$\frac{1}{h^2}\int_\Omega\left(\frac{R(t+h)}{R(t)}(x - \bar x(t)) + \bar x(t+h) - x\right)^2 d\mu_t(x)$$

converges to

$$\int_\Omega\left(\frac{R'(t)}{R(t)}(x - \bar x(t)) + \bar x'(t)\right)^2 d\mu_t(x) = \|\boldsymbol{v}_t\|_{L^2(\mu_t)}. \qquad \square$$

For computational simplicity we consider moving self-similar solutions only under a special reparametrization.

THEOREM 3.5. *If $\bar\mu$ is a probability measure on Ω with density*

$$u(x) = A[(R^2 - |x|^2)\vee 0]$$

and $\bar x(0)$, $\bar x(1) \in \Omega$ are assigned, a curve $\mu_t = (T^t)_\sharp\bar\mu$, parametrized so that $K = FV^{-1/2}$ is constant, is a moving self-similar solution (solving system (2.8) *together with its own tangent field) if and only if the vector x moves on the straight line segment from $\bar x(0)$ to $\bar x(1)$ with constant speed and R is a strictly concave function of t. This means*

$$\bar x'' = 0, \qquad R^{2d}(d(R')^2 + (d+4)(\bar x')^2) \text{ is constant and } R \text{ strictly concave.}$$

PROOF. We only need to check under which conditions the first equation is satisfied. We re-write in this case the quantity considered in Theorem 3.3: first we compute

$$u(x) = A[(R^2 - |x-\bar{x}|^2) \vee 0], \qquad A = \frac{(d+2)}{2R^{d+2}\omega_d}, \qquad \nabla u(x) = -\frac{(d+2)}{R^{d+2}\omega_d}(x-\bar{x}),$$
$$F = R^{-d}\frac{2(d+2)}{(d+4)\omega_d}, \qquad V = \frac{d}{d+4}(R')^2 + (\bar{x}')^2.$$

We have used the fact that u_t is symmetric around $\bar{x}(t)$ and hence there is no mixed term $(x-\bar{x}(t))\cdot\bar{x}'(t)$ in computing $V(t)$. Then we go on with $d\boldsymbol{v}/dt$ and $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$:

$$\frac{\partial \boldsymbol{v}}{\partial t} = (x-\bar{x})\frac{R''R - (R')^2}{R^2} - \bar{x}'\frac{R'}{R} + \bar{x}'', \qquad \nabla\boldsymbol{v} = \left(\frac{R'}{R}\right)I,$$
$$\boldsymbol{v}\cdot\nabla\boldsymbol{v} = \left(\frac{R'}{R}\right)^2(x-\bar{x}) + \frac{R'}{R}\bar{x}', \qquad \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = (x-\bar{x})\frac{R''}{R} + \bar{x}''.$$

Then we look at the condition to have $K'(t) = 0$, which is equivalent to $F^{-2}V$ being constant, and thus $R^{2d}(d(R')^2 + (d+4)(\bar{x}')^2)$ must be constant. Assuming K to be constant we try to satisfy the equation, and we write it in the following form that we can obtain after multiplying by $V^{1/2}$:

$$-2V\nabla u + F\left(\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2}\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) = 0.$$

This equation becomes

$$2\left(\frac{d}{d+4}(R')^2 + (\bar{x}')^2\right)\frac{(d+2)}{\omega_d R^{d+2}}(x-\bar{x}(t)) + R^{-d}\frac{2(d+2)}{(d+4)\omega_d}((x-\bar{x})\frac{R''}{R} + \bar{x}'') = 0.$$

To satisfy this equation it is necessary and sufficient that the two parts, the one involving $x-\bar{x}$ and the other with $\bar{x}''$, both vanish. After simplifying we get

$$R^{-2}(d(R')^2 + (d+4)(\bar{x}')^2) + \frac{R''}{R} = 0, \qquad \bar{x}'' = 0.$$

Hence we must have $\bar{x}(t) = (1-t)\bar{x}(0) + t\bar{x}(1)$ and $\bar{x}'(t) = e = \bar{x}(1) - \bar{x}(0)$. Now we recall that $R^{2d}(d(R')^2 + (d+4)(\bar{x}')^2)$ was assumed to be constant and so $d(R')^2 + (d+4)(\bar{x}')^2 = CR^{-2d}$. Hence we get $R'' = -CR^{-2d-1}$. Thus, u is a moving self-similar solution if and only if the following conditions simultaneously hold:

$$\begin{cases} d(R')^2 + (d+4)e^2 = CR^{-2d} & \text{for a certain } C, \\ R'' = -CR^{-2d-1} & \text{for the same } C, \\ \bar{x}(t) = \bar{x}(0) + te. \end{cases}$$

By differentiating the first equation we get $2dR'R'' = -2dCR^{-2d-1}R'$ and hence the second is automatically satisfied, provided we can ensure that $R' \neq 0$ a.e. This means that R being strictly concave is sufficient (it is not possible to have more than one time instance where R' vanishes), but it is also necessary from the second equation. The result is thus proved. □

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Scuola Normale Superiore
Piazza dei Cavalieri 7
56126 PISA, Italy
l.ambrosio@sns.it
f.santambrogio@sns.it