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Algebraic geometry. — *Canonical surfaces in* $\mathbb{P}^4$ *with* $p_g = p_a = 5$ *and* $K^2 = 11$, by
CHRISTIAN BÖHNING, communicated on 23 June 2006.

ABSTRACT. — In [Böh] a structure theorem for Gorenstein algebras in codimension 2 was obtained. In Section 1 of this article we give a geometric application and prove a structure theorem for good birational canonical projections of regular surfaces of general type with $p_g = 5$ to $\mathbb{P}^4$ (Theorem 1.6). In Section 2 we show how this can be used to analyze the moduli space of canonical surfaces with $q = 0$, $p_g = 5$ and $K^2 = 11$ (Theorem 2.4).

KEY WORDS: Surfaces of general type; canonical surfaces; canonical ring; Gorenstein algebras; canonical projections.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 14J39, 13D02.

0. INTRODUCTION

Open problems in the theory of algebraic surfaces of general type can roughly be put into two classes: first, there are questions concerning the existence of surfaces with prescribed invariants (the "geography") and, secondly, the problem of describing their moduli spaces and canonical resp. pluricanonical models has to be addressed (the "botany"). We want to consider the aforementioned issues, especially the latter, in the special case of surfaces with geometric genus $p_g = 5$, more precisely for canonical surfaces in $\mathbb{P}^4$ (i.e. those for which the 1-canonical map is a birational morphism onto the image in $\mathbb{P}^4$) with $q = 0$ and $p_g = 5$.

A Gorenstein algebra in codimension 2 is a finite R-algebra B (R some "nice" base ring) with $B \cong \mathrm{Ext}^2_R(B, R)$, possibly up to some twist if the base ring is graded (cf. Section 1 below for precise definitions).

Given a regular surface S of general type with canonical map $S \to Y \subset \mathbb{P}^4$ a birational morphism, the canonical ring $\mathcal{R} = \bigoplus_{m\geq 0} H^0(S, \mathcal{O}_S(mK))$ is a codimension 2 Gorenstein algebra over $\mathcal{A} = \mathbb{C}[x_0, \dots, x_4]$, the homogeneous coordinate ring of $\mathbb{P}^4$. In Section 1 we prove a converse to this assertion based on a structure theorem for Gorenstein algebras proven in [Böh]: Starting from some $\mathcal{A}$*-module* $\mathcal{R}$ with a Gorenstein symmetric resolution of length 2 such that some mild depth condition on a certain ideal of minors associated to a presentation matrix of $\mathcal{R}$ as $\mathcal{A}$-module is satisfied, one finds that $\mathcal{R}$ is automatically a Gorenstein algebra in codimension 2 (the point being, of course, that $\mathcal{R}$ has a ring structure). Then we can set $X = \mathrm{Proj}\,\mathcal{R}$, which turns out to be the canonical model of a surface S as above, provided X has only rational double points as singularities. This is our Theorem 1.6 on good birational canonical projections of regular surfaces of general type with $p_g = 5$ to $\mathbb{P}^4$. It is a generalization to higher codimension of a structure theorem for Gorenstein algebras in codimension 1 proven in [Cat2] for the purpose of studying the moduli of canonical surfaces in $\mathbb{P}^3$. The main difference is that within the codimension 1 setting the presence of a ring structure on $\mathcal{R}$ is equivalent to a *closed* condition on

Fitting ideals of a presentation matrix of $\mathcal{R}$ over $\mathcal{A}$, the so-called "ring condition" or "rank condition" or "condition of Rouché–Capelli", aptly abbreviated R.C., whereas in codimension 2 the ring condition is automatic, at least if we restrict attention to the case where the canonical image $Y \subset \mathbb{P}^4$ has isolated non-normal locus.

In Section 2 we use Theorem 1.6 to analyze canonical surfaces with $K^2 = 11$ and in particular we prove that "generically" the presentation matrices of the canonical rings $\mathcal{R}$ as $\mathcal{A}$-modules can be reduced to an explicit normal form (cf. Lemma 2.1; "generically" here means that if $\mathcal{A}_Y := \mathcal{A}/\mathrm{Ann}_{\mathcal{A}}\,\mathcal{R}$, then the subscheme of $\mathbb{P}^4$ defined by the zeroth Fitting ideal of $\mathcal{R}/\mathcal{A}_Y \cdot 1$ is a collection of three reduced points). The proof of Lemma 2.1 is an admittedly painstaking calculation, but it pays off immediately afterwards: Using Lemma 2.1 one can explicitly solve the Gorenstein symmetry condition imposed on the resolution of the canonical ring $\mathcal{R}$ for $K^2 = 11$, a problem raised already in [Cat3, after Remark 6.6]. We are then able to deduce that regular surfaces with $p_g = 5$ and $K^2 = 11$ whose canonical map is a birational morphism and whose canonical rings satisfy the above genericity assumption, form an irreducible unirational open set of dimension 38 inside their moduli space (Theorem 2.4).

One must mention that a theorem very similar to this last result was proven previously by Daniel Roßberg by completely different methods in his long article [Ross]: He constructs the canonical image Y as the degeneracy locus of a morphism between reflexive sheaves $\mathcal{F}$ and $\mathcal{G}$ with $\mathrm{rk}\,\mathcal{G} = \mathrm{rk}\,\mathcal{F} + 1$, and considers only those Y which are smooth except for a number of improper double points. He then deduces that the canonical surfaces with $p_g = 5$, $q = 0$, $K^2 = 11$ such that Y has only improper double points form a unirational open set of dimension 38 in their moduli space. Our approach is more algebraic and, in our opinion, of some interest (apart from its novelty) because it allows one to analyze also surfaces where Y could have more complicated singular locus, and it gives detailed information on the structure of the canonical rings.

We do not repeat here the history of the ideas underlying our approach since it can be found in [Cat3] and the introduction of [Böh].

Our commutative algebra notation agrees largely with [Ei], but the following point (which traditionally seems to cause notational confusion) should be noted: For $I \subset R$ an ideal in a Noetherian ring and M a finite R-module, we write $\mathrm{grade}(I, M)$ for the length of a maximal M-regular sequence contained in I ($= \min\{i : \mathrm{Ext}^i_R(R/I, M) \neq 0\}$), and also, if there is no risk of confusion, $\mathrm{grade}\,M := \mathrm{grade}(\mathrm{Ann}_R(M), R)$ and $\mathrm{grade}\,I := \mathrm{grade}(I, R)$. Furthermore if $R = (R, \mathfrak{m}, k)$ is a Noetherian local ring or graded ring with $\mathfrak{m}$ a unique maximal element among the graded proper ideals of R (e.g. a positively graded algebra over a field), we write $\mathrm{depth}\,R := \mathrm{grade}(\mathfrak{m}, R)$. This is in accordance with [B-He] and the terminology seems to go back to Rees.

Finally, I would like to thank Fabrizio Catanese for introducing me to the problem and for continuous stimulus and helpful suggestions.

1. The Structure Theorem for Good Birational Canonical Projections to $\mathbb{P}^4$

In this section we recall some facts for canonical surfaces in $\mathbb{P}^4$ needed in the following and prove the structure theorem 1.6.

DEFINITION 1.1. *Let S be a smooth surface and $\pi : S \to Y \subseteq \mathbb{P}^4$ a morphism given by a* 5*-dimensional base-point free linear subspace L of $H^0(S, \mathcal{O}_S(K))$ and such that π is birational onto its image Y in $\mathbb{P}^4$. Then Y is called a* canonical surface *in $\mathbb{P}^4$ (and π a* good birational canonical projection*).*

In the above situation, since K_S is nef, S is automatically a minimal model of a surface of general type.

Henceforth we assume that S is a *regular surface*, i.e. $q = h^1(S, \mathcal{O}_S) = 0$, basically because then the canonical ring $\mathcal{R} := \bigoplus_{n\geq 0} H^0(S, \mathcal{O}_S(nK))$ enjoys the following property which makes it convenient to study by homological methods:

PROPOSITION 1.2. *$\mathcal{R}$, viewed as a module over the homogeneous coordinate ring $\mathcal{A} = \mathbb{C}[x_0, \ldots, x_4]$ of $\mathbb{P}^4$ via π, is a Cohen–Macaulay (CM) module iff S is a regular surface.*

For a proof one can consult [Cil, Props. (1.1) and (5.1)].

On the other hand, the fact that π is a good birational canonical projection ($\mathcal{O}_S(K) \cong \pi^*\mathcal{O}_{\mathbb{P}^4}(1)$) implies that various remarkable duality statements hold for $\mathcal{R}$, which we shall frequently exploit and which can be best expressed in terms of properties of the minimal free resolution of $\mathcal{R}$. Precisely:

DEFINITION 1.3. *Let $R := k[x_1, \ldots, x_r]$ be a polynomial ring in r indeterminates over some field k, graded in the usual way, and let B be a graded R-algebra. Then B is said to be a* Gorenstein algebra of codimension c *(and with twist $d \in \mathbb{Z}$) over R if $B \cong \operatorname{Ext}^c_R(B, R(d))$ as B-modules.*

[The B-module structure on $\operatorname{Ext}^c_R(B, R(d))$ is induced from B by functoriality of $\operatorname{Ext}^c_R(\cdot, R(d))$: If $b \in B$ and $m_b : B \to B$ is multiplication by b on B, then the map $\operatorname{Ext}^c_R(m_b, R(d))$ is multiplication by b on $\operatorname{Ext}^c_R(B, R(d))$.]

THEOREM 1.4. *With the hypotheses and notation of Definition* 1.1*, $\mathcal{R}$ is a Gorenstein algebra of codimension* 2 *over $\mathcal{A}$ and as such has a minimal graded free resolution of the form*

$$\mathbf{R}_\bullet :\ 0 \to \bigoplus_{i=1}^{n+1} \mathcal{A}(-6 + r_i) \xrightarrow{\binom{-\beta^t}{\alpha^t}} \bigoplus_{j=1}^{n+1} \mathcal{A}(-6 + s_j) \oplus \bigoplus_{j=1}^{n+1} \mathcal{A}(-s_j)$$
$$\xrightarrow{(\alpha\ \beta)} \bigoplus_{i=1}^{n+1} \mathcal{A}(-r_i) \to \mathcal{R} \to 0.$$

PROOF (SKETCH). Setting $X := \operatorname{Proj} \mathcal{R}$, the canonical model of S, we see that π, being given by a base-point free linear subsystem of $|K_S|$, factors through X as in the diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \pi\ } & Y \subset \mathbb{P}^4 \\ & {\scriptstyle\kappa}\searrow \quad \nearrow{\scriptstyle\psi} & \\ & X & \end{array}$$

and ψ is a finite morphism onto Y. Hence by relative duality for finite morphisms (cf. e.g. [Lip, p. 48ff.]), $\psi_*\omega_X = \mathcal{H}om_{\mathcal{O}_Y}(\psi_*\mathcal{O}_X, \omega_Y)$, where $\omega_Y = \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{O}_Y, \omega_{\mathbb{P}^4})$ and ω_X are the Grothendieck dualizing sheaves of Y, X resp.; but $\mathcal{H}om_{\mathcal{O}_Y}(\psi_*\mathcal{O}_X, \omega_Y) = \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\psi_*\mathcal{O}_X, \omega_{\mathbb{P}^4})$ since Y has codimension 2 in $\mathbb{P}^4$ (cf. also [Har, p. 242]). Furthermore $\psi_*\omega_X = \tilde{\mathcal{R}}(1)$ (cf. [Cat2, p. 76, Prop. 2.7]) and $\psi_*\mathcal{O}_X = \tilde{\mathcal{R}}$. Thus we get

$$\tilde{\mathcal{R}} = \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\tilde{\mathcal{R}}, \mathcal{O}_{\mathbb{P}^4}(-6)). \tag{1}$$

Since $\mathcal{R}$ is CM we obtain a length 2 resolution

$$0 \to F_2 \to F_1 \to F_0 \to \mathcal{R} \to 0, \tag{2}$$

with $F_0,\ , F_1,\ F_2$ graded free $\mathcal{A}$-modules. Taking $\mathrm{Hom}_{\mathcal{A}}(\cdot, \mathcal{A}(-6))$ of (2) we get

$$0 \to F_0^\vee(-6) \to F_1^\vee(-6) \to F_2^\vee(-6) \to \mathrm{Ext}^2_{\mathcal{A}}(\mathcal{R}, \mathcal{A}(-6)) \to 0. \tag{3}$$

Since $\mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\tilde{\mathcal{R}}, \mathcal{O}_{\mathbb{P}^4}(-6))$ is the sheaf associated to $\mathrm{Ext}^2_{\mathcal{A}}(\mathcal{R}, \mathcal{A}(-6))$, and $\tilde{\mathcal{R}}$ the sheaf associated to $\mathcal{R}$, and we have resolutions (2) and (3) of length 2 over $\mathcal{A}$ for these two modules, it follows easily that $\mathrm{Ext}^2_{\mathcal{A}}(\mathcal{R}, \mathcal{A}(-6))$ equals the full module of sections of the sheaf $\mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\tilde{\mathcal{R}}, \mathcal{O}_{\mathbb{P}^4}(-6))$, and $\mathcal{R}$ the full module of sections of $\tilde{\mathcal{R}}$; thus from (1) we infer the isomorphism of $\mathcal{A}$-modules

$$\mathcal{R} = \mathrm{Ext}^2_{\mathcal{A}}(\mathcal{R}, \mathcal{A}(-6)), \tag{4}$$

which is also an isomorphism of $\mathcal{R}$-modules since it is functorial with respect to endomorphisms of $\mathcal{R}$ (which follows from the functoriality of the isomorphisms $\psi_*\omega_X = \mathcal{H}om_{\mathcal{O}_Y}(\psi_*\mathcal{O}_X, \omega_Y)$ and $\mathcal{H}om_{\mathcal{O}_Y}(\psi_*\mathcal{O}_X, \omega_Y) = \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\psi_*\mathcal{O}_X, \omega_{\mathbb{P}^4})$ above). The isomorphism (4) lifts to an isomorphism of minimal graded free resolutions (2) and (3). In particular, $\mathrm{rank}\, F_0 = \mathrm{rank}\, F_2$, and since $\mathrm{Ann}_{\mathcal{A}}\, \mathcal{R} \neq 0$ one has $\mathrm{rank}\, F_0 - \mathrm{rank}\, F_1 + \mathrm{rank}\, F_2 = 0$, whence there exists an integer n such that $\mathrm{rank}\, F_0 = \mathrm{rank}\, F_2 = n + 1$, $\mathrm{rank}\, F_1 = 2n + 2$. For the fact that now (2) can be symmetrized to give a resolution $\mathbf{R}_\bullet$ as in the statement of the theorem we refer to [Böh, Section 2] or [Gra, p. 938ff., Lemma 2.1 and Proposition 2.3]. □

Next, we certainly have $p_g(S) \geq 5$ for surfaces S as in Definition 1.1, and for simplicity we assume $p_g(S) = 5$ in what follows. As for K_S^2 of such surfaces, we list here:

- One can only expect to find canonical surfaces in $\mathbb{P}^4$ with $p_g = 5$ and $q = 0$ in the range $8 \leq K^2 \leq 54$. The lower bound follows from Castelnuovo's inequality $K^2 \geq 3p_g + q - 7$. The upper bound follows from the Bogomolov–Miyaoka–Yau inequality $K^2 \leq 3e(S)$ in combination with Noether's formula $K^2 + e(S) = 12(1 - q + p)$, where $e(S)$ is the topological Euler characteristic of S.
- For $K^2 = 8$ resp. 9 the solutions one gets are the complete intersections of type $(2, 4)$ resp. $(3, 3)$ (cf. [En, p. 284ff.]).
- Existence is known in the cases $K^2 = 10, 11, 12$; the case $K^2 = 10$ is treated in [Cil], subsequently also in [Cat3, p. 42ff.] and [Ross, p. 108ff.], by approaches different in taste each time. Moreover, in the latter case one has a satisfactory picture of the moduli space of these surfaces; for $K^2 = 11, 12$ a partial description of the moduli spaces is in [Ross].

Therefore let us also assume $K^2 \geq 10$ henceforth.

For the case $p_g = 5,\ q = 0,\ K^2 \geq 10$, the numbers $n,\ r_i,\ s_i,\ i = 1, \dots, n+1$, appearing in the resolution $\mathbf{R}_\bullet$ of Theorem 1.4 are readily calculated; this is done in [Cil, p. 304, Prop. (5.3)] (cf. also [Cat3, p. 41, Prop. 6.2]):

THEOREM 1.5. *For a canonical surface in $\mathbb{P}^4$ with $q = 0,\ p_g = 5,\ K^2 \geq 10$ one has a resolution of the canonical ring $\mathcal{R}$,*

$$(5)\qquad \mathbf{R}_\bullet : 0 \to \mathcal{A}(-6) \oplus \mathcal{A}(-4)^n \xrightarrow{\binom{-\beta^t}{\alpha^t}} \mathcal{A}(-3)^{2n+2} \xrightarrow{(\alpha\ \beta)} \mathcal{A} \oplus \mathcal{A}(-2)^n \to \mathcal{R} \to 0,$$

where $n := K^2 - 9$.

However, what is important here is that there is a converse to the story told so far, on which rests the analysis of canonical surfaces done in this work:

THEOREM 1.6. *Let $\mathcal{R}$ be some finite $\mathcal{A}$-module with minimal graded free resolution as in* (5). *Write $A := (\alpha\ \beta)$, $A' := A$ with first row erased, $I_n(A') =$ Fitting ideal of $n \times n$ minors of A', and assume*

- depth $I_n(A') \geq 4$.

Then $\mathcal{R}$ is a Gorenstein algebra; assume furthermore that

- $X :=$ Proj $\mathcal{R}$ *has only rational double points as singularities.*

Endow $Y :=$ Supp$(\mathcal{R}) \subseteq \mathbb{P}^4$ with its scheme structure given by $\mathcal{I}_Y = \mathrm{Ann}_{\mathcal{A}}\,\mathcal{R}$. Then X is the canonical model of a surface S of general type with $q = 0,\ p_g = 5,\ K^2 = n + 9$. More precisely, if we write $\mathcal{A}_Y := \mathcal{A}/\mathcal{I}_Y$, the morphism $\psi : X \to Y \subset \mathbb{P}^4$ induced by the inclusion $\mathcal{A}_Y \subset \mathcal{R}$ is a finite birational morphism, and is part of a diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \pi\ } & Y \subset \mathbb{P}^4 \\ & {\scriptstyle\kappa}\searrow \quad \nearrow{\scriptstyle\psi} & \\ & X & \end{array}$$

where S is the minimal desingularization of X, κ is the contraction morphism contracting exactly the (-2)-curves of S to rational double points on X, and the composite $\pi := \psi \circ \kappa$ is a birational morphism with $\pi^*\mathcal{O}_{\mathbb{P}^4}(1) = \mathcal{O}_S(K_S)$ (i.e. it is 1-canonical for S). Moreover Y is a canonical surface in $\mathbb{P}^4$.

PROOF. The fact that $\mathcal{R}$ is a Gorenstein algebra (commutative, associative with $1 \in \mathcal{R}$) follows from [Böh, Theorem 2.5].

Note that since the ideal of $(n+1) \times (n+1)$ minors of A, $I_{n+1}(A)$ (i.e. the zeroth Fitting ideal of $\mathcal{R}$), and $\mathrm{Ann}_{\mathcal{A}}\,\mathcal{R}$ have the same radical, the Eisenbud–Buchsbaum acyclicity criterion (cf. [Ei, Thm. 20.9, p. 500]) gives grade $I_{n+1}(A) =$ grade $\mathrm{Ann}_{\mathcal{A}}\,\mathcal{R} = \mathrm{codim}_{\mathcal{A}}\,\mathrm{Ann}_{\mathcal{A}}\,\mathcal{R} \geq 2$, whereas also grade $\mathcal{R} \equiv \mathrm{grade}(\mathrm{Ann}_{\mathcal{A}}\,\mathcal{R}, \mathcal{A}) \leq \mathrm{projdim}_{\mathcal{A}}\,\mathcal{R} = 2$ (cf. e.g. [B-He, p. 25]), whence Y, defined by the annihilator ideal $\mathrm{Ann}_{\mathcal{A}}\,\mathcal{R} \subset \mathcal{A}$, is in fact a two-dimensional algebraic subscheme of $\mathbb{P}^4$. Note in particular that Y might *a priori* very well be reducible or non-reduced with the subscheme structure given by $\operatorname{Ann}_{\mathcal{A}} \mathcal{R}$. We deliberately want to avoid assuming anything like $\operatorname{Ann}_{\mathcal{A}} \mathcal{R}$ being prime or radical, since this is awkward to verify (in machine computations of explicit examples).

$\mathcal{R}$ is CM because $\operatorname{grade} \mathcal{R} = \operatorname{projdim}_{\mathcal{A}} \mathcal{R}$ and thus $\mathcal{R}$ is a perfect module (cf. [B-He, p. 59, Thm. 2.1.5]). Next, the morphism $\psi : X \to Y$ induced by the inclusion $\mathcal{A}_Y \subset \mathcal{R}$ is finite since $\mathcal{R}$ is a finite $\mathcal{A}_Y$-module and thus $\psi_* \mathcal{O}_X = \tilde{\mathcal{R}}$ is a finite $\mathcal{O}_Y$-module over any affine open of Y. Now A' is a presentation matrix of $\mathcal{R}/(\mathcal{A}_Y \cdot 1)$, whence by Fitting's lemma, $I_n(A') \subset \operatorname{Ann}_{\mathcal{A}}(\mathcal{R}/\mathcal{A}_Y)$ and $(I_n(A') \cdot \mathcal{A}_Y)\mathcal{R} \subset \mathcal{A}_Y$. Since $\mathcal{R}$ is Cohen–Macaulay, we have $\operatorname{grade}(I_n(A'), \mathcal{R}) = \dim \mathcal{R} - \dim(\mathcal{R}/I_n(A')\mathcal{R}) \geq 2$, and there is thus a (homogeneous) element $d \in I_n(A') \cdot \mathcal{A}_Y \subset \mathcal{A}_Y$ which is a non-zerodivisor on $\mathcal{R}$ (therefore also on $\mathcal{A}_Y$) with $d\mathcal{R} \subset \mathcal{A}_Y \subset \mathcal{R}$. Thus one gets

$$\mathcal{R}[d^{-1}] = \mathcal{A}_Y[d^{-1}]. \tag{6}$$

(By the way, this shows that the algebra structure on $\mathcal{R}$ is uniquely determined since it is a subalgebra of $\mathcal{A}_Y[d^{-1}]$.) From (6) one sees that ψ gives an isomorphism on the degree zero components of the total rings of quotients of $\mathcal{R}$ resp. $\mathcal{A}_Y$ obtained by inverting all homogeneous non-zerodivisors in these rings; thus ψ is birational.

If $\tilde{\mathcal{R}}$ denotes the sheaf on Y associated to $\mathcal{R}$, we have the picture

$$X = \mathbf{Spec}\, \tilde{\mathcal{R}} \to Y \subset \mathbb{P}^4.$$

The fact that X has only rational double points as singularities implies that X is locally Gorenstein and the dualizing sheaf ω_X is invertible, $\omega_X = \mathcal{O}_X(K_X)$, where K_X is an associated (Cartier) divisor. Moreover, if ω_Y is the dualizing sheaf for Y, we have $\psi^! \omega_Y = \omega_X$ since ψ is finite. Moreover, by relative duality for the finite morphism ψ,

$$\psi_* \omega_X = \psi_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \psi^! \omega_Y) = \mathcal{H}om_{\mathcal{O}_Y}(\psi_* \mathcal{O}_X, \omega_Y) = \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\tilde{\mathcal{R}}, \omega_{\mathbb{P}^4}),$$

whence, as $\tilde{\mathcal{R}} = \mathcal{E}xt^2(\tilde{\mathcal{R}}, \mathcal{O}_{\mathbb{P}^4}(-6))$, we get

$$\psi_* \omega_X = \tilde{\mathcal{R}}(1) = \psi_*(\psi^* \mathcal{O}_{\mathbb{P}^4}(1)). \tag{7}$$

Thus, sheafifying on $X = \mathbf{Spec}\, \tilde{\mathcal{R}}$, we deduce from (7) that $\psi^* \mathcal{O}_{\mathbb{P}^4}(1) = \omega_X$, i.e. the morphism ψ is canonical. Since $\mathcal{R}$ equals the full module of sections of the sheaf $\tilde{\mathcal{R}}$ and the morphism ψ is finite, we get

$$\mathcal{R} = \bigoplus_{m \geq 0} H^0(\mathbb{P}^4, \psi_*(\psi^* \mathcal{O}_{\mathbb{P}^4}(m))) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X)).$$

(The easiest way to see that $\mathcal{R}$ is equal to the full module of sections of $\tilde{\mathcal{R}}$ is perhaps to look at the exact sequence relating local and global cohomologies

$$0 \to H^0_{\mathfrak{m}}(\mathcal{R}) \to \mathcal{R} \to \bigoplus_m H^0(\mathbb{P}^4, \tilde{\mathcal{R}}(m)) \to H^1_{\mathfrak{m}}(\mathcal{R}) \to 0,$$

where $\mathfrak{m} = (x_0, \ldots, x_4)$ is the irrelevant maximal ideal of $\mathcal{A}$, and to note that since $\mathcal{R}$ is Cohen–Macaulay, $\operatorname{depth} \mathcal{R} = \dim \mathcal{R} = 3$ and thus $H^0_{\mathfrak{m}}(\mathcal{R}) = H^1_{\mathfrak{m}}(\mathcal{R}) = 0$.)

By our assumption that X has only rational double points as singularities, we see that our X is a disjoint union of normal projective surfaces (reduced and irreducible) with only rational double points as singularities such that ω_X is ample on each irreducible component: In fact, as $\mathcal{R}$ is a Cohen–Macaulay ring, X is pure two-dimensional by Macaulay's unmixedness theorem (cf. [Ei, Cor. 18.14 and Ex. 18.6]), reduced by assumption, and if two irreducible components met, then removing all the rational double points would disconnect them; this contradicts Hartshorne's connectedness theorem because X is Cohen–Macaulay (cf. [Ei, Thm. 18.12]).

Let S be the surface obtained by passing to the minimal desingularization of (each connected component of) X: It comes equipped with a morphism $\kappa : S \to X$ which contracts exactly the curves with self-intersection -2 on S. Let $\pi := \psi \circ \kappa$ be the composition. Clearly it is birational onto the image Y. Then the assumption that X has only rational double points as singularities implies that also $\pi^*\mathcal{O}_{\mathbb{P}^4}(1) = \omega_S$ and

$$\mathcal{R} = \bigoplus_{m \geq 0} H^0(S, \mathcal{O}_S(mK_S)).$$

Now we are almost done: It only remains to see that Y is irreducible and to calculate the invariants of S (which might still be a union of several connected components). This can be done in one stroke now:

The invariants $p_g(S)$, $q(S)$, K_S^2 are immediately found from the resolution (5): If S had several connected components, their geometric genera would have to add up to $p_g(S) = \mathcal{R}_1 = 5$. This is impossible since the morphism π is 1-canonical and birational onto $Y \subset \mathbb{P}^4$.

For the higher plurigenera one has $P_m = \binom{m}{2}K_S^2 + \chi(\mathcal{O}_S)$, $m \geq 2$, by Kodaira's formula (cf. [Bom, p. 185]); on the other hand, writing $\mathcal{R}_m$ for the mth graded piece of $\mathcal{R}$, and $\bigoplus_j \mathcal{A}(-a_{0,j}) := \mathcal{A} \oplus \mathcal{A}(-2)^n$, $\bigoplus_j \mathcal{A}(-a_{1,j}) := \mathcal{A}(-3)^{2n+2}$, $\bigoplus_j \mathcal{A}(-a_{2,j}) := \mathcal{A}(-6) \oplus \mathcal{A}(-4)^n$, one has

$$\dim_{\mathbb{C}} \mathcal{R}_m = \sum_{i=0}^{2} (-1)^i \sum_j \binom{m - a_{i,j} + 4}{4}$$

from the Hilbert resolution of $\mathcal{R}$ ($\binom{k}{l} = 0$ for $k < l$). Comparing these one concludes that $K^2 + 6 - q = 15 + n$, $3K^2 + 6 - q = 33 + 3n$, whence the invariants are the ones given in the theorem ($q = 0$ is clear since $\mathcal{R}$ is CM by Prop. 1.2). □

REMARK 1.7 (cf. [Cil, §4]). With the set-up of Theorem 1.6, $V(I_n(A')) =$ non-normal locus of Y. In fact, $\psi : X \to Y$ is the normalization map, and therefore the sheaf of ideals $\mathcal{A}nn_{\mathcal{O}_{\mathbb{P}^4}}(\psi_*\mathcal{O}_X/\mathcal{O}_Y) = \mathcal{A}nn_{\mathcal{O}_{\mathbb{P}^4}}(\tilde{\mathcal{R}}/\mathcal{O}_Y)$ defines the non-normal locus of Y. But since A' is a presentation matrix for $\mathcal{R}/\mathcal{A}_Y$, it is $\sqrt{\mathrm{Ann}_{\mathcal{A}}(\mathcal{R}/\mathcal{A}_Y)} = \sqrt{I_n(A')}$ (cf. e.g. [Ei, Prop. 20.6, p. 498]) and the assertion follows.

REMARK 1.8. If $Y \subset \mathbb{P}^4$ has only improper double points as singularities (i.e. points with tangent cone consisting of two planes spanning $\mathbb{P}^4$), then Y is sometimes said to have *ordinary singularities*. We state here (cf. [Cil, p. 306ff.]):

THEOREM 1.9. *Let $\pi : S \to Y \subset \mathbb{P}^4$ be a canonical surface with $q = 0$, $p_g = 5$. If Y has ordinary singularities, the number $\delta(Y) := \binom{K_S^2-8}{2}$ is the number of improper double points of Y (a very special case of the "double point formula of Severi").*

REMARK 1.10. Given a matrix $A = (\alpha\ \beta)$ with entries in the first row cubic forms on $\mathbb{P}^4$ and linear entries otherwise, satisfying $\alpha\beta^t = \beta\alpha^t$, in machine computations (e.g. with Macaulay 2, cf. [G-S]) one will usually check the following properties of A in order for the hypotheses of Theorem 1.6 to be satisfied: First, $\operatorname{codim} I_{n+1}(A) = 2$ and $\operatorname{codim} I_n(A') \geq 4$. Secondly, to check that $X = \operatorname{Proj} \mathcal{R}$ has only rational double points as singularities, it suffices to check that the subscheme of $\mathbb{P}^4$ defined by $I_{n+1}(A)$ is regular away from a finite number of improper double points (i.e. it is in particular reduced, whence coincides with the subscheme defined by $\operatorname{Ann}_{\mathcal{A}} \mathcal{R}$): Indeed, $\psi : X \to Y$ is the normalization map (the fact that $X = \operatorname{Proj} \mathcal{R}$ is normal follows in this case because X is isomorphic to Y away from a codimension 2 subset and hence is non-singular in codimension 1 and Cohen–Macaulay: thus it is normal by Serre's criterion). If one knows the singular points of Y explicitly, the fact that they are improper double points follows by a tangent cone computation which is in general quite feasible.

2. ANALYSIS OF THE CASE $K^2 = 11$

Let $\pi : S \to Y$ be a canonical surface in $\mathbb{P}^4$ with $q = 0,\ p_g = 5$, $K^2 = 11$. According to Theorem 1.5, one has a resolution

$$\mathbf{R}_\bullet : 0 \to \mathcal{A}(-6) \oplus \mathcal{A}(-4)^2 \xrightarrow{\binom{-\beta^t}{\alpha^t}} \mathcal{A}(-3)^6 \xrightarrow{(\alpha\ \beta)} \mathcal{A} \oplus \mathcal{A}(-2)^2 \to \mathcal{R} \to 0 \tag{1}$$

of the canonical ring $\mathcal{R}$ of S. We want to solve the ring condition (= Gorenstein symmetry condition) explicitly in this case. More notation:

$$A := (\alpha\ \beta) =: \left(\begin{array}{ccc|ccc} A_1 & A_2 & A_3 & B_1 & B_2 & B_3 \\ a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & b_6 \end{array}\right), \tag{2}$$

where the A_i, B_i, $i = 1, 2, 3$, are cubic forms, the a_j, b_j, $j = 1, \ldots, 6$, are linear forms; $A' := A$ with first row erased, $I_2(A)$, $I_2(A') :=$ Fitting ideals of 2×2 minors of A, A' respectively; $\mathcal{I}_Y := \operatorname{Ann}_{\mathcal{A}} \mathcal{R}$, $\mathcal{A}_Y := \mathcal{A}/\mathcal{I}_Y$. Furthermore we will assume that

(A) the zeroth Fitting ideal of $\mathcal{R}/\mathcal{A}_Y$, i.e. $I_2(A')$, defines *scheme-theoretically* three reduced points in $\mathbb{P}^4$.

This is of course equivalent to saying that the saturation of $I_2(A')$ is the homogeneous ideal of three reduced points in $\mathbb{P}^4$. This is a natural condition from the point of view of Theorem 1.9: if Y has ordinary singularities, then Y has exactly three improper double points; the three points are exactly the non-normal points of Y by Remark 1.7.

The following is the key result:

LEMMA 2.1. *If assumption* (A) *holds, then acting on the tableau in* (2) *with elements* $\begin{pmatrix} 1 & 0 \\ 0 & \varphi \end{pmatrix}$, $\varphi \in \mathrm{Gl}_2(\mathbb{C})$, *from the left, and elements of* $\mathrm{Sp}_6(\mathbb{C})$ *from the right, one can eventually* *obtain the normal form*

$$\tilde{A} = \left(\begin{array}{ccc|ccc} \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 & \tilde{B}_1 & \tilde{B}_2 & \tilde{B}_3 \\ 0 & \tilde{a}_2 & \tilde{a}_3 & 0 & \tilde{b}_2 & \tilde{b}_3 \\ \tilde{a}_4 & -\tilde{a}_2 & 0 & \tilde{b}_4 & -\tilde{b}_2 & 0 \end{array}\right) =: (\tilde{\alpha}\ \tilde{\beta}) \tag{3}$$

such that Gorenstein symmetry still holds: $\tilde{\alpha}\tilde{\beta}^t = \tilde{\beta}\tilde{\alpha}^t$. *The* $\tilde{a}_i$, $\tilde{b}_i$ *are linear forms such that*

$$V(\tilde{a}_2, \tilde{a}_3, \tilde{b}_2, \tilde{b}_3) = \{\textit{first non-normal point of } Y\},$$
$$V(\tilde{a}_4, \tilde{a}_2, \tilde{b}_4, \tilde{b}_2) = \{\textit{second non-normal point of } Y\},$$
$$V(\tilde{a}_4, \tilde{a}_3, \tilde{b}_4, \tilde{b}_3) = \{\textit{third non-normal point of } Y\}.$$

(The $\tilde{A}_j$, $\tilde{B}_j$, $j \in \{1, 2, 3\}$, *are of course cubics, linear combinations of the* A_j, B_j*.)*

Before giving the proof we make a preliminary observation:

REMARK 2.2. Write α_μ resp. β_ν for the μth resp. νth column of α resp. β. We want to make a list of some invertible row and column operations on A that preserve the Gorenstein symmetry:

(i) Elementary operations on rows: indeed, $\forall g = \begin{pmatrix} 1 & 0 \\ 0 & \varphi \end{pmatrix}$, $\varphi \in \mathrm{Gl}_2(\mathbb{C}) : \alpha\beta^t = \beta\alpha^t \Rightarrow (g\alpha)(g\beta)^t = (g\beta)(g\alpha)^t$.

(ii) For $\lambda \in \mathbb{C}$ and μ a fixed but arbitrary column index, adding $\lambda\beta_\mu$ to α_μ: Note that both sides of each of the equations $\sum_i \alpha_{hi}\beta_{li} = \sum_i \beta_{hi}\alpha_{li}$ are just changed by a summand $\lambda\beta_{h\mu}\beta_{l\mu}$. This operation is of course as well applicable with the rôles of α and β interchanged.

(iii) For $\lambda \in \mathbb{C}$ and $\mu,\ \nu$ column indices, adding $\lambda\beta_\nu$ to α_μ and at the same time adding $\lambda\beta_\mu$ to α_ν: Both sides of each of the equations $\sum_i \alpha_{hi}\beta_{li} = \sum_i \beta_{hi}\alpha_{li}$ change by a summand $\lambda(\beta_{h\nu}\beta_{l\mu} + \beta_{h\mu}\beta l\nu)$ [(ii) is thus a special case of (iii) with $\mu = \nu$]; the same operation also with the rôles of $\alpha,\ \beta$ interchanged.

(iv) For $\lambda \in \mathbb{C}$ and $\mu \neq \nu$ column indices, adding $\lambda\alpha_\nu$ to α_μ and simultaneously subtracting $\lambda\beta_\mu$ from β_ν: This is O.K. since it corresponds to changing the left side of $\sum_i \alpha_{hi}\beta_{li} = \sum_i \beta_{hi}\alpha_{li}$ by a summand $\lambda(\alpha_{h\nu}\beta_{l\mu} - \alpha_{h\nu}\beta_{l\mu}) = 0$, and the right side by a summand $\lambda(\beta_{h\mu}\alpha_{l\nu} - \beta_{h\mu}\alpha_{l\nu}) = 0$; the same operation also with the rôles of $\alpha,\ \beta$ interchanged.

(v) For $\mu \neq \nu$, interchanging columns $\alpha_\mu,\ \alpha_\nu$ and at the same time interchanging columns $\beta_\mu,\ \beta_\nu$, which clearly preserves the symmetry.

(vi) For a column index μ, multiplying column α_μ by -1 and then interchanging columns $-\alpha_\mu$ and β_μ (i.e. the substitution $\alpha_\mu \mapsto \beta_\mu,\ \beta_\mu \mapsto -\alpha_\mu$): Namely, $\sum_i \alpha_{hi}\beta_{li} = \sum_i \beta_{hi}\alpha_{li} \Leftrightarrow \sum_{i\neq\mu} \alpha_{hi}\beta_{li} - \beta_{h\mu}\alpha_{l\mu} = \sum_{i\neq\mu} \beta_{hi}\alpha_{li} - \alpha_{h\mu}\beta_{l\mu}$.

Call these operations (Op). Note that (Op)(ii)–(vi) correspond to multiplication on A from the right by symplectic 6×6 matrices. In fact, more systematically, one sees that since symplectic matrices $\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \in \mathrm{Gl}_{2n+2}(\mathbb{C})$, $S_1,\ S_2,\ S_3,\ S_4$ $(n+1) \times (n+1)$ matrices, can be characterized by the equations $S_1S_2^t = S_2S_1^t,\ S_3S_4^t = S_4S_3^t,\ S_1S_4^t - S_2S_3^t = I_{n+1}$, if $A = (\alpha\ \beta)$ is an $(n+1) \times (n+1)$ matrix with $\alpha\beta^t$ symmetric (as in Thm. 1.5) then also $(\alpha S_1 + \beta S_3)(\alpha S_2 + \beta S_4)^t$ is symmetric (this is also immediate because the symmetry condition can be rephrased as saying that, for each choice of homogeneous coordinate vector $(x_0 : \ldots : x_4)$ in $\mathbb{P}^4$, the rows of $(\alpha\ \beta)$ span an isotropic subspace for the standard symplectic form on $\mathbb{C}^{2n+2}$, and a matrix is symplectic iff its transpose is).

To make the proof of Lemma 2.1 more transparent we state the following auxiliary result separately; it will be invoked later a couple of times. The condition that $I_2(A')$ defines *scheme-theoretically* three reduced points was posed in order to have this result at our disposal. Otherwise (A) could be replaced by the requirement that Y has exactly three non-normal points.

LEMMA 2.3. *A matrix A' satisfying condition* (A) *cannot have a column with all entries equal to zero.*

PROOF. Assume that

$$A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & b_1 & b_2 & b_3 \\ 0 & a_5 & a_6 & b_4 & b_5 & b_6 \end{array}\right).$$

Then the three reduced points are defined scheme-theoretically by the vanishing of the maximal minors of a 2×5 matrix. Codimension 4 is the expected codimension for the degeneracy locus of a 2×5 matrix of linear forms; but then the degree of the subscheme defined by the vanishing of the maximal minors must be 5, a contradiction. (This is a special case of Porteous' formula; see [Mi, Lemma 1.1.1].) □

PROOF OF LEMMA 2.1. First a general remark: Given a matrix of linear forms, call an arbitrary linear combination of the rows with not all coefficients zero a *generalized row*. Then the locus where the rows are linearly dependent is the union, over all generalized rows, of the linear spaces cut out by the linear forms which are the entries of the generalized row. Therefore we can assume $A' = \left(\begin{array}{ccc|ccc} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & b_6 \end{array}\right)$ to be such that one of the non-normal points is given by the vanishing of the linear forms in the upper row of A', the second one by the vanishing of the linear forms in the lower row, and the third as the zero set of the linear forms obtained by adding up the two rows.

The rest of the proof is a game on the tableau A', using (Op) and symmetry, and deriving Koszul sequences from the fact that the rows of A' resp. their sum define three distinct points. To ease notation, we will treat the $a_i, b_i, i = 1, \ldots, 6$, and A' as dynamical variables. For clarity's sake, we will box certain assumptions in the course of the following argument.

Using (Op)(v)–(vi), then (iv) and finally (iii) one gets

$$A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & b_6 \end{array}\right). \tag{4}$$

$\boxed{a_4 = 0}$: This cannot occur thanks to Lemma 2.3.

$\boxed{a_4 \neq 0}$: Use (Op)(v), (vi), (iv), (iii) in this order to put a zero in place of b_6 $(a_4,\ a_5,\ a_6,\ b_5,\ b_6$ are dependent!):

$$A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & 0 \end{array}\right). \tag{5}$$

We claim that now $\boxed{a_2,\ b_1,\ b_2,\ b_3 \text{ are dependent}}$: For if they are independent we can also assume $a_4,\ a_5,\ a_6,\ b_5$ to be independent (otherwise interchange rows and use (Op)(v), (vi)). Symmetry gives $b_1a_4 + b_2a_5 + b_3a_6 + b_5 \cdot (-a_2) = 0$, which is a Koszul relation saying there are antisymmetric matrices $S,\ \tilde{S}$ of scalars such that

$$\begin{pmatrix} a_4 \\ a_5 \\ a_6 \\ b_5 \end{pmatrix} = \tilde{S} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ -a_2 \end{pmatrix}, \qquad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ -a_2 \end{pmatrix} = S \begin{pmatrix} a_4 \\ a_5 \\ a_6 \\ b_5 \end{pmatrix},$$

$\tilde{S}S = I$, and S, $\tilde{S}$ are invertible. Now interchange the 4th and 5th columns of A' and multiply by $\left(\begin{array}{c|cc} S^t & \cdots & 0 \\ \hline \vdots & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$ on the right. (This will in general destroy the symmetry but preserve the points that are defined by the rows of A' and their sum; this operation is only used to derive a contradiction.) The second row of the transformed matrix is then $(b_1, b_2, b_3, -a_2, b_4, 0)$, and one sees that it either defines $\emptyset$ or the same point as the first row, a contradiction because we assumed the points defined by the rows of A' to be distinct. Therefore $a_2,\ b_1,\ b_2,\ b_3$ are dependent.

We claim further that then $\boxed{a_2,\ b_2,\ b_3 \text{ are independent}}$. Suppose not. Since the possibility of a zero column was excluded by Lemma 2.3, we can then use (Op)(iii) and if necessary (vi) to get $A' = \left(\begin{array}{ccc|ccc} 0 & 0 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & 0 \end{array}\right)$. Here $a_3,\ b_1,\ b_2,\ b_3$ are independent. We have two cases:

(1) $a_4,\ a_5,\ a_6$ are independent. Then symmetry implies that there exist antisymmetric matrices $T,\ \tilde{T}$ of scalars such that

$$\begin{pmatrix} a_4 \\ a_5 \\ a_6 \end{pmatrix} = T \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \tilde{T} \begin{pmatrix} a_4 \\ a_5 \\ a_6 \end{pmatrix};$$

but then $T,\ \tilde{T}$ are invertible, contradicting the fact that a skew-symmetric matrix of odd size has determinant zero.

(2) $a_4,\ a_5,\ a_6$ are dependent. Since no zero column can occur (Lemma 2.3), we can use (Op)(iv) to write $A' = \left(\begin{array}{ccc|ccc} 0 & 0 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & 0 & b_4 & b_5 & 0 \end{array}\right)$; but the symmetry $a_4b_1 = -a_5b_2$ tells us that we are left with discussing the case $A' = \left(\begin{array}{ccc|ccc} 0 & 0 & a_3 & -a_5 & a_4 & b_3 \\ a_4 & a_5 & 0 & b_4 & b_5 & 0 \end{array}\right)$. But then the points defined by the second row and the sum of the rows coincide, or the linear forms in the sum of the rows define $\emptyset$, a contradiction.

Using the last two boxed assumptions and (Op)(iii) and then (iv), we can pass from the shape of A' in (5) to

$$A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & 0 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & 0 \end{array}\right). \tag{6}$$

Now we play the game again, but this time it is quicker. We claim:

$\boxed{a_5,\ a_6,\ b_5 \text{ are dependent}}$: If not, the symmetry $a_5b_2 + a_6b_3 + b_5(-a_2) = 0$ gives as above the existence of 3×3 invertible skew-symmetric matrices, a contradiction. But we also claim: $\boxed{a_5,\ b_5 \text{ are independent}}$: Otherwise we get, using (Op)(ii) and possibly (vi), $A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & 0 & b_2 & b_3 \\ a_4 & 0 & a_6 & b_4 & b_5 & 0 \end{array}\right)$, and using the symmetry $a_2b_5 = a_6b_3$, we must look at $A' = \left(\begin{array}{ccc|ccc} 0 & a_6 & a_3 & 0 & b_2 & b_5 \\ a_4 & 0 & a_6 & b_4 & b_5 & 0 \end{array}\right)$. But then either the points defined by the second row of A' and the sum of its rows resp. coincide, or the linear forms in the sum of the rows define $\emptyset$, a contradiction. Using the previous two boxed assumptions and (Op)(iv) and then (iii), we can pass from (6) to

$$A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & 0 & b_2 & b_3 \\ a_4 & a_5 & 0 & b_4 & b_5 & 0 \end{array}\right). \tag{7}$$

Invoking the symmetry $a_2b_5 = a_5b_2$ a last time, we are through:

$$A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & 0 & b_2 & b_3 \\ a_4 & -a_2 & 0 & b_4 & -b_2 & 0 \end{array}\right).$$

This concludes the proof of Lemma 2.1. □

We now proceed to apply the normal form obtained in Lemma 2.1 to solve the Gorenstein symmetry condition imposed on the matrix A and to analyze the moduli space of canonical surfaces $\pi : S \to Y \subset \mathbb{P}^4$ with $q = 0$, $p_g = 5$ and $K^2 = 11$. Combining what has been said so far with Theorem 1.6, we conclude the following: The datum (D) of

> *a matrix*
>
> $$A = \left(\begin{array}{ccc|ccc} A_1 & A_2 & A_3 & B_1 & B_2 & B_3 \\ 0 & a_2 & a_3 & 0 & b_2 & b_3 \\ a_4 & -a_2 & 0 & b_4 & -b_2 & 0 \end{array}\right)$$
>
> *with the $A_i,\ B_i,\ i = 1, \dots, 3$, cubic forms, $a_2,\ a_3,\ a_4,\ b_2,\ b_3,\ b_4$ linear forms on $\mathbb{P}^4$ satisfying the symmetry $A_2b_2 + A_3b_3 + B_2(-a_2) + B_3(-a_3) = 0$, $A_1b_4 + A_2(-b_2) + B_1(-a_4) + B_2a_2 = 0$, plus the open conditions that $I_2(A')$ defines scheme-theoretically three reduced points in $\mathbb{P}^4$, and that with $\mathcal{R} :=$ coker A, $\mathrm{Ann}_{\mathcal{A}}\,\mathcal{R}$ be of codimension 2 and $X = \mathrm{Proj}\,\mathcal{R}$ have only rational double points as singularities, modulo graded automorphisms of $\mathcal{A}\oplus\mathcal{A}(-2)^2$ resp. $\mathcal{A}(-3)^6$ (acting on A from the right resp. left) which preserve the normal form of A just described, modulo automorphisms of $\mathbb{P}^4$,*

is equivalent to the datum (D′) of

> *a canonical surface $\pi : S \to Y \subset \mathbb{P}^4$ with $q = 0,\ p_g = 5,\ K^2 = 11$ such that, denoting by $\mathcal{R}$ the canonical ring of S, the zeroth Fitting ideal of $\mathcal{R}/\mathcal{A}\cdot 1_{\mathcal{R}}$ as a module over the homogeneous coordinate ring of $\mathbb{P}^4$ defines scheme-theoretically three reduced points in $\mathbb{P}^4$, modulo isomorphism.*

We now describe the set of isomorphism classes of surfaces in (D′) inside their moduli space $\mathfrak{M}_{K^2,\chi} = \mathfrak{M}_{11,6}$:

First one notes that the symmetry condition in (D) amounts to the existence of skew-symmetric 4×4 matrices $P = (P_{ij})$ and $Q = (Q_{ij})$ of quadratic forms such that $(-B_2, -B_3, A_2, A_3)^t = P(a_2, a_3, b_2, b_3)^t$ and $(-B_1, -B_2, A_1, A_2)^t = Q(a_4, -a_2, b_4, -b_2)^t$. Of course there is some ambiguity in the choice of (P_{ij}), (Q_{ij}), for the Koszul complexes $\mathbf{K}_\bullet(a_2, a_3, b_2, b_3)$ and $\mathbf{K}_\bullet(a_4, -a_2, b_4, -b_2)$ associated to these regular sequences

$$\mathcal{A}(-4) \overset{d_3}{\rightarrowtail} \mathcal{A}(-3)^4 \xrightarrow{d_2} \mathcal{A}(-2)^6 \xrightarrow{d_1} \mathcal{A}(-1)^4 \xrightarrow{d_0} \mathcal{A} \twoheadrightarrow \mathcal{A}/(a_2, a_3, b_2, b_3),$$
$$\mathcal{A}(-4) \overset{d_3'}{\rightarrowtail} \mathcal{A}(-3)^4 \xrightarrow{d_2'} \mathcal{A}(-2)^6 \xrightarrow{d_1'} \mathcal{A}(-1)^4 \xrightarrow{d_0'} \mathcal{A} \twoheadrightarrow \mathcal{A}/(a_4, -a_2, b_4, -b_2)$$

show that e.g. the vector $(P_{ij})_{i<j} \in \mathcal{A}(-2)^6_4$ is only determined up to addition of $d_2(\underline{l})$ where $\underline{l} \in \mathcal{A}(-3)^4_4$ is a vector of linear forms, and two $\underline{l}$'s give rise to the same $(P_{ij})_{i<j}$ iff they differ by $d_3(s)$ where $s \in \mathcal{A}(-4)_4$ is a complex scalar. In other words, $\dim_{\mathbb{C}}(\ker(d_1)_4) = 19$ and effectively, instead of the $(P_{ij})_{i<j}$, one chooses $(\overline{P}_{ij})_{i<j} \in \mathcal{A}(-2)^6_4/d_2(\mathcal{A}(-3)^4_4/d_3(\mathcal{A}(-4)_4))$. Similarly for the (Q_{ij}).

Next it is clear that whereas now $\boxed{P_{24}}$ and $\boxed{Q_{13}}$ are subject to no further relations, for the $\{(P_{ij})_{i<j}\} - \{P_{24}\}$ and $\{(Q_{ij})_{i<j}\} - \{Q_{13}\}$ the relations

$$A_2 = -P_{13}a_2 - P_{23}a_3 + P_{34}b_3, \qquad B_2 = -P_{12}a_3 - P_{13}b_2 - P_{14}b_3,$$
$$A_2 = -Q_{14}a_4 + Q_{24}a_2 - Q_{34}b_4, \qquad B_2 = Q_{12}a_4 - Q_{23}b_4 + Q_{24}b_2$$

imply

$$Q_{14}a_4 + (P_{13} + Q_{24})(-a_2) + (-P_{23})a_3 + Q_{34}b_4 + P_{34}b_3 = 0, \tag{8}$$
$$Q_{12}a_4 + P_{12}a_3 + (-Q_{23})b_4 + (P_{13} + Q_{24})b_2 + P_{14}b_3 = 0. \tag{9}$$

We claim that we can assume that the sequences $(a_4, -a_2, a_3, b_4, b_3)$ and $(a_4, a_3, b_4, b_2, b_3)$ are both regular, whence (8) and (9) would be Koszul relations. According to the normal form of the matrix A given in (D), a_4, a_3, b_4, b_3 are independent (and define one of the non-normal points of Y). Assume both $-a_2$ and b_2 were expressible in terms of the latter. Then $V(a_2, a_3, b_2, b_3)$ and $V(a_4, a_3, b_4, b_3)$ would not give distinct points, a contradiction. Therefore at least one of the sequences $(a_4, -a_2, a_3, b_4, b_3)$ and $(a_4, a_3, b_4, b_2, b_3)$ is regular. But if one of them, $(a_4, -a_2, a_3, b_4, b_3)$ say, is not regular, then replacing a_2 with $a_2 + b_2$ (which corresponds to applying once (Op)(ii) to the matrix A) the sequence $(a_4, -(a_2 + b_2), a_3, b_4, b_3)$ will be regular. Similarly if $(a_4, a_3, b_4, b_2, b_3)$ fails to be regular.

Therefore considering (8) and (9) as Koszul relations, one gets two skew-symmetric 5×5 matrices $L = (L_{kl})$ and $M = (M_{kl})$ of linear forms such that

$$(Q_{14}, P_{13} + Q_{24}, -P_{23}, Q_{34}, P_{34})^t = L(a_4, -a_2, a_3, b_4, b_3)^t, \tag{10}$$
$$(Q_{12}, P_{12}, -Q_{23}, P_{13} + Q_{24}, P_{14})^t = M(a_4, a_3, b_4, b_2, b_3)^t. \tag{11}$$

Set $\underline{a} := (a_4, -a_2, a_3, b_4, b_3)$ and $\underline{a}' := (a_4, a_3, b_4, b_2, b_3)$. Again looking at Koszul complexes

$$\mathcal{A}(-5) \overset{D_4}{\rightarrowtail} \mathcal{A}(-4)^5 \xrightarrow{D_3} \mathcal{A}(-3)^{10} \xrightarrow{D_2} \mathcal{A}(-2)^{10} \xrightarrow{D_1} \mathcal{A}(-1)^5 \xrightarrow{D_0} \mathcal{A} \twoheadrightarrow \mathcal{A}/\underline{a},$$
$$\mathcal{A}(-5) \overset{D_4'}{\rightarrowtail} \mathcal{A}(-4)^5 \xrightarrow{D_3'} \mathcal{A}(-3)^{10} \xrightarrow{D_2'} \mathcal{A}(-2)^{10} \xrightarrow{D_1'} \mathcal{A}(-1)^5 \xrightarrow{D_0'} \mathcal{A} \twoheadrightarrow \mathcal{A}/\underline{a}'$$

one sees that whereas e.g. the (L_{kl}) are not unique, the $(\overline{L}_{kl})_{k<l} \in \mathcal{A}(-2)_3^{10}/D_2(\mathcal{A}(-3)_3^{10})$ are, and $\dim_{\mathbb{C}}(\ker(D_1))_3 = 10$. Likewise for the (M_{kl}).

Now equations (10) and (11) should be interpreted as saying that after one of P_{13} and Q_{24}, $\boxed{P_{13}}$ say, is chosen freely, the other P's and Q's in (10) and (11) are determined by $L,\ M,\ \underline{a},\ \underline{a}'$.

Furthermore one remarks that then the six $\boxed{(L_{kl})_{k<l,\, k\neq 2,\, l\neq 2}}$ and the six $\boxed{(M_{kl})_{k<l,\, k\neq 4,\, l\neq 4}}$ satisfy no further relations, but the other ones enter in the following relation resulting from equating the second resp. fourth vector components of (10) resp. (11):

$$(12) \qquad (M_{14} - L_{12})a_4 + (M_{24} + L_{23})a_3 + (M_{34} + L_{24})b_4 + (-M_{45} + L_{25})b_3 = 0.$$

The sequence (a_4, a_3, b_4, b_3) is regular by the characterization of the normal form of A given in (D). One therefore infers the existence of a skew-symmetric 4×4 matrix $S = (S_{rs})$ of complex scalars such that

$$(13) \qquad (M_{14} - L_{12}, M_{24} + L_{23}, M_{34} + L_{24}, -M_{45} + L_{25})^t = S(a_4, a_3, b_4, b_3)^t$$

and one notes that the (S_{rs}) are then uniquely determined from equation (12). Moreover upon choosing $\boxed{M_{14},\ M_{24},\ M_{34},\ M_{45}}$ arbitrarily, we can recover $L_{12},\ L_{23},\ L_{24},\ L_{25}$ from S and (a_4, a_3, b_4, b_3) using (13); and the six scalars $\boxed{(S_{rs})_{r<s}}$ are not subject to any other relation in the present set-up.

To get back to the study of the moduli space of surfaces in (D′), fit together the $a_t, b_t, t = 2, \dots, 4$, and all the boxed objects above into one big affine space of parameters:

$$\mathcal{P} = \left\{ \begin{array}{c|l} & t \in \{2,3,4\};\ k,l,\kappa,\lambda \in \{1,\dots,5\},\ \kappa < \lambda, \\ (a_t, b_t, P_{24}, Q_{13}, & k < l,\ k \neq 2,\ l \neq 2;\ r,s \in \{1,\dots,4\},\ r < s; \\ P_{13}, L_{kl}, M_{\kappa\lambda}, S_{rs}) & \text{and } P_{24},\ Q_{13},\ P_{13} \text{ quadratic, } a_t,\ b_t,\ L_{kl}, \\ & M_{\kappa\lambda} \text{ linear in the hom. coord. } (x_0 : \dots : x_4), \\ & S_{rs} \text{ complex scalars} \end{array} \right\}.$$

Counting one finds that there are 3 quadratic forms, 22 linear forms and 6 scalars in $\mathcal{P}$, depending on 45, 110 and 6 parameters respectively, whence we have $\mathcal{P} = \mathbb{A}^{161}$.

According to the above discussion, for each choice in an open set of $\mathcal{P}$ one gets a matrix A meeting the requirements in (D) and a ring $\mathcal{R}$ which is the canonical ring of a surface of general type S as in (D′). In other words, the parameter space for the canonical rings of the surfaces in (D′) is a projection of an open set of $\mathcal{P}$. One has to show that this open set is non-empty; this is possible, making general choices in $\mathcal{P}$ and verifying that one gets a matrix A fulfilling the open conditions in (D) e.g. with the help of a computer algebra package like Macaulay 2 (cf. [G-S]); it is convenient to choose

$$A' = \left(\begin{array}{ccc|ccc} 0 & x_2 & x_0 - x_4 & 0 & x_3 + x_4 & x_3 \\ x_3 - x_1 & -x_2 & 0 & -x_4 & -x_3 - x_4 & 0 \end{array}\right),$$

an example already showing up in [Ross]. It is easy to verify with Macaulay 2 that in this case the saturation of $I_2(A')$ is radical and defines the points $(1 : 0 : 0 : 0 : 0)$, $(0 : 1 : 0 : 0 : 0)$, $(0 : 0 : 1 : 0 : 0)$ in $\mathbb{P}^4$. One then completes the matrix A' to a matrix A by making generic choices for the cubic forms A_i, B_i, $i = 1, 2, 3$ (such that the symmetry holds). Another check with Macaulay 2 shows that such A will have the required properties listed in Remark 1.10. We give the necessary computations in the Appendix below.

In particular, by the preceding remark one finds that the surfaces in (D′) form an irreducible open set $\mathfrak{U}$ inside their moduli space, and $\mathfrak{U}$ is unirational (since $\mathcal{P}$ is rational).

Moreover, the set of matrices A as in (D) can be identified with an open subset

$$\mathfrak{V} \subset \mathbb{A}^d$$

where

$$\begin{aligned} d = {} & 161 \dim \mathcal{P} - 38(\dim_{\mathbb{C}}(\ker(d_1)_4) + \dim_{\mathbb{C}}(\ker(d_1')_4)) \\ & -20(\dim_{\mathbb{C}}(\ker(D_1)_3) + \dim_{\mathbb{C}}(\ker(D_1')_3)) = 103. \end{aligned}$$

To calculate the dimension of $\mathfrak{U}$ we note that we have three groups acting on the set $\mathfrak{V}$ of normal forms of matrices A in (D):

(1) The group Γ of graded automorphisms of the ring $\mathcal{A}$, i.e. $\Gamma = \mathrm{Gl}_5(\mathbb{C})$ and $\dim \Gamma = 25$.

(2) $\Lambda = \left\{ \text{graded auto.'s of } \mathcal{A} \oplus \mathcal{A}(-2)^2 \text{ of the form } \begin{pmatrix} s_1 & q_1 & q_2 \\ 0 & s_2 & 0 \\ 0 & 0 & s_2 \end{pmatrix} \right\}$

with $s_1, s_2 \in \mathbb{C}\backslash\{0\}$ and $q_1,\ q_2$ quadratic. Here $\dim \Lambda = 32$.

(3) $\Theta = \left\{ \text{group of invertible matrices } \begin{pmatrix} \lambda_1 & 0 & 0 & \mu_1 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & \mu_3 \\ \mu_4 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & \mu_5 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & \mu_6 & 0 & 0 & \lambda_6 \end{pmatrix} \right\} \cap \mathrm{Sp}_6(\mathbb{C})$,

where $\lambda_i \in \mathbb{C}$, $\mu_i \in \mathbb{C}$, $i = 1, \ldots, 6$, and $\mathrm{Sp}_6(\mathbb{C})$ denotes the group of symplectic 6×6 matrices. We have $\dim \Theta = 9$.

Denoting by ϱ the homomorphism

$$\varrho : \Gamma \to \mathrm{Aut}(\Lambda \times \Theta), \qquad \gamma \mapsto ((L, T) \mapsto (\gamma(L), \gamma(T))),$$

we find that the semi-direct product

$$(\Lambda \times \Theta) \rtimes_{\varrho} \Gamma$$

acts on $\mathfrak{V}$ from the left in the following way: for $A \in \mathfrak{V}$ and $((L,T),\gamma) \in (\Lambda \times \Theta) \rtimes_\varrho \Gamma$ we have

$$((L,T),\gamma)\cdot A = L\gamma(A)T^{-1}.$$

The subgroup I of the semi-direct product consisting of those $((L,T),\gamma)$ with

$$L = \begin{pmatrix} s^3 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix}, \quad T = 6\times 6 \text{ identity matrix},$$

$$\gamma = \text{the automorphism of } \mathcal{A} \text{ that multiplies } x_i,\ i = 0,\dots,4,$$
$$\text{by } s^{-1} \text{ for some } s \in \mathbb{C}^*,$$

acts trivially on $\mathfrak{V}$. We put

$$G := ((\Lambda\times\Theta)\rtimes_\varrho \Gamma)/I.$$

Then $\dim G = 65$. Then G acts on $\mathfrak{V}$ with finite stabilizers: to see this it suffices to note that, the surfaces under consideration being of general type, there are only finitely many projectivities carrying $Y \subset \mathbb{P}^4$ onto itself, and that secondly, the equation

$$LAT^{-1} = A$$

for $A \in \mathfrak{V}$, $(L,T) \in \Lambda\times\Theta$ can hold only if

$$L = \pm 3\times 3 \text{ identity matrix}, \quad T = \pm 6 \times 6 \text{ identity matrix}$$

(the latter follows from a direct calculation using the facts that the a_2, a_3, b_2, b_3 resp. a_4, $-a_2$, b_4, $-b_2$ occurring in A are independent linear forms and that T is symplectic, whence if T is a homothety, it must be $\pm$ the identity).

Thus an upper bound for the dimension of $\mathfrak{U}$ is

$$103 \dim \mathfrak{V} - 65 \dim G = 38.$$

On the other hand, $\dim \mathfrak{U} \geq 10\chi - 2K^2 = 38$ by general principles (see [Cat1b, p. 484]). Thus $\dim \mathfrak{U} = 38$ and one obtains the following theorem:

THEOREM 2.4. *Regular surfaces S of general type with $p_g = 5$, $K^2 = 11$ such that the canonical map is a birational morphism onto the image $Y \subset \mathbb{P}^4$ and such that assumption* (A) *above is satisfied, form an irreducible unirational open set $\mathfrak{U}$ of dimension* 38 *inside their moduli space.*

3. APPENDIX

Here we describe the Macaulay 2 computation [G-S] to prove existence of matrices A fulfilling the conditions of datum (D) on page 50 above. We do it by checking the required properties listed in Remark 1.10. For the sake of brevity, we do not reproduce our entire Macaulay 2 session here, but use a mixture of code and words to explain it. We first input the example of the matrix A'

$$A' = \left(\begin{array}{ccc|ccc} 0 & x_2 & x_0 - x_4 & 0 & x_3 + x_4 & x_3 \\ x_3 - x_1 & -x_2 & 0 & -x_4 & -x_3 - x_4 & 0 \end{array}\right)$$

given on page 53 above (we call it `Asmall`) and check that the saturation of $I_2(A')$ is radical. The code is:

```
R=ZZ/31991[x_0..x_4];
Asmall=matrix{{0,x_2,x_0-x_4,0,x_3+x_4,x_3},
      {-x_1+x_3,-x_2,0,-x_4,-x_3-x_4,0}}
Ismall=minors(2,Asmall);
Ismallsat=saturate Ismall;
Ismallsat==radical Ismallsat
```

Thus the matrix A' scheme-theoretically defines the points $P_1 = (1:0:0:0:0)$, $P_2 = (0:1:0:0:0)$ and $P_3 = (0:0:1:0:0)$ in $\mathbb{P}^4$. Then we have to choose a random vector $(A_1, A_2, A_3, B_1, B_2, B_3)$ of cubic forms such that, after concatenating it with the matrix A', we get a matrix $A = (\alpha\ \beta)$ which satisfies the symmetry $\alpha\beta^t = \beta\alpha^t$. Such a vector of cubics is in the kernel of the following matrix M:

$$M = \begin{pmatrix} 0 & x_3 + x_4 & x_3 & 0 & -x_2 & -x_0 + x_4 \\ -x_4 & -x_3 - x_4 & 0 & x_1 - x_3 & x_2 & 0 \end{pmatrix}.$$

Thus we create A as follows:

```
M=matrix{{0,x_3+x_4,x_3,0,-x_2,-x_0+x_4},
  {-x_4,-x_3-x_4,0,x_1-x_3,x_2,0}}
G=generators (kernel M);
Aux1=random(R^1,R^{-2,-2,-2});
Auxx1=transpose Aux1;
Aux2=random (R^1,R^{-1,-1,-1,-1,-1,-1,-1,-1});
Auxx2=transpose Aux2;
Aux3=Auxx1||Auxx2;
C=G*Aux3;
D=transpose C;
A=D||Asmall;
alpha=submatrix(A,{0,1,2},{0,1,2});
beta=submatrix(A,{0,1,2},{3,4,5});
alpha*(transpose beta)==beta*(transpose alpha)
```

One then checks that the subscheme defined by the maximal minors of A is of codimension 2 and its singular locus coincides with the points $P_1,\ P_2,\ P_3$:

```
Isurface1=saturate minors(3,A);
codim Isurface1
Sing=Isurface1+minors(2, jacobian Isurface1);
Sing=saturate Sing;
Ismallsat==radical Sing
```

It remains to investigate the nature of the singularities of the subscheme defined by $I_3(A)$ at the points P_1, P_2, P_3. In fact, they all turn out to be improper double points. For this we compute the tangent cone at these points. Let us look for example at P_1. Unfortunately, if we put $x_0 = 1$ in the polynomials that are the generators of `Isurface1` and then take leading terms, it is not necessarily true that the resulting polynomials determine the tangent cone at P_1 (cf. [CLS, Chapter 9, §7, p. 485]). However, this procedure does yield the correct result if we first compute a Gröbner basis of `Isurface1` with respect to a monomial order such that any monomial involving x_0 is greater than any monomial involving only $x_1, \dots, x_4$ (cf. [CLS, Ch. 9, §7, Prop. 4]). For example the lexicographic order with $x_0 > x_1 > x_2 > x_3 > x_4$ will do:

```
Rnew=ZZ/31991[x_0,x_1,x_2,x_3,x_4,MonomialOrder=>Lex];
Isurface2=substitute(Isurface1,Rnew);
Gro=transpose gens gb Isurface2;
```

The elements of the desired Gröbner basis are now stored in the matrix `Gro` (it had 178 entries in our test computation which might seem slightly large, but the rest of the computational steps went through). We now have to put $x_0 = 1$ in the polynomials which are the entries of `Gro` and create the ideal `tangentconepoint1` generated by the leading terms of the resulting polynomials. This is most efficiently done by successively differentiating the entries of `Gro` with respect to x_0: If e is one of the entries and $\partial^{j+1}e/\partial x_0^{j+1} = 0$, but $\partial^j e/\partial x_0^j \neq 0$, then we store $\partial^j e/\partial x_0^j$ among the generators of the ideal `tangentconepoint1` (in our test computation, differentiating the entries of `Gro` four times with respect to x_0 already yielded the zero matrix). We can then extract the information we want by typing

```
codim tangentconepoint1
degree tangentconepoint1
genera tangentconepoint1
tangentconepoint1==top tangentconepoint1
tangentconepoint1==radical tangentconepoint1
```

One finds that `tangentconepoint1` defines a pure two-dimensional reduced subscheme of degree 2 in $\mathbb{P}^4$ whose general hyperplane section has arithmetic genus -1: This hyperplane section thus consists of two skew lines in $\mathbb{P}^4$ and `tangentconepoint1` defines two planes that span $\mathbb{P}^4$ and intersect in the point P_1.

The computations for the points P_2 and P_3 are of course completely analogous.

REFERENCES

[Böh] C. BÖHNING, *L. Szpiro's conjecture on Gorenstein algebras in codimension 2*. J. Algebra 288 (2005), 545–555.

[Bom] E. BOMBIERI, *Canonical models of surfaces of general type*. Publ. Math. I.H.E.S. 42 (1973), 171–219.

[B-He] W. BRUNS - J. HERZOG, *Cohen–Macaulay Rings*. Cambridge Univ. Press, 1998.

[Cat1] F. CATANESE, *On the moduli spaces of surfaces of general type*. J. Differential Geom. 19 (1984), 483–515.

[Cat2] F. CATANESE, *Commutative algebra methods and equations of regular surfaces*. In: Algebraic Geometry (Bucharest, 1982), Lecture Notes in Math. 1056, Springer, 1984, 68–111.

[Cat3] F. CATANESE, *Homological algebra and algebraic surfaces*. In: Proc. Sympos. Pure Math. 62, Amer. Math. Soc. 1997, 3–56.

[Cil] C. CILIBERTO, *Canonical surfaces with $p_g = p_a = 5$ and $K^2 = 10$*. Ann. Scuola Norm. Sup. Pisa 9 (1982), 287–336

[CLS] D. COX - J. LITTLE - D. O'SHEA, *Ideal, Varieties, and Algorithms, An Introduction to Computational Algebraic Geometry and Commutative Algebra*. 2nd ed., Springer, New York, 1997.

[Ei] D. EISENBUD, *Commutative Algebra with a View towards Algebraic Geometry*. Grad. Texts in Math. 150, Springer, New York, 1995.

[En] F. ENRIQUES, *Le superficie algebriche*. Zanichelli, Bologna, 1949.

[Gra] M. GRASSI, *Koszul modules and Gorenstein algebras*. J. Algebra 180 (1996), 918–953.

[G-S] D. GRAYSON - M. STILLMAN, *Macaulay 2, A system for computation in algebraic geometry and commutative algebra*. Source code and documentation available at http://www.math.uiuc.edu/Macaulay2.

[Har] R. HARTSHORNE, *Algebraic Geometry*. Grad. Texts in Math. 52, Springer, New York, 1977.

[Lip] J. LIPMAN, *Dualizing sheaves, differentials and residues on algebraic varieties*. Astérisque 117 (1984).

[Mi] J. C. MIGLIORE, *Introduction to Liaison Theory and Deficiency Modules*. Progr. Math. 165, Birkhäuser, 1998.

[Ross] D. ROSSBERG, *Kanonische Flächen mit $p_g = 5,\ q = 0$ und $K^2 = 11, 12$*. Dissertation, Universität Bayreuth, Bayreuth, 1997; Bayreuth. Math. Schr. 52 (1997), 75–172.

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Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstr. 3-5
D-37073 GÖTTINGEN, Germany
boehning@uni-math.gwdg.de