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Partial differential equations. — *The balance between diffusion and absorption in semilinear parabolic equations*, by ANDREY SHISHKOV and LAURENT VÉRON, communicated on 23 June 2006.

ABSTRACT. — Let $h : [0, \infty) \to [0, \infty)$ be continuous and nondecreasing, $h(t) > 0$ if $t > 0$, and $m, q > 0$. We investigate the behavior as $k \to \infty$ of the fundamental solutions $u = u_k$ of $\partial_t u - \Delta u^m + h(t)u^q = 0$ in $\Omega \times (0, T)$ satisfying $u_k(x, 0) = k\delta_0$. The main question is whether the limit is still a solution of the above equation with an isolated singularity at $(0, 0)$, or a solution of the associated ordinary differential equation $u' + h(t)u^q = 0$ which blows up at $t = 0$.

KEY WORDS: Parabolic equations; Saint-Venant principle; very singular solutions; asymptotic expansions.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35K60.

1. INTRODUCTION

Let m and q be positive parameters and $h : [0, \infty) \to [0, \infty)$ nondecreasing continuous. If one considers a reaction-diffusion equation such as

$$\partial_t u - \Delta u^m + h(t)u^q = 0 \tag{1.1}$$

($u > 0$ for simplicity) in a cylindrical domain $Q^T = \mathbb{R}^N \times (0, T)$ ($N \geq 1$), the behaviour of u is subject to two competing features: diffusion associated to the partial differential operator, here $-\Delta$, and absorption which is represented by the term $h(t)u^q$. When $q > 1$ and $h(t) > 0$ for $t > 0$, the absorption term is strong enough to make any positive solution satisfy a universal bound

$$0 \leq u(x, t) \leq U_h(t) = \left((q-1)\int_0^t h(s)\,ds\right)^{-1/(q-1)} \tag{1.2}$$

for every $(x, t) \in Q^T$. In addition, the function U_h which appears above is a particular solution of (1.1). The associated diffusion equation

$$\partial_t v - \Delta v^m = 0 \tag{1.3}$$

admits fundamental solutions $v = v_k$ ($k > 0$) which satisfy $v_k(x, 0) = k\delta_0$ if $m > (N-2)_+/N$. If

$$\int_0^T \int_{B_R} h(t)v_k^q\,dx\,dt < \infty, \quad B_R := \{|x| < R\}, \tag{1.4}$$

for any $R \in (0, \infty]$, it is known that (1.1) admits fundamental solutions $u = u_k$ in Q^T which satisfy the initial condition $u_k(x, 0) = k\delta_0$. The maximum principle holds and therefore the mapping $k \mapsto u_k$ is increasing. If $h > 0$ on $(0, \infty)$ then due to universal bound (1.2) the limit $u_\infty = \lim_{k\to\infty} u_k$ exists, and u_∞ is a solution of (1.1) in Q^T. A natural question is whether u_∞ admits a singularity only at the origin $(0, 0)$ or at other points too. Actually, in the last case it will imply $u_\infty \equiv U$ since the following alternative occurs: either

(i) $u_\infty = U_h$ (*complete initial blow-up*), or
(ii) u_∞ is a solution singular at $(0, 0)$ and such that $\lim_{t\to 0} u(x, t) = 0$ for all $x \neq 0$ (*single-point initial blow-up*).

This phenomenon was observed for the first time by Marcus and Véron. They considered the semilinear equation

$$\partial_t u - \Delta u + h(t)u^q = 0 \tag{1.5}$$

and proved [8, Prop. 5.2]

THEOREM 1.1. *If $h(t) = e^{-\kappa/t}$ $(\kappa > 0)$, then the complete initial blow-up occurs.*

However they raised the question whether this type of degeneracy of absorption is sharp or not. The method of [8] relies on the construction of subsolutions associated to very singular solutions of equations

$$\partial_t u - \Delta u + c_\epsilon t^\alpha u^q = 0 \tag{1.6}$$

for suitable $\alpha > 0$ and $c_\epsilon > 0$, and on the study of asymptotics of these solutions. One of the main results of the present paper states that if the degeneracy of the absorption terms is slightly smaller in comparison to Theorem 1.1, then localization occurs.

THEOREM 1.2. *If $h(t) = \exp(-\omega(t)/t)$, where ω is continuous, nondecreasing and satisfies*

$$\int_0^1 \frac{\sqrt{\omega(s)}}{s}\, ds < \infty, \tag{1.7}$$

then u_∞ has single-point initial blow-up at $(0, 0)$.

The method of proof is totally different from the one of Marcus and Véron and based upon local energy estimates in the spirit of the famous Saint-Venant principle (see [5, 12, 13]). Using appropriate test functions we prove by induction that the energy of the fundamental solutions u_k remains uniformly locally bounded in $\overline{Q^T} \setminus \{(0, 0)\}$.

In the case of the equation

$$\partial_t u - \Delta u + h(t)(e^u - 1) = 0 \tag{1.8}$$

the same type of phenomenon occurs, but at a different scale of degeneracy. We prove the following

THEOREM 1.3. (i) *If* $h(t) = e^{-e^{\kappa/t}}$ *for some* $\kappa > 0$, *then the complete initial blow-up occurs.*

(ii) *If* $h(t) = e^{-e^{\omega(t)/t}}$ *for some* $\omega \in C(0,\infty)$ *positive, nondecreasing and satisfying* (1.7), *then* u_∞ *has single-point initial blow-up at* $(0,0)$.

In this paper we also extend the study of equation (1.1) to the case $m \neq 1$. The situation is completely different for $m > 1$, the porous media equation with slow diffusion, and for $(N-2)_+/N < m < 1$, the fast diffusion equation. Concerning the porous media equation, we prove

THEOREM 1.4. *If* $q > m > 1$ *and* h *is nondecreasing with* $h(t) = O(t^{(q-m)/(m-1)})$ *as* $t \to 0$, *then* $u_\infty \equiv U_h$.

We give two proofs. The first one, valid only in the subcritical case $1 < m < q < m + 2/N$, is based upon the construction of suitable subsolutions, as in the semilinear case. The second one, based upon scaling transformations, is valid in all the cases $q + 1 > 2m > 2$ where the u_k exists. It reduces to proving that the equation

$$-\Delta\Psi - \Psi^{1/m} + \Psi^{q/m} = 0 \quad \text{in } \mathbb{R}^N$$

admits only one positive solution, the constant 1. The localization counterpart is as follows:

THEOREM 1.5. *Assume* $q > m > 1$ *in equation* (1.1). *If* $h(t) = t^{(q-m)/(m-1)}\omega(t)^{-1}$ *with* $\omega(t) \to 0$ *as* $t \to 0$, *and*

$$\int_0^1 \omega(s)^\theta \, \frac{ds}{s} < \infty \tag{1.9}$$

where

$$\theta = \frac{m^2 - 1}{[N(m-1) + 2(m+1)](q-1)},$$

then u_∞ *has single-point initial blow-up at* $(0,0)$.

Actually, the method is applicable to a much more general class of equations.

In the fast diffusion case there is always localization.

THEOREM 1.6. *Assume* $(N-2)_+/N < m < 1$ *and* $q > 1$ *in equation* (1.1). *Then*

$$u_\infty(x,t) \le \min\{U_h(t), C_*(t/|x|^2)^{1/(1-m)}\} \tag{1.10}$$

where

$$C_* = \left(\frac{(1-m)^3}{2m(mN + 2 - N)}\right)^{1/(1-m)}.$$

This type of problem has an elliptic counterpart which is initiated in [10] where the following question is considered: suppose Ω is a C^2 bounded domain in $\mathbb{R}^N$, $q > 1$ and

$h \in C(0, \infty)$ is positive. What is the limit as $k \to \infty$ of the solutions (when they exist) $u = u_k$ of the problem

$$\begin{cases} -\Delta u + h(\rho(x))u^q = 0 & \text{in } \Omega, \\ u = k\delta_0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$? It is proved in [10] that, if $h(t) = e^{-1/t}$, then u_∞ ($:= \lim_{k\to\infty} u_k$) is the maximal solution of the equation in Ω, that is, it satisfies

$$\begin{cases} -\Delta u + h(\rho(x))u^q = 0 & \text{in } \Omega, \\ \lim\limits_{\rho(x)\to 0} u(x) = \infty. \end{cases} \tag{1.12}$$

On the contrary, if $h(t) = t^\alpha$ for $\alpha > 0$ and $1 < q < (N + 1 + \alpha)/(N - 1)$, it is proved in [11] that u_∞ has an isolated singularity at 0, and vanishes everywhere outside 0. In a forthcoming article we shall study this localization of singularity phenomenon for the complete nonlinear elliptic problem, replacing the powers by more general functions, and the ordinary Laplacian by the p-Laplacian operator.

Our paper is organized as follows. In §2 we study sufficient conditions for complete initial blow-up for a semilinear heat equation. In §3 we prove a sharp sufficient condition of existence of single-point initial blow-up for the heat equation with power nonlinear absorption. In §4 the local energy method from §3 is adapted to the heat equation with a nonpower absorption nonlinearity. §5 deals with the porous media equation with power nonlinear absorption, and §6 with a fast diffusion equation with nonlinear absorption.

2. Complete initial blow-up for a semilinear heat equation

We recall the standard result concerning the existence of a fundamental solution $u = u_k$ ($k > 0$) to the problem

$$\begin{cases} \partial_t u - \Delta u + g(x, t, u) = 0 & \text{in } Q^T = \mathbb{R}^N \times (0, T), \\ u(x, 0) = k\delta_0. \end{cases} \tag{2.1}$$

If v is defined in Q^T, we denote by $\tilde{g}(v)$ the function $(x, t) \mapsto g(x, t, v(x, t))$. By a solution we mean a function $u \in L^1_{\mathrm{loc}}(\overline{Q^T})$ such that $\tilde{g}(u) \in L^1_{\mathrm{loc}}(\overline{Q^T})$ and

$$\iint_{Q^T} (-u\partial_t\phi - u\Delta\phi + \tilde{g}(u)\phi)\, dx\, dt = k\phi(0, 0) \tag{2.2}$$

for any $\phi \in C_0^{2,1}(\mathbb{R}^N \times [0, T) \times \mathbb{R})$. We denote by $E(x, t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$ the fundamental solution of the heat equation in Q^∞, by $B_R(a)$ the open ball of center a and radius R, and $B_R(0) = B_R$. The following result is classical:

THEOREM 2.1. *Let $g \in C(\mathbb{R}^N \times [0, T] \times \mathbb{R})$ with $g(x, t, r) \ge 0$ on $\mathbb{R}^N \times [0, T] \times \mathbb{R}_+$, and assume that $g = g_1 + g_2$ where g_1 and g_2 are respectively nondecreasing and locally Lipschitz continuous with respect to the r-variable. Let $k > 0$ be such that*

$$\int_0^T \int_{B_R} g(x, t, kE(x, t))\, dx\, dt < \infty \tag{2.3}$$

for any $R > 0$. Then there exists a solution $u = u_k$ to problem (2.1)*. Furthermore, if $g_2 = 0$, then u_k is unique.*

The function $g(x,t,r) = e^{-\kappa/t}|r|^{q-1}r$ with $\kappa > 0$ and $q > 1$ satisfies (2.3). Thus the problem

$$\begin{cases} \partial_t u - \Delta u + e^{-\kappa/t}|u|^{q-1}u = 0 & \text{in } Q^\infty, \\ u(x,0) = k\delta_0, \end{cases} \tag{2.4}$$

admits a unique solution. The next result is proved in [8], but we recall the proof both for the sake of completeness and to present the key elements of the method in a simple case.

THEOREM 2.2. *For $k > 0$, let u_k denote the solution of* (2.4) *in Q^∞. Then $u_k \uparrow U_S$ as $k \to \infty$, where*

$$U_S(t) = \left((q-1)\int_0^t e^{-\kappa/s}\,ds\right)^{1/(1-q)} \quad \forall t > 0. \tag{2.5}$$

PROOF. CASE 1: $1 < q < 1 + 2/N$. For any $\epsilon > 0$, $u_k = u$ satisfies

$$\partial_t u - \Delta u + e^{-\kappa/\epsilon}u^q \ge 0 \tag{2.6}$$

on Q^ϵ. Therefore if $v = v_k$ is the solution of

$$\begin{cases} \partial_t v - \Delta v + e^{-\kappa/\epsilon}v^q = 0 & \text{in } Q^\infty, \\ v(x,0) = k\delta_0, \end{cases} \tag{2.7}$$

then $u_k \ge v_k$. Letting $k \to \infty$ yields

$$\lim_{k\to\infty} u_k := u_\infty \ge v_\infty = \lim_{k\to\infty} v_k \quad \text{in } Q^\epsilon. \tag{2.8}$$

If we write $v_\infty(x,t) = e^{\kappa/\epsilon(q-1)}t^{-1/(q-1)}f(x/\sqrt{t})$, then f is radial and satisfies

$$\begin{cases} f'' + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right)f' + \dfrac{1}{q-1}f - f^q = 0 & \text{on } (0,\infty), \\ f'(0) = 0, \quad \lim_{r\to\infty} r^{2/(q-1)}f(r) = 0. \end{cases}$$

Furthermore the asymptotics of f is given in [2],

$$f(r) = Cr^{2/(q-1)-N}e^{-r^2/4}(1+o(1)) \quad \text{as } r \to \infty,$$

for some $C = C(N,q) > 0$. Therefore

$$f(r) \ge \tilde{C}(r+1)^{2/(q-1)-N}e^{-r^2/4} \quad \forall r \ge 0, \tag{2.9}$$

for some $\tilde{C} = \tilde{C}(N,q) > 0$. If we take $t = \epsilon$, we derive from (2.8) that

$$u_\infty(x,t) \ge e^{\kappa/t(q-1)}t^{-1/(q-1)}f(x/\sqrt{t}) \quad \text{in } \mathbb{R}^N. \tag{2.10}$$

Let $0 < \ell < 2\sqrt{\kappa/(q-1)}$. Inequalities (2.9) and (2.10) imply

$$u_\infty(x,t) \geq \tilde{C} t^{-1/(q-1)} e^{(\kappa/(q-1)-\ell^2/4)t^{-1}} \qquad \forall x \in \bar{B}_\ell. \tag{2.11}$$

Therefore $\lim_{t\to 0} u_\infty(x,t) = \infty$ for all $x \in \bar{B}_\ell$. We pick some point x_0 in B_ℓ. Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (2.4) with initial value $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)$ $(0 < \sigma < \ell - |x_0|)$, the previous inequality implies

$$u_\infty(x,t) \geq u_\infty(x - x_0, t).$$

Reversing the roles of 0 and x_0 yields

$$u_\infty(x,t) = u_\infty(x - x_0, t).$$

If we iterate this process we derive

$$u_\infty(x,t) = u_\infty(x - y, t) \qquad \forall y \in \mathbb{R}^N. \tag{2.12}$$

Since $u_{k\delta_y}$ is radial with respect to y, (2.12) implies that $u_\infty(x,t)$ is independent of x and therefore it is a solution of

$$\begin{cases} z' + e^{-\kappa/t} z^q = 0 & \text{on } (0,\infty), \\ \lim\limits_{t\to 0} z(t) = \infty. \end{cases} \tag{2.13}$$

Thus $u_\infty = U_S$ where U_S is defined by (2.5).

CASE 2: $q \geq 1 + 2/N$. Let $\alpha > 0$ be such that $q < q_{c,\alpha} = 1 + 2(1+\alpha)/N$. We write $e^{-\kappa/t} = t^\alpha \tilde{h}(t)$ with $\tilde{h}(t) = t^{-\alpha} e^{-\kappa/t}$. The function $\tilde{h}$ is increasing on $(0, \kappa/\alpha]$ and we extend it by $\tilde{h}(0) = 0$. Let $0 < \epsilon \leq \kappa/\alpha$. Then the solution $u = u_k$ of (2.4) satisfies

$$\partial_t u - \Delta u + \tilde{h}(\epsilon) t^\alpha u^q \geq 0 \qquad \text{in } \mathbb{R}^N \times (0, \epsilon].$$

As in Case 1, u is bounded from below on $\mathbb{R}^N \times (0,\epsilon]$ by $\tilde{h}(\epsilon)^{-1/(q-1)} v_\infty$ where $v_\infty = v$ is the very singular solution of

$$\partial_t v - \Delta v + t^\alpha v^q = 0. \tag{2.14}$$

Then $v_\infty(x,t) = t^{-(1+\alpha)/(q-1)} f_\alpha(|x|/\sqrt{t})$, and $f_\alpha = f$ satisfies

$$\begin{cases} f'' + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right) f' + \dfrac{1+\alpha}{q-1} f - f^q = 0 & \text{on } (0,\infty), \\ f'(0) = 0, \quad \lim\limits_{r\to\infty} r^{2(1+\alpha)/(q-1)} f(r) = 0. \end{cases}$$

The asymptotics of f_α is given in [9]:

$$f_\alpha(r) = C r^{2(1+\alpha)/(q-1)-N} e^{-r^2/4}(1 + o(1)) \qquad \text{as } r \to \infty,$$

thus

$$f_\alpha(r) \geq \tilde{C}(1+r)^{2(1+\alpha)/(q-1)-N} e^{-r^2/4} \qquad \forall r \in \mathbb{R}_+.$$

Consequently,

$$u(x,t) \geq \tilde{C} e^{(\kappa/(q-1)-\ell^2/4)t^{-1}} \quad \forall x \in \bar{B}_\ell. \tag{2.15}$$

Taking again $0 < \ell < 2\sqrt{\kappa/(q-1)}$, we derive

$$\lim_{t\to 0} u(x,t) = \infty \quad \forall x \in \bar{B}_\ell.$$

As in Case 1, this leads to $u_\infty(x,t) = u_\infty(x-y,t)$ for any $y \in \mathbb{R}^N$, and finally $u_\infty(x,t) = U_S(t)$. □

Next we consider the Cauchy problem for the diffusion equation with an exponential type absorption term

$$\begin{cases} \partial_t u - \Delta u + h(t)e^u = 0 & \text{in } Q^\infty, \\ u(x,0) = k\delta_0, \end{cases} \tag{2.16}$$

where $h \in C(\mathbb{R}_+)$ is nonnegative. Theorem 2.1 yields the following existence result:

PROPOSITION 2.3. *Assume h satisfies*

$$\lim_{t\to 0} t^{N/2} \ln h(t) = -\infty. \tag{2.17}$$

Then for any $k > 0$ problem (2.16) *admits a unique solution $u = u_k$. Furthermore*

$$u_k(x,t) \leq V_S(t) := -\ln\left(\int_0^t h(s)\,ds\right) \quad \forall (x,t) \in Q^\infty. \tag{2.18}$$

Notice that estimate (2.18) is a consequence of the fact that V_S satisfies the associated O.D.E.

$$y' + h(t)e^y = 0 \quad \text{in } (0,\infty),$$

with infinite initial value. Our main result concerning nonexistence of localized singularities for equation (2.16) is

THEOREM 2.4. *Let $h(t) = e^{-e^{\sigma/t}}$ for some $\sigma > 0$ and any $t > 0$. Then $u_k \uparrow V_S$ as $k \to \infty$.*

PROOF. STEP 1. *Construction of an approximate very singular solution.* For $n > 1$ and $c_n > 0$ to be defined later, let $v = V_n$ be the very singular solution of

$$\partial_t v - \Delta v + c_n t^{\alpha_n} v^n = 0. \tag{2.19}$$

A necessary and sufficient condition for the existence of a V_n is

$$n < 1 + N(\alpha_n + 1)/2.$$

This function is obtained in the form

$$V_n(x,t) = t^{-(1+\alpha_n)/(n-1)} F(x/\sqrt{t}),$$

where F solves

$$\Delta F + \frac{1}{2}\xi \cdot DF + \frac{1+\alpha_n}{n-1}F - c_n F^n = 0.$$

We fix

$$\frac{1+\alpha_n}{n-1} = 1 + \frac{N}{2}, \quad \text{i.e.} \quad \alpha_n = (2+N)(n-1)/2 - 1, \tag{2.20}$$

and set

$$f_n = c_n^{1/(n-1)} F.$$

Then f_n solves

$$\Delta f_n + \frac{1}{2}\xi \cdot Df_n + \frac{N+2}{2} f_n - f_n^n = 0.$$

We prove that f_n has an asymptotic expansion essentially independent of n, in the following form:

$$f_n(\xi) \geq \delta(|\xi|^2 + 1)e^{-|\xi|^2/4}, \quad \text{so} \tag{2.21}$$

$$V_n(x,t) \geq \delta c_n^{-1/(n-1)} t^{-2-N/2}(|x|^2 + t)e^{-|x|^2/4t}.$$

In order to see that, we put

$$\tilde{f}_n = \left(\frac{2}{N+2}\right)^{1/(n-1)} f_n.$$

Then

$$\Delta \tilde{f}_n + \frac{1}{2}\xi \cdot D\tilde{f}_n + \frac{N+2}{2}\tilde{f}_n - \frac{N+2}{2}\tilde{f}_n^n = 0.$$

By the maximum principle $0 \leq \tilde{f}_n \leq 1$ so that $0 \leq \tilde{f}_n^{n'} \leq \tilde{f}_n^n$ for $n' > n$. Thus

$$\Delta \tilde{f}_n + \frac{1}{2}\xi \cdot D\tilde{f}_n + \frac{N+2}{2}\tilde{f}_n - \frac{N+2}{2}\tilde{f}_n^{n'} \geq 0,$$

which implies that $\tilde{f}_n$ is a subsolution of the equation for $\tilde{f}_{n'}$ and therefore, if $n' > n$, then

$$\tilde{f}_n \leq \tilde{f}_{n'} \;\Leftrightarrow\; f_n \leq \left(\frac{N+2}{2}\right)^{(n'-n)/(n-1)(n'-1)} f_{n'}. \tag{2.22}$$

In the particular case $n = n^* = (N+4)/(N+2)$, the equation falls within the scope of the Brezis–Peletier–Terman study since it can also be written in the form

$$\Delta f_{n^*} + \frac{1}{2}\xi \cdot Df_{n^*} + \frac{1}{n^*-1} f_{n^*} - f_{n^*}^{n^*} = 0,$$

and their asymptotic expansion applies (with $2/(n^*-1) - N = 2$) as $|\xi| \to \infty$:

$$f_{n^*}(\xi) = C|\xi|^2 e^{-|\xi|^2/4}(1+o(1)), \quad \text{so} \quad f_{n^*}(\xi) \geq \delta_*(|\xi|^2+1)e^{-|\xi|^2/4} \quad \forall \xi. \tag{2.23}$$

Combining (2.22) with $n = n^*$ and n' replaced by n, and (2.23), we get

$$f_n(\xi) \geq \delta_* \left(\frac{2}{N+2}\right)^{(n-n^*)/(n-1)(n^*-1)} (|\xi|^2+1)e^{-|\xi|^2/4} \quad \forall \xi. \tag{2.24}$$

Since $n \mapsto (2/(N+2))^{(n-n^*)/(n-1)(n^*-1)}$ is bounded from below independently of $n > n^*$, we get (2.21).

STEP 2. *Some estimates from below for a related problem.* In order to have $v_n \leq u$ in the range of values of u, which is

$$u(t) \leq V_S(t) = -\ln\left(\int_0^t h(s)\,ds\right) \quad \forall t > 0, \tag{2.25}$$

we need $v = v_n$ to be a subsolution near $t = 0$ of the equation that u satisfies. Furthermore this can be done up to some bounded function. It is sufficient to have

$$c_n t^{\alpha_n}(x^n + 1) \geq h(t)e^x \quad \forall t \in (0, \tau_n],\ x \in [0, V_S(t)] \tag{2.26}$$

where τ_n has to be defined. In particular, at the end points of the interval,

$$\begin{cases} \text{(i) } c_n t^{\alpha_k} \geq h(t), \\ \text{(ii) } c_n t^{\alpha_n}\left(\ln^n\left(\dfrac{1}{\int_0^t a(s)ds}\right) + 1\right) \geq \dfrac{h(t)}{\int_0^t h(s)\,ds}. \end{cases} \tag{2.27}$$

We write (2.26) in the form

$$\frac{e^x}{1+x^n} \leq \frac{c_n t^{\alpha_n}}{h(t)}, \tag{2.28}$$

and set

$$\phi(x) = \frac{e^x}{1+x^n}.$$

Then

$$\phi'(x) = e^x \frac{1 + x^n - nx^{n-1}}{(1+x^n)^2}.$$

The sign of ϕ' is the same as the one of $\psi(x) = 1 + x^n - nx^{n-1}$, a function which is decreasing then increasing, positive near 0, vanishes somewhere between 0 and 1 and again between $n-1$ and n. The first maximum of ϕ is less than $e/2$. This is not important in (2.28) since we can always assume that the minimum of $c_n t^{\alpha_n}/h(t)$ is larger than $e/2$. Therefore, it is sufficient to have

$$\frac{e^{V_S(t)}}{1 + V_S(t)^n} \leq \frac{c_n t^{\alpha_n}}{h(t)} \tag{2.29}$$

in order to have (2.28). This is exactly (2.27)(ii). If we express $h(t)$ in the form

$$h(t) = -\omega'(t)e^{-\omega(t)},$$

then (2.27)(ii) is equivalent to

$$c_n t^{\alpha_n}(\omega(t)^n + 1) \geq -\omega'(t). \tag{2.30}$$

Since

$$\omega(t)^n + 1 \geq 2^{1-n}(\omega(t)+1)^n,$$

we consider the following O.D.E. on $\mathbb{R}_+$:

$$c_n t^{\alpha_n} = 2^{1-n}\frac{-\eta'}{(\eta+1)^n},$$

the maximal solution of which is

$$\eta(t) = \frac{1}{2}\left(\frac{1}{c_n(n-1)}\right)^{1/(n-1)} t^{-(\alpha_n+1)/(n-1)} = \frac{1}{2}\left(\frac{1}{c_n(n-1)}\right)^{1/(n-1)} t^{-1-N/2}.$$

If we write ω in the form $\omega(t) = e^{\alpha(t)}$, with $\alpha(0) = \infty$, $\alpha' < 0$, then (2.27)(ii) becomes

$$c_n t^{\alpha_n}(e^{n\alpha(t)} + 1) \geq -\alpha'(t)e^{\alpha(t)},$$

and this inequality is ensured provided

$$\text{(2.31)} \qquad c_n t^{\alpha_n} e^{(n-1)\alpha(t)} \geq -\alpha'(t), \quad \text{i.e.} \quad c_n \geq -\alpha'(t)e^{(1-n)\alpha(t)-\alpha_n \ln t} = -t\alpha'(t)e^{(1-n)(\alpha(t)+2^{-1}(N+2)\ln t)},$$

by replacing α_n by its value. Next we fix

$$\text{(2.32)} \qquad \alpha(t) = \alpha_\sigma(t) = \sigma/t \quad \forall t > 0$$

where $\sigma > 0$ is a parameter, thus

$$-t\alpha'(t)e^{(1-n)(\alpha(t)+2^{-1}(N+2)\ln t)} = e^{(1-n)\sigma/t-(2^{-1}(n-1)(N+2)+1)\ln t} = e^{\rho(t)}.$$

In order to have (2.31) it is sufficient to have the monotonicity of the function ρ and

$$\rho'(t) = \frac{\sigma(n-1)}{t^2} - \frac{n(N+2)-N}{2t}.$$

Then there exists $\gamma > 0$, independent of k and σ, such that $\rho'(t) > 0$ on $(0, \sigma\gamma]$. Consequently, inequality (2.31) is ensured on $(0, \epsilon] \subset (0, \sigma\gamma]$ as soon as

$$\text{(2.33)} \qquad c_n \geq e^{\rho(\epsilon)} = e^{(1-n)\sigma/\epsilon - 2^{-1}(n(N+2)-N)\ln\epsilon}.$$

STEP 3. *Complete initial blow-up for a related problem.* Assume now

$$\text{(2.34)} \qquad h(t) = \tilde{\sigma}t^{-2}e^{\tilde{\sigma}t^{-1}-e^{\tilde{\sigma}/t}}$$

for some $\tilde{\sigma} > 0$. For $n > 2$, we fix $\epsilon < \tilde{\sigma}\gamma$ and take $c_n = e^{\rho(\epsilon)}$. On $(0, \epsilon]$ we have

$$c_n t^{\alpha_n}(e^{n\alpha(t)} + 1) \geq -\alpha'(t)e^{\alpha(t)}.$$

Therefore, if $u = u_k$ is the solution of (2.16) with $h(t)$ given by (2.34), then $u(t) \leq V_S(t)$, where V_S is given by (2.25), and

$$\partial_t u - \Delta u + c_n t^{\alpha_n}(u^n + 1) \geq 0 \quad \text{in } Q^\epsilon.$$

Therefore u is larger than the solution $v = \tilde{v}_k$ of

$$\partial_t v - \Delta v + c_n t^{\alpha_n}(v^n + 1) = 0 \quad \text{in } Q^\epsilon$$

with $\tilde{v}_k(0) = k\delta_0$. Furthermore $\tilde{v}_k \geq v_k - c_n t^{\alpha_n+1}/(\alpha_n + 1)$, where $v = v_k$ solves

$$\partial_t v - \Delta v + c_n t^{\alpha_n} v^n = 0 \quad \text{in } Q^\epsilon$$

with $v_k(0) = k\delta_0$. If we let $k \to \infty$, we derive from (2.21) and by replacing $c_n = e^{\rho(\epsilon)}$ by its precise value $e^{(1-n)\sigma/\epsilon - 2^{-1}(n(N+2)-N)\ln\epsilon}$, that

$$u_\infty(x,t) \geq V_n(x,t) - \frac{c_n t^{\alpha_n+1}}{\alpha_n + 1} \geq \delta t^{-2-N/2}(|x|^2 + t)e^{\frac{\sigma}{\epsilon} + \frac{(n(N+2)-N)\ln\epsilon}{n-1} - \frac{|x|^2}{4t}}$$

on $(0, \epsilon]$. In particular

$$u_\infty(x,\epsilon) \geq \delta\epsilon^{-2-N/2}(|x|^2 + \epsilon)e^{\frac{\sigma}{\epsilon} + \frac{(n(N+2)-N)\ln\epsilon}{n-1} - \frac{|x|^2}{4\epsilon}}. \tag{2.35}$$

Taking $|x|^2 < \sigma/4$ yields

$$\lim_{\epsilon\to 0} \epsilon^{-2-N/2}(|x|^2 + \epsilon)e^{\frac{\sigma}{\epsilon} + \frac{(n(N+2)-N)\ln\epsilon}{n-1} - \frac{|x|^2}{4\epsilon}} = \infty.$$

Thus

$$\lim_{\epsilon\to 0} u_\infty(x,\epsilon) = \infty \quad \forall x \in B_{\sqrt{\sigma}/2}.$$

As in the proof of Theorem 2.2, this implies $u_\infty = V_S$.

STEP 4. *End of proof.* Since for any $\sigma > \tilde{\sigma} > 0$ there exists an interval $(0, \theta]$ on which

$$\tilde{\sigma} t^{-2} e^{\sigma' t^{-1} - e^{\sigma'/t}} \geq e^{-e^{\sigma/t}},$$

any solution of (2.16) with $h(t)$ given by (2.34) is a subsolution in Q^θ of the same equation with $h(t) = e^{-e^{-\sigma/t}}$. This implies the claim. □

3. SINGLE-POINT INITIAL BLOW-UP FOR A SEMILINEAR HEAT EQUATION

We consider the following Cauchy problem:

$$\begin{cases} \partial_t u - \Delta u + h(t)|u|^{q-1}u = 0 & \text{in } Q^\infty, \\ u(x,0) = k\delta_0. \end{cases} \tag{3.1}$$

The first result dealing with the localization of the blow-up that we prove is the following.

THEOREM 3.1. *Assume $h(t) = e^{-\omega(t)/t}$ where $\omega \in C([0,\infty))$ is a positive, nondecreasing function which satisfies $\omega(s) \geq s^{\alpha_0}$ for some $\alpha_0 \in [0,1)$ and any $s > 0$, and the following Dini-like condition:*

$$\int_0^1 \frac{\sqrt{\omega(s)}}{s}\, ds < \infty. \tag{3.2}$$

Then u_k always exists and $u_\infty := \lim_{k\to\infty} u_k$ has a pointwise singularity at $(0,0)$.

PROOF. The proof is based on the study of the asymptotic properties as $k \to \infty$ of solutions $u = u_k$ of the regularized Cauchy problem

$$
\begin{cases} u_t - \Delta u + h(t)|u|^{q-1}u = 0 & \text{in } Q^T, \\ u(x,0) = u_{0,k}(x) = M_k^{1/2} k^{-N/2} \delta_k(x) & \forall x \in \mathbb{R}^N, \end{cases} \tag{3.3}
$$

where $\delta_k \in C(\mathbb{R}^N)$, $\operatorname{supp} \delta_k \subset \{|x| \le k^{-1}\}$, $\delta_k \rightharpoonup \delta(x)$ weakly in the sense of measures as $k \to \infty$ and $\{M_k\}$ is some sequence tending to ∞ as $k \to \infty$ fast enough so that

$$
M_k^{1/2} k^{-N/2} \to \infty \quad \text{as } k \to \infty. \tag{3.4}
$$

Without loss of generality we will suppose that

$$
\|\delta_k\|^2_{L_2(\mathbb{R}^N)} \le c_0 k^N \quad \forall k \in \mathbb{N}, \quad c_0 = \text{const.} \tag{3.5}
$$

Our method of analysis is some variant of the local energy estimates method (also called Saint-Venant principle), particularly developed in [12, 13, 15–17] (see also review in [5]). Let us introduce the families of subdomains

$$
\begin{aligned}
\Omega(\tau) &= \mathbb{R}^N \cap \{|x| > \tau\} \quad \forall \tau > 0, \\
Q^r(\tau) &= \Omega(\tau) \times (0,r) \quad \forall r \in (0,T), \\
Q_r(\tau) &= \Omega(\tau) \times (r,T) \quad \forall r \in (0,T).
\end{aligned}
$$

STEP 1. *The local energy framework.* We fix an arbitrary $k \in \mathbb{N}$ and consider the solution $u = u_k$ of (3.3), but for convenience we will denote it by u. Firstly we deduce some integral vanishing properties of u in the subdomains $Q_r := \mathbb{R}^N \times (r,T)$. Multiplying (3.3) by $u(x,t)\exp(-\frac{t-r}{1+T-r})$ and integrating over Q_r, we get

$$
\begin{aligned}
&\left(2\exp\left(\frac{T-r}{1+T-r}\right)\right)^{-1} \int_{\mathbb{R}^N} |u(x,T)|^2\,dx \\
&\qquad + \int_{Q_r} (|D_x u|^2 + h(t)|u|^{q+1}) \exp\left(-\frac{t-r}{1+T-r}\right) dx\,dt \\
&\qquad + \frac{1}{1+T-r} \int_{Q_r} |u|^2 \exp\left(-\frac{t-r}{1+T-r}\right) dx\,dt \\
&\quad = 2^{-1} \int_{\Omega(\tau)} |u(x,r)|^2\,dx + 2^{-1} \int_{\mathbb{R}^N \setminus \Omega(\tau)} |u(x,r)|^2\,dx,
\end{aligned} \tag{3.6}
$$

where $\tau > 0$ is an arbitrary parameter. Using Hölder's inequality, it is easy to check that

$$
\int_{\mathbb{R}^N \setminus \Omega(\tau)} |u(x,r)|^2\,dx \le c\tau^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} \left(\int_{\mathbb{R}^N \setminus \Omega(\tau)} |u(x,r)|^{q+1} h(r)\,dx\right)^{\frac{2}{q+1}}. \tag{3.7}
$$

Here and below we denote by c, c_i different positive constants which do not depend on the parameters k, τ, r, but the precise value of which may change from one occurrence to another. Let us now consider the energy functions

$$
\begin{aligned}
&I_1(r) = \int_{Q_r} |D_x u|^2 \, dx\, dt, \qquad I_2(r) = \int_{Q_r} h(t)|u(x,t)|^{q+1} \, dx\, dt,\\
&I_3(r) = \int_{Q_r} |u|^2 \, dx\, dt.
\end{aligned} \tag{3.8}
$$

It is easy to check that

$$
-\frac{dI_2(r)}{dr} = \int_{\mathbb{R}^N} h(r)|u(x,r)|^{q+1} \, dx \geq \int_{\mathbb{R}^N \setminus \Omega(\tau)} h(r)|u(x,r)|^{q+1} \, dx \quad \forall \tau > 0.
$$

Therefore it follows from (3.6) and (3.7) that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |u(x,T)|^2 \, dx + I_1(r) + I_2(r) + I_3(r)\\
&\leq c\tau^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_2'(r))^{\frac{2}{q+1}} + c\int_{\Omega(\tau)} |u(x,r)|^2 \, dx \quad \forall \tau > 0,\ \forall r : 0 < r < T.
\end{aligned} \tag{3.9}
$$

Next we introduce additional energy functions

$$
\begin{aligned}
f(r,\tau) &= \int_{\Omega(\tau)} |u(x,r)|^2 \, dx, \qquad E_1(r,\tau) = \int_{Q^r(\tau)} |D_x u|^2 \, dx\, dt,\\
& \qquad\qquad\qquad\qquad\qquad\quad\;\; E_2(r,\tau) = \int_{Q^r(\tau)} |u|^2 \, dx\, dt.
\end{aligned} \tag{3.10}
$$

Now we deduce some vanishing estimates for these energy functions. Let μ be some nondecreasing smooth function defined on $(0,\infty)$, with $\mu(\tau) > 0$ for $\tau > 0$ (a more precise definition will be given later on). Then multiplying (3.3) by $u(x,t)\exp(-\mu(\tau)^2 t)$ and integrating over $Q^r(\tau)$ with $\tau > k^{-1}$ (remember that $\operatorname{supp} u_{0,k} \subset \{|x| < k^{-1}\}$) we deduce easily

$$
\begin{aligned}
2^{-1} f_{\mu,r}(\tau) + J_{\mu,r}(\tau) &:= 2^{-1} \int_{\Omega(\tau)} |u(x,r)|^2 \exp(-\mu(\tau)^2 r) \, dx\\
&\quad + \int_{Q^r(\tau)} (|\nabla_x u|^2 + \mu(\tau)^2 |u|^2) \exp(-\mu(\tau)^2 t) \, dx\, dt\\
&\leq \mu(\tau)^{-1} \int_{\partial\Omega(\tau)\times(0,r)} (|\nabla_x u|^2 + \mu(\tau)^2 |u|^2) \exp(-\mu(\tau)^2 t) \, ds\, dt \quad \forall \tau > k^{-1}.
\end{aligned} \tag{3.11}
$$

Clearly

$$
\begin{aligned}
\frac{dJ_{\mu,r}(\tau)}{d\tau} &= -\int_{\partial\Omega(\tau)\times(0,r)} (|\nabla_x u|^2 + \mu(\tau)^2 |u|^2) \exp(-\mu(\tau)^2 t) \, ds\, dt\\
&\quad + \int_{Q^r(\tau)} 2\mu\mu'(\tau)|u|^2 \exp(-\mu(\tau)^2 t) \, dx\, dt\\
&\quad - 2\int_{Q^r(\tau)} \mu\mu'(\tau) t (|\nabla_x u|^2 + \mu(\tau)^2 |u|^2) \exp(-\mu(\tau)^2 t) \, dx\, dt.
\end{aligned}
$$

Since $\mu'(\tau) > 0$, it follows from (3.11) that

$$\text{(3.12)} \quad 2^{-1} f_{\mu,r}(\tau) + J_{\mu,r}(\tau) \le \mu(\tau)^{-1}\Big[-\frac{d}{d\tau}J_{\mu,r}(\tau) + 2\int_{Q^r(\tau)} \mu(\tau)\mu'(\tau)|u|^2 \exp(-\mu(\tau)^2 t)\,dx\,dt\Big].$$

If we suppose

$$\text{(3.13)} \qquad 1 - \frac{2\mu'(\tau)}{\mu(\tau)^2} \ge 2^{-1},$$

we derive from (3.12)

$$f_{\mu,r}(\tau) + J_{\mu,r}(\tau) \le -2\mu(\tau)^{-1}\frac{dJ_{\mu,r}(\tau)}{d\tau}.$$

It is easy to check that this last inequality is equivalent to

$$\frac{\mu(\tau)}{2}\exp\Big(\int_{\tau_1}^{\tau}\frac{\mu(s)}{2}\,ds\Big) f_{\mu,r}(\tau) \le -\frac{d}{d\tau}\Big(J_{\mu,r}(\tau)\exp\Big(\int_{\tau_1}^{\tau}\frac{\mu(s)}{2}\,ds\Big)\Big) \quad \forall \tau > \tau_1 > k^{-1}.$$

By integrating this inequality and using the monotonicity of $f_{\mu,r}(\tau)$ we get

$$f_{\mu,r}(\tau_2)\int_{\tau_1}^{\tau_2}\frac{\mu(\tau)}{2}\exp\Big(\int_{\tau_1}^{\tau}\frac{\mu(s)}{2}\,ds\Big)d\tau + J_{\mu,r}(\tau_2)\exp\Big(\int_{\tau_1}^{\tau_2}\frac{\mu(s)}{2}\,ds\Big) \le J_{\mu,r}(\tau_1) \quad \forall \tau_2 > \tau_1 > k^{-1}.$$

Since

$$\frac{\mu(\tau)}{2}\exp\Big(\int_{\tau_1}^{\tau_2}\frac{\mu(s)}{2}\,ds\Big) = \frac{d}{d\tau}\Big(\exp\Big(\int_{\tau_1}^{\tau}\frac{\mu(s)}{2}\,ds\Big)\Big),$$

it follows from the last inequality that

$$\text{(3.14)} \quad f_{\mu,r}(\tau_2)\Big[\exp\Big(\int_{\tau_1}^{\tau_2}\frac{\mu(s)}{2}\,ds\Big) - 1\Big] + J_{\mu,r}(\tau_2)\exp\Big(\int_{\tau_1}^{\tau_2}\frac{\mu(s)}{2}\,ds\Big) \le J_{\mu,r}(\tau_1) \quad \forall \tau_2 > \tau_1 > k^{-1}.$$

Now we have to define $\mu(\tau)$. Let $\varepsilon > 0$ and set

$$\text{(3.15)} \qquad \mu(\tau) = \varepsilon r^{-1}(\tau - k^{-1}) \quad \forall \tau > k^{-1}.$$

One can easily verify that condition (3.13) is equivalent to

$$\text{(3.16)} \qquad \tau \ge k^{-1} + 2\varepsilon^{-1/2} r^{1/2}.$$

Now (3.14) implies two inequalities:

$$(3.17)\qquad A(\tau_2) := \int_{Q^r(\tau_2)} \left(|\nabla_x u|^2 + \frac{\varepsilon^2(\tau_2 - k^{-1})^2}{r^2} |u|^2 \right) dx\, dt$$
$$\le A(\tau_1) \exp\left[-\frac{\varepsilon((\tau_2 - k^{-1})^2 - (\tau_1 - k^{-1})^2)}{4r} + \frac{\varepsilon^2(\tau_2 - k^{-1})}{r} \right]$$
$$\forall \tau_2 > \tau_1 > k^{-1} + 2\varepsilon^{-1/2} r^{1/2},$$

and

$$(3.18)\qquad f(r, \tau_2)$$
$$\le A(\tau_1) \left[\exp\left(\frac{\varepsilon((\tau_2 - k^{-1})^2 - (\tau_1 - k^{-1})^2)}{4r} \right) - 1 \right]^{-1} \exp\left(\frac{\varepsilon^2(\tau_2 - k^{-1})^2}{r} \right)$$
$$\forall \tau_2 > \tau_1 > k^{-1} + 2\varepsilon^{-1/2} r^{1/2}.$$

In particular, for $\varepsilon = 8^{-1}$ we infer from (3.17) and (3.18) that

$$(3.19)\qquad \int_{Q^r(\tau)} \left(|\nabla_x u|^2 + \frac{(\tau - k^{-1})^2}{64 r^2} |u|^2 \right) dx\, dt$$
$$\le e \exp\left(-\frac{(\tau - k^{-1})^2}{64 r} \right) \int_{Q^r(\tau_0^{(k)})} \left(|\nabla_x u|^2 + \frac{|u|^2}{2r} \right) dx\, dt \quad \forall \tau \ge \tau_0^{(k)}(r) := k^{-1} + 4\sqrt{2}\sqrt{r},$$

and

$$(3.20)\qquad f(r, \tau) \le \frac{e^2}{e - 1} \exp\left(-\frac{(\tau - k^{-1})^2}{64 r} \right) \int_{Q^r(\tau_0^{(k)})} \left(|\nabla_x u|^2 + \frac{u^2}{2r} \right) dx\, dt$$
$$\forall \tau \ge \widetilde{\tau}_0^{(k)}(r) := k^{-1} + 8\sqrt{r}.$$

In order to have an estimate from above of the last factor on the right-hand side of (3.19), (3.20), we return to the equation satisfied by u, multiply it by the test function $u_k(x, t) \exp(-t)$ and integrate over $Q^r = \mathbb{R}^N \times (0, r)$. After standard computations we obtain, using (3.5),

$$(3.21)\qquad \int_{\mathbb{R}^N} |u_k(x, r)|^2\, dx + \int_{Q^r} (|\nabla_x u_k|^2 + |u_k|^2 + h(t)|u_k|^{q+1})\, dx\, dt$$
$$\le \bar{c}\, \|u_{0,k}\|^2_{L_2(\mathbb{R}^N)} \le c M_k \to \infty \quad \text{as } k \to \infty,\ \forall r \le T.$$

Due to (3.20), (3.21) it follows from (3.9) that

$$(3.22)\qquad \int_{\mathbb{R}^N} |u(x, T)|^2\, dx + I_1(r) + I_2(r) + I_3(r)$$
$$\le c_1 \tau^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_2'(r))^{\frac{2}{q+1}} + c_2 M_k r^{-1} \exp\left(-\frac{(\tau - k^{-1})^2}{64 r} \right) \quad \forall \tau \ge \widetilde{\tau}_0^{(k)}(r).$$

Relationships (3.19), (3.20) due to (3.21) yield

$$f(r,\tau)+E_1(r,\tau)+\frac{(\tau-k^{-1})^2}{64r^2}E_2(r,\tau) \le c_2\,M_k r^{-1}\exp\left(-\frac{(\tau-k^{-1})^2}{64r}\right) \quad \forall\tau>\widetilde{\tau}_0^{(k)}(r). \tag{3.23}$$

STEP 2. *The first round of computations.* Next we construct some sequences $\{\tau_j\},\ \{r_j\},\ j=k,k-1,\dots,1$. First we specify the choice of M_k from condition (3.3), namely let

$$M_k=e^{e^k}. \tag{3.24}$$

Then we choose $\tau_k,\ r_k$ such that

$$c_2\,r_k^{-1}\exp\left(-\frac{\tau_k^2}{64r_k}\right)M_k=M_k^{\varepsilon_0},\qquad 0<\varepsilon_0<e^{-1}, \tag{3.25}$$

where c_2 is from (3.22), (3.23). As a consequence of (3.25) and (3.24) we get

$$\tau_k=8r_k^{1/2}[(1-\varepsilon_0)e^k+\ln r_k^{-1}+\ln c_2]^{1/2}. \tag{3.26}$$

In inequality (3.22) we fix $\tau=\tau_k+k^{-1}$; then due to definition (3.25) it follows from (3.22) that

$$\int_{\mathbb{R}^N}|u(x,T)|^2\,dx+I_1(r)+I_2(r)+I_3(r) \le c_1(k^{-1}+\tau_k)^{\frac{N(q-1)}{q+1}}h(r)^{-\frac{2}{q+1}}(-I_2'(r))^{\frac{2}{q+1}}+M_k^{\varepsilon_0} \quad \forall r:0<r\le r_k. \tag{3.27}$$

$I_1(r),\ I_2(r),\ I_3(r)$ are nonincreasing functions which satisfy, due to the global *a priori* estimate (3.21),

$$I_1(0)+I_2(0)+I_3(0)\le cM_k. \tag{3.28}$$

Let us define the number r_k by

$$r_k=\sup\{r:I_1(r)+I_2(r)+I_3(r)\ge 2M_k^{\varepsilon_0}\}. \tag{3.29}$$

Then (3.27) implies the following differential inequality:

$$I_1(r)+I_2(r)+I_3(r)+\int_{\mathbb{R}^N}|u(x,T)|^2\,dx \le 2c_1(\tau_k+k^{-1})^{\frac{N(q-1)}{q+1}}h(r)^{-\frac{2}{q+1}}(-I_2'(r))^{\frac{2}{q+1}} \quad \forall r\le r_k. \tag{3.30}$$

Solving it, we get

$$I_1(r)+I_2(r)+I_3(r)\le c_3(\tau_k+k^{-1})^N H(r)^{-\frac{2}{q-1}} \quad \forall r\le r_k, \tag{3.31}$$

where

$$H(r) = \int_0^r h(s)\,ds \quad \text{and} \quad c_3 = \left(\frac{2}{q-1}\right)^{2/(q-1)} (2c_1)^{(q+1)/(q-1)}$$

Next we will use more specific functions

$$h(t) = \exp\left(-\frac{\omega(t)}{t}\right),$$

where $\omega(t)$ is nondecreasing and satisfies the following technical assumption:

$$t^{\alpha_0} \le \omega(t) \le \omega_0 = \text{const} \quad \forall t : 0 < t < t_0,\ 0 \le \alpha_0 < 1. \tag{3.32}$$

It is easy to show by integration by parts the following relation:

$$\int_0^r \exp\left(-\frac{a\omega(t)}{t}\right)dt \ge \frac{1-\delta(r)}{(1-\alpha_0)a} \cdot \frac{r^2}{\omega(r)} \exp\left(-\frac{a\omega(r)}{r}\right) \quad \forall r > 0,$$

where $\delta(r) \to 0$ if $r \to 0$. Therefore

$$H(r) \ge \bar{c}\frac{r^2}{\omega(r)} h(r), \qquad \bar{c} = \text{const} > 0. \tag{3.33}$$

As a consequence we derive from (3.31), using (3.26),

$$\begin{aligned} I_1(r) + I_2(r) + I_3(r) \le c_4[8r_k^{1/2}((1-\varepsilon_0)e^k + \ln r_k^{-1} + \ln c_2)^{1/2} + k^{-1}]^N \\ \times \frac{\omega(r)^{\frac{2}{q-1}}}{r^{\frac{4}{q-1}}} \exp\left(\frac{2\omega(r)}{(q-1)r}\right) \quad \forall r \le r_k. \end{aligned} \tag{3.34}$$

Comparing (3.29) and estimate (3.34) we deduce that r_k satisfies

$$r_k \le b_k, \tag{3.35}$$

where b_k is the solution of the equation

$$\begin{aligned} c_4[8b_k^{1/2}((1-\varepsilon_0)e^k + \ln b_k^{-1} + \ln c_2)^{1/2} + k^{-1}]^N \omega(b_k)^{\frac{2}{q-1}} b_k^{-\frac{4}{q-1}} \exp\left(\frac{2\omega(b_k)}{(q-1)b_k}\right) \\ = 2M_k^{\varepsilon_0} = 2\exp(\varepsilon_0 e^k). \end{aligned}$$

This equation may be rewritten in the form

$$\begin{aligned} \ln c_4 + \frac{2}{q-1}\ln\left(\frac{\omega(b_k)}{b_k}\right) + \frac{2}{q-1} \cdot \frac{\omega(b_k)}{b_k} \\ + N\ln[8b_k^{\frac{N(q-1)-4}{2(q-1)N}}((1-\varepsilon_0)e^k + \ln b_k^{-1} + \ln c_2)^{1/2} + k^{-1}b_k^{-\frac{2}{(q-1)N}}] \\ = \ln 2 + \varepsilon_0 e^k \quad \forall k \in \mathbb{N}. \end{aligned} \tag{3.36}$$

Since $s^{-1}\ln s \to 0$ as $s \to \infty$, it follows from equality (3.36) that

$$(1+c\gamma(k))\varepsilon_0 e^k \geq A_k + \frac{2}{q-1}\frac{\omega(b_k)}{b_k} \tag{3.37}$$
$$:= N\ln[8b_k^{\frac{N(q-1)-4}{2(q-1)N}}((1-\varepsilon_0)e^k + \ln b_k^{-1} + \ln c_2)^{\frac{1}{2}} + k^{-1}b_k^{-\frac{2}{N(q-1)}}] + \frac{2}{q-1}\frac{\omega(b_k)}{b_k}$$
$$\geq (1-\gamma(k))\varepsilon_0 e^k \quad \forall k \in \mathbb{N},$$

where $0 < \gamma(k) < 1$, $\gamma(k) \to 0$ as $k \to \infty$. Keeping in mind condition (3.32), we obtain easily

$$\frac{\omega(b_k)}{b_k} \geq b_k^{-(1-\alpha_0)}, \quad |A_k| \leq c(|\ln b_k| + k) \quad \forall k \in \mathbb{N}. \tag{3.38}$$

Due to properties (3.38), it follows from (3.37) that

$$ce^k > \frac{\omega(b_k)}{b_k} \geq d_1 e^k \quad \forall k \in \mathbb{N},\ d_1 > 0. \tag{3.39}$$

As a consequence of (3.39), (3.38) we also obtain

$$\ln b_k^{-1} \leq ck \quad \forall k \in \mathbb{N}. \tag{3.40}$$

Now using estimate (3.39) we are able to obtain a suitable upper estimate of τ_k. Thanks to (3.35), (3.39) and (3.40) we deduce from (3.26) that

$$\tau_k \leq cb_k^{1/2}\exp\left(\frac{k}{2}\right) \leq c\exp\left(\frac{k}{2}\right)\left(\frac{\omega(b_k)}{d_1\exp k}\right)^{1/2} = \frac{c}{d_1^{1/2}}\omega(b_k)^{1/2}.$$

Using again estimate (3.39) and the monotonicity of $\omega(s)$, we deduce from the above relation that

$$\tau_k \leq c\left[\omega\left(\frac{\omega_0}{d_1 e^k}\right)\right]^{1/2}, \quad \omega_0 \text{ is from (3.32).} \tag{3.41}$$

Therefore, from inequalities (3.23) and (3.34), definitions (3.25), (3.29) and property (3.35), we derive the following estimates:

$$I_1(r_k) + I_2(r_k) + I_3(r_k) \leq 2M_k^{\varepsilon_0} \quad \text{where } r_k \text{ is from (3.35), (3.29),} \tag{3.42}$$

$$f(r_k, \tau_k + k^{-1}) + E_1(r_k, \tau_k + k^{-1}) + \frac{\tau_k^2}{64r_k^2}E_2(r_k, \tau_k + k^{-1}) \leq M_k^{\varepsilon_0}, \tag{3.43}$$

where τ_k is from (3.26), (3.41). Because $\varepsilon_0 < e^{-1}$, it follows from definition (3.24) of the sequence M_k that

$$3M_k^{\varepsilon_0} < cM_{k-1} \quad \forall k \geq k_0(c), \tag{3.44}$$

where $c > 0$ is an arbitrary constant. Therefore, adding estimates (3.42) and (3.43), we obtain, thanks to (3.44) and the fact that $\tau_k \gg r_k$ (which follows from (3.25)), the inequality

$$f(r_k, \tau_k + k^{-1}) + \sum_{i=1}^{3} I_i(r_k) + \sum_{i=1}^{2} E_i(r_k, \tau_k + k^{-1}) < cM_{k-1} \quad \forall k \geq k_0(c). \tag{3.45}$$

STEP 3. *The second round of computations.* Next we introduce the terms r_{k-1}, τ_{k-1}. Firstly we return to inequality (3.14). Fixing here the function

$$\mu(t) = \varepsilon r^{-1}(\tau - k^{-1} - \tau_k) \quad \forall \tau > k^{-1} + \tau_k \tag{3.46}$$

instead of (3.15) and using estimates (3.16)–(3.20), we obtain

$$\begin{aligned} &\int_{Q^r(\tau)} \left(|\nabla_x u|^2 + \frac{(\tau - k^{-1} - \tau_k)^2 |u|^2}{64r^2} \right) dx\, dt \\ &\quad \leq e \exp\left(-\frac{(\tau - k^{-1} - \tau_k)^2}{64r} \right) \int_{Q^r(\tau_0^{(k-1)}(r))} \left(|\nabla_x u|^2 + \frac{|u|^2}{2r} \right) dx\, dt \\ &\qquad \forall \tau > \tau_0^{(k-1)}(r) := k^{-1} + \tau_k + 4\sqrt{2}\sqrt{r}, \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} f(r, \tau) &\leq \frac{e^2}{e-1} \exp\left(-\frac{(\tau - k^{-1} - \tau_k)^2}{64r} \right) \int_{Q^r(\tau_0^{(k-1)}(r))} \left(|\nabla_x u|^2 + \frac{|u|^2}{2r} \right) dx\, dt \\ &\qquad \forall \tau \geq \tilde{\tau}_0^{(k-1)} := k^{-1} + \tau_k + 8\sqrt{r}. \end{aligned} \tag{3.48}$$

The integral term on the right-hand side of (3.47), (3.48) is now estimated by using estimate (3.45) obtained in the first round of computations. So, we have

$$\begin{aligned} &\int_{Q^r(\tau_0^{(k-1)}(r))} \left(|\nabla_x u|^2 + \frac{u^2}{2r} \right) dx\, dt \\ &\quad \leq (2r)^{-1} \left[\sum_{i=1}^{3} I_i(r_k) + \sum_{i=1}^{2} E_i(r_k, \tau_k + k^{-1}) \right] \leq c(2r)^{-1} M_{k-1} \\ &\qquad \forall k > k_0(c),\ \forall r \geq r_k. \end{aligned} \tag{3.49}$$

Using this estimate we deduce from (3.47) and (3.48) that

$$\begin{aligned} &f(r, \tau) + E_1(r, \tau) + \frac{(\tau - \tau_k - k^{-1})^2}{64r^2} E_2(r, \tau) \\ &\quad \leq c_2 r^{-1} M_{k-1} \exp\left(-\frac{(\tau - \tau_k - k^{-1})^2}{64r} \right) \quad \forall \tau \geq \tilde{\tau}_0^{(k-1)}(r). \end{aligned} \tag{3.50}$$

This estimate is similar to estimate (3.23) from the first round. Now we have to deduce the analogue of (3.31). For this we return to the initial relation (3.9), where we now estimate the last term on the right-hand side by (3.48), using additionally (3.49). As a result we have

$$\sum_{i=1}^{3} I_i(r) \le c_1 \tau^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_2'(r))^{\frac{2}{q+1}} + c_2 M_{k-1} r^{-1} \exp\left(-\frac{(\tau - \tau_k - k^{-1})^2}{64r}\right) \quad \forall r \ge r_k,\ \forall \tau \ge \widetilde{\tau}_0^{(k-1)}(r), \tag{3.51}$$

which is analogous to estimate (3.22) from the first round. Next we define the numbers τ_{k-1} and r_{k-1} by equalities analogous to (3.26) and (3.29),

$$c_2 r_{k-1}^{-1} M_{k-1} \exp\left(-\frac{\tau_{k-1}^2}{64 r_{k-1}}\right) = M_{k-1}^{\varepsilon_0}, \quad 0 < \varepsilon_0 < e^{-1}, \tag{3.52}$$

$$r_{k-1} = \sup\{r : I_1(r) + I_2(r) + I_3(r) \ge 2M_{k-1}^{\varepsilon_0}\}. \tag{3.53}$$

Now combining inequalities (3.30) and (3.44), and using definitions (3.52), (3.53), we obtain the following differential inequality:

$$\sum_{i=1}^{3} I_i(r) \le 2c_1(\tau_{k-1} + \tau_k + k^{-1})^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_2'(r))^{\frac{2}{q+1}} \quad \forall r \le r_{k-1}. \tag{3.54}$$

Solving it, we obtain an estimate similar to (3.31). Using property (3.33) we arrive at

$$\sum_{i=1}^{3} I_i(r) \le c_4(\tau_{k-1} + \tau_k + k^{-1})^N \frac{\omega(r)^{\frac{2}{q-1}}}{r^{\frac{4}{q-1}}} \exp\left(\frac{2\omega(r)}{(q-1)r}\right) \quad \forall r \le r_{k-1}. \tag{3.55}$$

As in the first round, from (3.52) we express τ_{k-1} as a function $\tau_{k-1}(r_{k-1})$ (the analogue of (3.26)):

$$\tau_{k-1} = 8 r_{k-1}^{1/2} [(1 - \varepsilon_0) e^{k-1} + \ln r_{k-1}^{-1} + \ln c_2]^{1/2}. \tag{3.56}$$

Inserting this expression into (3.55) and then comparing the resulting inequality with definition (3.53), we deduce an estimate similar to (3.35),

$$r_{k-1} \le b_{k-1}, \tag{3.57}$$

where b_{k-1} is the solution of the equation

$$c_4[8 b_{k-1}^{1/2}((1-\varepsilon_0)e^{k-1} + \ln b_k^{-1} + \ln c_2)^{1/2} + \tau_k + k^{-1}]^N \times \frac{\omega(b_{k-1})^{\frac{2}{q-1}}}{b_{k-1}^{\frac{4}{q-1}}} \exp\left(\frac{2\omega(b_{k-1})}{(q-1)b_{k-1}}\right) = 2M_{k-1}^{\varepsilon_0} = 2\exp(\varepsilon_0 e^{k-1}). \tag{3.58}$$

From (3.50), and due to definition (3.52), it follows that

$$f(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) + \frac{\tau_{k-1}^2}{64 r_{k-1}} E_2(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) + E_1(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) \le M_{k-1}^{\varepsilon_0}. \tag{3.59}$$

From (3.55), due to (3.56)–(3.58), it follows that

$$I_1(r_{k-1}) + I_2(r_{k-1}) + I_3(r_{k-1}) \le 2M_{k-1}^{\varepsilon_0}. \tag{3.60}$$

Summing (3.59), (3.60) and using property (3.44), we deduce a new global *a priori* estimate (the analogue of (3.45)) which is the main starting information for the next round of computations:

$$f(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) + \sum_{i=1}^{3} I_i(r_{k-1}) + \sum_{i=1}^{2} E_i(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) \le cM_{k-2}. \tag{3.61}$$

We are now ready for the next round of computations, introducing the function

$$\mu(t) = \varepsilon r^{-1}(\tau - k^{-1} - \tau_k - \tau_{k-1}) \quad \forall \tau > k^{-1} + \tau_k + \tau_{k-1}$$

instead of (3.46), and estimate (3.61) instead of (3.45). We perform j rounds of such computations. As a result we obtain

$$f\left(r_{k-j}, \sum_{l=0}^{j} \tau_{k-l} + k^{-1}\right) + \sum_{i=1}^{3} I_i(r_{k-j}) + \sum_{i=1}^{2} E_i\left(r_{k-j}, \sum_{l=0}^{j} \tau_{k-l} + k^{-1}\right) \le cM_{k-j-1}, \tag{3.62}$$

which was our main aim.

STEP 4. *The control of* r_{k-j}, $\sum_{l=0}^{j} \tau_{k-l}$ *as* $j = 1, \dots, k$ *with arbitrary* $k \in \mathbb{N}$. It is clear that r_{k-j}, τ_{k-j} are defined by the conditions (see (3.52), (3.53))

$$c_2 r_{k-j}^{-1} M_{k-j} \exp\left(-\frac{\tau_{k-j}^2}{64 r_{k-j}}\right) = M_{k-j}^{\varepsilon_0}, \quad 0 < \varepsilon_0 < e^{-1}. \tag{3.63}$$

$$r_{k-j} = \sup\{r : I_1(r) + I_2(r) + I_3(r) \ge 2M_{k-j}^{\varepsilon_0}\}. \tag{3.64}$$

Similarly to (3.56)–(3.58) we deduce that

$$\tau_{k-j} = 8 r_{k-j}^{1/2} [(1 - \varepsilon_0) e^{k-j} + \ln r_{k-j}^{-1} + \ln c_2]^{1/2}, \tag{3.65}$$

$$r_{k-j} \le b_{k-j}, \tag{3.66}$$

where b_{k-j} satisfies

$$c_4\Big[8b_{k-j}^{1/2}((1-\varepsilon_0)e^{k-j}+\ln b_{k-j}^{-1}+\ln c_2)^{1/2}+\sum_{i=0}^{j-1}\tau_{k-i}+k^{-1}\Big]^N \times \frac{\omega(b_{k-j})^{\frac{2}{q-1}}}{b_{k-j}^{\frac{4}{q-1}}}\exp\bigg(\frac{2\omega(b_{k-j})}{(q-1)b_{k-j}}\bigg)=2M_{k-j}^{\varepsilon_0}=2\exp(\varepsilon_0 e^{k-j}). \tag{3.67}$$

In the first round of computations we have obtained the upper estimate (3.41) for τ_k. Let us suppose by induction that the following estimate is true:

$$\tau_{k-i}\leq c\bigg[\omega\bigg(\frac{\omega_0}{d_1e^{k-i}}\bigg)\bigg]^{1/2}\quad \forall i\leq j-1. \tag{3.68}$$

We have to prove that estimate (3.68) holds also for $i=j$. Obviously condition (3.67) is equivalent to (see (3.36))

$$\ln c_4+\frac{2}{q-1}\ln\bigg(\frac{\omega(b_{k-j})}{b_{k-j}}\bigg)+\frac{2}{q-1}\cdot\frac{\omega(b_{k-j})}{b_{k-j}}+A_k^{(j)}=\ln 2+\varepsilon_0e^{k-j}, \tag{3.69}$$

where

$$A_k^{(j)}=N\ln\bigg[b_{k-j}^{\frac{N(q-1)-4}{2(q-1)N}}((1-\varepsilon_0)e^{k-j}+\ln b_{k-j}^{-1}+\ln c_2)^{1/2}+\frac{k^{-1}+\sum_{i=0}^{j-1}\tau_{k-i}}{b_{k-j}^{\frac{2}{(q-1)N}}}\bigg].$$

Because of the induction assumption (3.68),

$$\sum_{i=0}^{j-1}\tau_{k-i}\leq c\sum_{i=0}^{j-1}\bigg[\omega\bigg(\frac{\omega_0}{d_1e^{k-i}}\bigg)\bigg]^{1/2}\leq c\int_0^1\frac{\omega(s)^{1/2}}{s}\,ds=:cL,$$

therefore

$$|A_k^{(j)}|\leq c(|\ln b_{k-j}|+(k-j)+\ln L). \tag{3.70}$$

From (3.69) due to (3.70) we derive easily

$$ce^{k-j}\geq\frac{\omega(b_{k-j})}{b_{k-j}}\geq d_1e^{k-j}\quad \forall j:k-j\geq k_0=k_0(L), \tag{3.71}$$

where $k_0<\infty$ does not depend on k. From (3.71) it follows in particular that

$$\ln b_{k-j}^{-1}\leq c(k-j)\quad \forall j:k-j\geq k_0. \tag{3.72}$$

Thanks to (3.66) and properties (3.71), (3.72), we derive from (3.65) that

$$\begin{aligned}\tau_{k-j}&\leq 8b_{k-j}^{1/2}((1-\varepsilon_0)e^{k-j}+\ln b_{k-j}^{-1}+\ln c_2)^{1/2}\\ &\leq cb_{k-j}^{1/2}\exp\bigg(\frac{k-j}{2}\bigg)\leq\frac{c}{d_1^{1/2}}\omega(b_{k-j})^{1/2}\quad \forall j:k-j\geq k_0(L).\end{aligned} \tag{3.73}$$

Using again estimate (3.71) and monotonicity of $\omega(s)$ we deduce from (3.73) that

$$\tau_{k-j} \le c\left[\omega\left(\frac{\omega_0}{d_1 e^{k-j}}\right)\right]^{1/2} \quad \forall j : k-j \ge k_0(L). \tag{3.74}$$

Thus, we have proved by induction estimate (3.68), for arbitrary $k-j \ge k_0(L)$ with $r_i,\ \tau_i$ satisfying (3.66), (3.67) and (3.74).

STEP 5. *Completion of the proof.* We now fix $n > k_0(L)$ and take $j = k-n$ in (3.62). This leads to

$$f\left(r_n, \sum_{l=0}^{k-n} \tau_{k-l} + k^{-1}\right) + \sum_{i=1}^{3} I_i(r_n) + \sum_{i=1}^{2} E_i\left(r_n, \sum_{l=0}^{k-n} \tau_{k-l} + k^{-1}\right) \le cM_{n-1} \quad \forall n > k_0(L). \tag{3.75}$$

Next we have

$$\sum_{l=0}^{k-n} \tau_{k-l} \le \sum_{i=n}^{\infty} \tau_i \le c\sum_{i=n}^{\infty}\left[\omega\left(\frac{\omega_0}{d_1 e^i}\right)\right]^{1/2} \le c\int_0^{\frac{\omega_0}{d_1 \exp(n-1)}} \frac{\omega(s)^{1/2}}{s}\, ds \to 0 \quad \text{as } n \to \infty. \tag{3.76}$$

Therefore, for arbitrarilyy small $\delta > 0$, we can find and fix $n = n(\delta) < \infty$ such that (3.75) implies the uniform (with respect to $k \in \mathbb{N}$) a priori estimate,

$$\sup_{t>0} \int_{|x|>\delta} |u_k(x,t)|^2\, dx + \int_0^T \int_{|x|>\delta} (|\nabla_x u_k|^2 + |u_k|^2)\, dx\, dt \le C = C(\delta) < \infty \quad \forall k \in \mathbb{N}. \tag{3.77}$$

Since $u_k(x,0) = 0$ for all $|x| > k^{-1}$ and $k \in \mathbb{N}$, it follows from (3.77) that $u_\infty(x,0) = 0$ for all $x \neq 0$, which ends the proof. □

4. REGIONAL INITIAL BLOW-UP FOR AN EQUATION WITH EXPONENTIAL ABSORPTION

The local energy method we have used in the proof of Theorem 3.1 is based on the sharp interpolation theorems for Sobolev function spaces, which are a natural tool for the study of solutions of equations with power nonlinearities. Here we propose an adaptation of the method to equations with nonpower nonlinearities.

Thus, we consider the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u + h(t)(e^u - 1) = 0 & \text{in } Q^\infty, \\ u(x,0) = k\delta_0. \end{cases} \tag{4.1}$$

THEOREM 4.1. *Assume* $h(t) = e^{-e^{\omega(t)/t}}$ *where* $\omega \in C([0,\infty))$ *satisfies the same asumptions as in Theorem* 3.1. *Then the solution* u_k *always exists and* $u_\infty := \lim_{k\to\infty} u_k$ *has a pointwise singularity at* $(0,0)$.

PROOF. We will consider the family $u_k(x,t)$ of solutions of regularized problems:

$$\begin{cases} u_t - \Delta u + h(t)(e^u - 1) = 0 \quad \text{in } Q^T, \\ u(x,0) = u_{0,k}(x) = M_k^{1/2} k^{-N/2} \delta_k(x) \quad \forall x \in \mathbb{R}^N, \end{cases} \tag{4.2}$$

where δ_k is nonnegative, continuous with compact support in $B_{k^{-1}}$, satisfies estimate (3.5) and converges weakly to δ_0 as $k \to \infty$, and $\{M_k\}$ satisfies condition (3.2). Let us introduce the energy functions (we omit the subscript k on u_k):

$$\begin{aligned} &I_{1,0}(r) = \int_{Q_r} |\nabla_x u|^2 \, dx\, dt, \qquad I_q(r) = (q!)^{-1} \int_{Q_r} h(t)|u|^{q+1} \, dx\, dt, \\ &I_{3,0}(r) = \int_{Q_r} |u|^2 \, dx\, dt. \end{aligned} \tag{4.3}$$

Multiplying (4.2) by $u(x,t)\exp(-\frac{t-r}{1+T-r})$, integrating over Q_r and using the equality

$$s(e^s - 1) = \sum_{q=1}^{\infty} \frac{s^{q+1}}{q!},$$

we obtain easily

$$\begin{aligned} I_{1,0}(r) + \sum_{l=1}^{\infty} I_l(r) + I_{3,0}(r) &\le c(q!)^{\frac{2}{q+1}} \tau^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_q'(r))^{\frac{2}{q+1}} \\ &\quad + c \int_{\Omega(\tau)} |u(x,r)|^2 \, dx \quad \forall \tau > 0,\ \forall r : 0 < r < T,\ \forall q \in \mathbb{N}. \end{aligned} \tag{4.4}$$

We introduce the additional energy functions

$$\begin{aligned} f(r,\tau) \text{ from (3.10)}, \qquad &E_{1,0}(r,\tau) = \int_{Q^r(\tau)} |D_x u|^2 \, dx\, dt, \\ &E_{2,0}(r,\tau) = \int_{Q^r(\tau)} |u|^2 \, dx\, dt. \end{aligned} \tag{4.5}$$

Instead of (3.21) we derive the following global *a priori* estimate:

$$\begin{aligned} \int_{\mathbb{R}^N} |u_k(x,r)|^2 \, dx + \int_{Q^r} \left(|\nabla_x u|^2 + |u_k|^2 + h(t) \sum_{l=1}^{\infty} \frac{|u_k|^{l+1}}{l!} \right) dx\, dt \\ \le \bar{c}\, \|u_{0,k}\|^2_{L_2(\mathbb{R}^N)} \le c M_k \quad \forall r < T. \end{aligned} \tag{4.6}$$

Using estimate (4.6) instead of (3.21) in a similar way to the proof of Theorem 3.1, we obtain the following inequality, analogous to (3.23):

$$\begin{aligned} f(r,\tau) + E_{1,0}(r,\tau) + \frac{(\tau - k^{-1})^2}{64 r^2} E_{2,0}(r,\tau) \\ \le c_2 M_k r^{-1} \exp\left(-\frac{(\tau - k^{-1})^2}{64 r} \right) \quad \forall \tau \ge \tilde{\tau}_0^{(k)}(r) = k^{-1} + 8\sqrt{r}. \end{aligned} \tag{4.7}$$

Using this estimate we deduce from (4.4) that

$$I_{1,0}(r) + \sum_{l=1}^{\infty} I_l(r) + I_{3,0}(r) \le c(q!)^{\frac{2}{q+1}} \tau^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_q'(r))^{\frac{2}{q+1}} + c_2 M_k r^{-1} \exp\left(-\frac{(\tau - k^{-1})^2}{64r}\right) \quad \forall \tau \ge \tilde{\tau}_0^{(k)}(r),\ \forall q \in \mathbb{N}. \tag{4.8}$$

Next, we define the numbers $\tau_k,\ r_k$. Firstly, set

$$r_k := \sup\left\{r : I_{1,0}(r) + \sum_{l=1}^{\infty} I_l(r) + I_{3,0} \ge 2M_k^{\varepsilon_0}\right\}, \quad 0 < \varepsilon_0 < e^{-1}. \tag{4.9}$$

Then we fix the sequence $\{M_k\}$ by (3.24) again and τ_k by (3.25), (3.26). Thanks to these definitions we derive the following series of inequalities from relations (4.8):

$$I_{1,0}(r) + \sum_{l=1}^{\infty} I_l(r) + I_{3,0}(r) \le 2c_1 (q!)^{\frac{2}{q+1}} (\tau_k + k^{-1})^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_q'(r))^{\frac{2}{q+1}} \quad \forall q \in \mathbb{N},\ \forall r \le r_k. \tag{4.10}$$

Solving these differential inequalities we obtain the estimates

$$I_{1,0}(r) + \sum_{l=1}^{\infty} I_l(r) + I_{3,0}(r) \le c_3 (\tau_k + k^{-1})^N (q!)^{\frac{2}{q-1}} H(r)^{-\frac{2}{q-1}} \quad \forall r \le r_k,\ \forall q \in \mathbb{N}, \tag{4.11}$$

where $H(r)$ is from (3.31). We now have to optimize estimate (4.11) with respect to the parameter q. By integration by parts, it is easy to check the following inequality:

$$H(r) \ge \bar{c} \frac{r^2}{\omega(r)} \exp\left(-\frac{\omega(r)}{r}\right) h(r) \quad \forall r > 0,\ \bar{c} > 0. \tag{4.12}$$

Using the Stirling formula $q! \sim (q/e)^q$ and estimate (4.12), we deduce from (4.11) that

$$I_{1,0}(r) + \sum_{l=1}^{\infty} I_l(r) + I_{3,0}(r) \le c_4 (\tau + k^{-1})^N F_q(r) \quad \forall r \le r_k, \tag{4.13}$$

where

$$F_q(r) = q^2 \omega(r)^{\frac{2}{q-1}} r^{-\frac{4}{q-1}} \exp\left(\frac{2}{q-1} \cdot \frac{\omega(r)}{r}\right) \exp\left[\frac{2}{q-1} \exp\left(\frac{\omega(r)}{r}\right)\right].$$

Fixing here the optimal value of the parameter q:

$$q = \tilde{q} := \left[2 \exp\left(\frac{\omega(r)}{r}\right)\right],$$

where $[a]$ denotes the integer part of a, we obtain easily

$$F_{\tilde{q}} \leq c \exp\left(\frac{2\omega(r)}{r}\right).$$

Therefore it follows from (4.13) that

$$I_{1,0}(r) + \sum_{l=1}^{\infty} I_l(r) + I_{3,0}(r) \leq c_5(\tau_k + k^{-1})^N \exp\left(\frac{2\omega(r)}{r}\right) \quad \forall r \leq r_k. \tag{4.14}$$

Comparing now definition (4.9) of r_k and estimate (4.14), and using additionally the expression (3.26) of τ_k, we obtain

$$r_k \leq b_k, \tag{4.15}$$

where b_k is defined by the equation

$$\begin{aligned} c_5[8b_k^{1/2}((1-\varepsilon_0)e^k + \ln b_k^{-1} + \ln c_2)^{1/2} + k^{-1}]^N \exp\left(\frac{2\omega(b_k)}{b_k}\right) \\ = 2M_k^{\varepsilon_0} = 2\exp(\varepsilon_0 e^k), \quad 0 < \varepsilon_0 < e^{-1}. \end{aligned} \tag{4.16}$$

By an analysis similar to Step 2 in the proof of Theorem 3.1, we obtain estimates (3.37)–(3.40) for b_k. Then we prove the validity of estimate (3.41) for τ_k. As a consequence of estimates (4.7), (4.14), thanks to definitions (3.26), (4.9) of $\tau_k,\ r_k$ and the previous estimates of $\tau_k,\ r_k$, we get

$$I_{1,0}(r) + \sum_{l=1}^{\infty} I_l(r) + I_{3,0}(r) \leq 2M_k^{\varepsilon_0},$$

$$f(r_k, \tau_k + k^{-1}) + E_{1,0}(r_k, \tau_k + k^{-1}) + \frac{\tau_k^2}{64 r_k^2} E_{2,0}(r_k, \tau_k + k^{-1}) \leq M_k^{\varepsilon_0}.$$

Summing these inequalities, and using the definition of $\{M_k\}$ and the property $\tau_k \gg r_k$, we obtain an analogue of estimate (3.45), namely,

$$\begin{aligned} f(r_k, \tau_k + k^{-1}) + I_{1,0}(r_k) + \sum_{l=1}^{\infty} I_l(r_k) + I_{3,0}(r_k) + E_{1,0}(r_k, \tau_k + k^{-1}) \\ + E_{2,0}(r_k, \tau_k + k^{-1}) \leq c M_{k-1}. \end{aligned} \tag{4.17}$$

Using (4.17) as a global *a priori* estimate instead of (4.6) and performing a second round of computations similar to (3.46)–(3.57) we derive a second global *a priori* estimate analogous to (3.61),

$$\begin{aligned} f(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) + I_{1,0}(r_{k-1}) + \sum_{l=1}^{\infty} I_l(r_{k-1}) + I_{3,0}(r_{k-1}) \\ + E_{1,0}(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) + E_{2,0}(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) \leq c M_{k-2}. \end{aligned}$$

Repeating such rounds j times we derive a corresponding analogue of relation (3.62). It is easy to see that estimate (3.76) for the constructed shifts τ_{k-i} remains valid. This fact, similar to what was used in the proof of Theorem 3.1, yields the conclusion. □

5. The Porous Media Equation with Absorption

In this section we consider the following problem dealing with fundamental solutions of the porous media equation with time dependent absorption:

$$\begin{cases} \partial_t u - \Delta(|u|^{m-1}u) + h(t)|u|^{q-1}u = 0 \quad \text{in } Q^T, \\ u(x,0) = k\delta_0. \end{cases} \tag{5.1}$$

It is standard to assume that $h \geq 0$ is a continuous function and m, q are positive real numbers. By a solution we mean a function $u \in L^1_{\text{loc}}(Q^T)$ such that $u^m \in L^1_{\text{loc}}(Q^T)$, $hu^q \in L^1_{\text{loc}}(Q^T)$ and

$$\iint_{Q^T} (-u\partial_t\phi - |u|^{m-1}u\Delta\phi + h(t)|u|^{q-1}u\phi)\, dx\, dt = k\phi(0,0) \tag{5.2}$$

for any $\phi \in C_0^{2,1}(\mathbb{R}^N \times [0,T))$. If $h \equiv 0$ and $m > (N-2)_+/N$ this problem admits a solution for any $k > 0$. When $m > 1$ this solution has the form

$$B_k(x,t) = t^{-\ell}\left(C_k - \frac{(m-1)\ell}{2mN}\frac{|x|^2}{t^{2\ell/N}}\right)_+^{1/(m-1)}, \tag{5.3}$$

where

$$\ell = \frac{N}{N(m-1)+2} \quad \text{and} \quad C_k = a(m,N)k^{2(m-1)\ell/N}. \tag{5.4}$$

Since B_k is a supersolution for problem (5.1), a sufficient condition for existence (and uniqueness) of u_k is

$$\iint_{Q^T} B_k(x,t)^q h(t)\, dx\, dt < \infty. \tag{5.5}$$

By the change of variable $y = t^{\ell/N}x$ this condition is independent of $k > 0$ and we have

PROPOSITION 5.1. *Assume $m > 1$, $q > 0$. If*

$$\int_0^1 h(t)t^{\ell-\ell q}\, dt < \infty, \tag{5.6}$$

then problem (5.1) *admits a unique positive solution $u = u_k$. In the particular case where $h(t) = O(t^\alpha)$ ($\alpha \geq 0$), the condition is*

$$\alpha > \frac{N(q-m)-2}{N(m-1)+2}. \tag{5.7}$$

We recall that if $q > 1$ and $m > (N-2)_+/N$, any solution of the porous media equation with absorption is bounded from above by the maximal solution U_h given by

$$U_h(t) = \left((q-1)\int_0^t h(s)\, ds\right)^{-1/(q-1)}. \tag{5.8}$$

THEOREM 5.2. *Assume $q+1>2m>2$ and $h\in C((0,\infty))$ is nondecreasing, positive and satisfies $h(t)=O(t^{(q-m)/(m-1)})$ as $t\to 0$. Then for any $k>0$, u_k exists and* $\lim_{k\to\infty} u_k := u_\infty = U_h$.

PROOF. We first notice that

$$q+1>2m>2, \quad \text{i.e.} \quad q>m>1 \quad \text{and} \quad \frac{q-m}{m-1}>\frac{N(q-m)-2}{N(m-1)+2}.$$

STEP 1. *Case $q<m+2/N$.* In this range of values we know [14] that there exists a nonnegative very singular solution $v=v_\infty$ to

$$\partial_t v-\Delta v^m+v^q=0 \quad \text{in } Q^T, \tag{5.9}$$

and $v_\infty=\lim_{k\to} v_k$, where the v_k are solutions of the same equation with initial data $k\delta_0$. Furthermore, v_∞ is unique [6], radial with respect to x and has the following form:

$$v_\infty(x,t)=t^{-1/(q-1)}F(|x|/t^{(q-m)/2(q-1)}),$$

where F solves

$$\begin{cases} (F^m)''+\dfrac{N-1}{\eta}(F^m)'+\dfrac{q-m}{2(q-1)}\eta F'+\dfrac{1}{q-1}F-F^q=0 \quad \text{in } (0,\infty), \\ F'(0)=0 \quad \text{and} \quad \lim\limits_{\eta\to\infty}\eta^{2/(q-m)}F(\eta)=0. \end{cases} \tag{5.10}$$

Actually F has compact support in $[0,\xi_0]$ for some $\xi_0>0$. Let $\gamma=(q-m)/(m-1)$. Then for any $\epsilon>0$, $u=u_\infty$ satisfies, for some $c>0$,

$$\partial_t u-\Delta u^m+c\epsilon^\gamma u^q\geq 0 \quad \text{in } Q^\epsilon.$$

If we set $w_\epsilon(x,t)=a^\theta v_\infty(x,at)$ with $\theta=1/(m-1)$ and $a=\epsilon^{-1}c^{-(q-1)/(q-m)}$, then

$$\partial_t w_\epsilon-\Delta w_\epsilon^m+c\epsilon^\gamma w_\epsilon^q=0 \quad \text{in } Q^T.$$

By comparison $u_\infty\geq w_\epsilon$ in Q^ϵ. If we take in particular $t=\epsilon$, this implies

$$\begin{aligned} u_\infty(x,t) &\geq c^{-1/(q-m)}t^{-1/(m-1)}v_\infty(x,c^{-(m-1)/(q-m)}) \\ &= c^{-1}t^{-1/(m-1)}F(c^{(m-1)/2(q-1)}|x|). \end{aligned} \tag{5.11}$$

If $|x|<\xi_c=c^{-(m-1)/2(q-1)}\xi_0$, we derive that $\lim_{t\to 0}u_\infty(x,t)=\infty$, locally uniformly in B_{ξ_c}. This implies $u_\infty=U_h$.

STEP 2. *Case $q\geq m+2/N$.* We give an alternative proof valid for all q. We first observe that it is sufficient to prove the result when $h(t)$ is replaced by t^γ. If we look for a family of transformations $u\mapsto T_\ell(u)$ of the form

$$T_\ell(u)(x,t)=\ell^\alpha u(\ell^\beta x,\ell t) \quad \forall (x,t)\in Q^\infty,\ \forall \ell>0$$

which leaves the equation

$$\partial_t u-\Delta|u|^{m-1}u+t^\gamma|u|^{q-1}u=0 \tag{5.12}$$

invariant, we find $\alpha = (1+\gamma)/(q-1)$ and $\beta = (q-m-\gamma(m-1))/2(q-1)$. Due to the value of γ, we have $\beta = 0$. Because of uniqueness and the value of the initial mass,

$$T_\ell(u_k) = u_{\ell^\alpha k} \quad \forall \ell > 0,\ \forall k > 0, \quad \text{so} \quad T_\ell(u_\infty) = u_\infty \forall \ell > 0. \tag{5.13}$$

Therefore

$$\ell^\alpha u_\infty(x, \ell t) = u_\infty(x,t) \quad \forall (x,t) \in Q^\infty,\ \forall \ell > 0.$$

In particular, if we take $\ell = t^{-1}$,

$$u_\infty(x,t) = t^{-\alpha} u_\infty(x,1) = t^{-\alpha}\phi(x).$$

Plugging this decomposition into (5.12) yields

$$-\alpha t^{-\alpha-1}\phi - t^{-\alpha m}\Delta\phi^m + t^{\gamma-\alpha q}\phi^q = 0,$$

where all the exponents of t coincide since

$$\alpha m = \frac{m}{m-1}, \quad \alpha q - \gamma = \frac{m}{m-1}, \quad \alpha + 1 = \frac{m}{m-1}.$$

Therefore ϕ is a positive and radial (as the u_k are) solution of

$$-\alpha\phi - \Delta\phi^m + \phi^q = 0 \quad \text{in } \mathbb{R}^N.$$

Setting $\psi = \phi^m$ yields

$$-\Delta\psi - \frac{1}{m-1}\psi^{1/m} + \psi^{q/m} = 0 \quad \text{in } \mathbb{R}^N. \tag{5.14}$$

Clearly $\psi = \psi_0 = (m-1)^{-m/(q-1)}$ is a solution. By a standard variation of the Keller–Osserman estimate, any solution is bounded from above by ψ_0. Putting $\tilde\psi(x) = A\psi(a)$, it is easy to find $A > 0$ and $a > 0$ such that

$$-\Delta\tilde\psi - \tilde\psi^{1/m} + \tilde\psi^{q/m} = 0 \quad \text{in } \mathbb{R}^N, \tag{5.15}$$

with $0 \le \tilde\psi \le 1$. Writing $\tilde\psi$ as a solution of an ODE, we derive

$$\tilde\psi(r) = \tilde\psi(0) + \int_0^r s^{1-n} \int_0^s (\tilde\psi^{q/m} - \tilde\psi^{1/m})\sigma^{n-1}\, ds \quad \forall r > 0.$$

If $\tilde\psi^{q/m}$ is not constantly 1, the right-hand side of the above inequality is decreasing with respect to r, and the only possible nonnegative limit is 0, by the La Salle principle. Thus

$$\tilde\psi'' + \frac{N-1}{r}\tilde\psi' + \frac{1}{2}\tilde\psi^{1/m} \le 0$$

for $r \ge r_0$ large enough. If $N = 2$, we set $\tau = \ln r$, $\Psi(\tau) = \tilde\psi(r)$ and get

$$\Psi'' + \frac{1}{2}e^{2\tau}\Psi^{1/m} \le 0$$

for $\tau \geq \ln r_0$, and the concavity of Ψ yields a contradiction. If $N \geq 3$, we set $\tau = r^{N-2}/(N-2)$ and $\Psi(\tau) = r^{N-2}\tilde{\psi}(r)$. Then Ψ satisfies

$$\Psi'' + c_N \tau^{(4-N)/(N-2)-1/m} \Psi^{1/m} \leq 0.$$

Again the concavity yields a contradiction. In any case we obtain $\Psi = 1$, or equivalently $\psi = \psi_0$, and finally, $u_\infty = t^{-1/(m-1)}\psi_0^{1/m}$. □

THEOREM 5.3. *Assume $q > m > 1$ and $h \in C(0,\infty)$ is nondecreasing, positive. If $h(t) = t^{(q-m)/(m-1)}\omega(t)^{-1}$ with $\omega(t) \to 0$ as $t \to 0$, and*

$$\int_0^1 \omega(s)^\theta \frac{ds}{s} < \infty, \tag{5.16}$$

where

$$\theta = \frac{m^2-1}{[N(m-1)+2(m+1)](q-1)},$$

then $u_\infty := \lim_{k\to\infty} u_k$ has a pointwise singularity at $(0,0)$.

PROOF. The structure of the proof is similar to the one of Theorem 3.1. We study the asymptotic behaviour as $k \to \infty$ of solutions $u = u_k(x,t)$ of the regularized Cauchy problem

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + h(t)|u|^{q-1}u = 0 \quad \text{in } Q^T, \\ u(x,0) = u_{0,k}(x) = M_k^{\frac{1}{m+1}} k^{-\frac{mN}{m+1}} \delta_k(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{5.17}$$

where δ_k is as in Theorem 3.1. Let us rewrite problem (5.17) in the form

$$\begin{cases} (|v|^{p-1}v)_t - \Delta v + h(t)|v|^{g-1}v = 0 \quad \text{in } Q^T, \\ v = v_k = |u|^{m-1}u, \quad p = 1/m, \quad g = q/m, \\ |v(x,0)|^{p-1}v(x,0) = |v_{0,k}|^{p-1}v_{0,k} := u_{0,k}(x) = M_k^{\frac{p}{p+1}} k^{-\frac{N}{p+1}} \delta_k(x). \end{cases} \tag{5.18}$$

Without loss of generality we may suppose

$$\|\delta_k(x)\|^{\frac{p+1}{p}}_{L^{\frac{p+1}{p}}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\delta_k(x)|^{\frac{p+1}{p}} dx \leq c_0 k^{\frac{N}{p}} \quad \forall k \in \mathbb{N}. \tag{5.19}$$

Now the sequence $\{M_k\}$ is such that

$$M_k^{\frac{p}{p+1}} k^{-\frac{N}{p+1}} \to \infty \quad \text{as } k \to \infty. \tag{5.20}$$

STEP 1. *The local energy framework.* Consider the following energy functions:

$$\begin{aligned} &I_1(\tau) = \int_{Q_r} |\nabla_x v|^2 \, dx\, dt, \qquad I_2(\tau) = \int_{Q_r} h(t)|v|^{g+1} \, dx\, dt, \\ &I_3(\tau) = \int_{Q_r} |v|^{p+1} \, dx\, dt. \end{aligned} \tag{5.21}$$

Analogously to (3.9) we deduce the inequality

$$(5.22)\qquad \int_{\mathbb{R}^N} |v(x,T)|^{p+1}\,dx + I_1(r) + I_2(r) + I_3(r)$$
$$\le c\tau^{\frac{N(g-p)}{g+1}} h(r)^{-\frac{p+1}{g+1}} (-I_2'(r))^{\frac{p+1}{g+1}} + c\int_{\Omega(\tau)} |v(x,r)|^{p+1}\,dx \qquad \forall \tau > 0,\ \forall r : 0 < r < T.$$

This inequality will control the spreading of energy with respect to the r-variable (the time direction). As to the vanishing property of energy in variable τ, we will use the finite speed of propagation of support for the porous media equation with slow diffusion. In the domain $Q^{(r)}(\tau)$ we will use the energy function $E_1(r,\tau) = \int_{Q^{(r)}(\tau)} |\nabla_x v|^2\,dx\,dt$ from (3.12). Since $\operatorname{supp} v(\cdot,0) = \operatorname{supp} v_k(\cdot,0) = \operatorname{supp} v_{0,k} = \{x : |x| < k^{-1}\}$, multiplying equation (5.18) by $v(x,t)$ and integrating over $Q^{(r)}(\tau),\ \tau \ge k^{-1}$, we obtain after simple computations (see, for example, [1, 4]) the differential inequality

$$(5.23)\quad \int_{\Omega(\tau)} |v(x,r)|^{p+1}\,dx + E_1(r,\tau) \le c r^{\frac{(p+1)(1-\theta_1)}{p+1-(1-\theta_1)(1-p)}} \left(-\frac{d}{d\tau}E_1(r,\tau)\right)^{\frac{p+1}{p+1-(1-\theta_1)(1-p)}}$$
$$\forall \tau \ge k^{-1},\ \forall r > 0,$$

where
$$\theta_1 = \frac{N(1-p)+(p+1)}{N(1-p)+2(p+1)}, \qquad 1-\theta_1 = \frac{p+1}{N(1-p)+2(p+1)}.$$

Solving this inequality and keeping in mind that $E_1(r,\tau) \ge 0$ for all $r > 0$ and $\tau > 0$, we deduce easily that

$$(5.24)\qquad v(x,r) \equiv 0 \quad \forall x : |x| > k^{-1} + c_0 r^{1-\theta_1} E_1(r,k^{-1})^{\frac{(1-\theta_1)(1-p)}{1+p}}$$
$$=: k^{-1} + c_0\chi(r),\ \forall r > 0.$$

Here the constant $c_0 > 0$ depends on the parameters of the problem under consideration, but not on r and k. Analogously to (3.25) we deduce the following global *a priori* estimate:

$$(5.25)\qquad \int_{Q^{(r)}} (|\nabla_x v|^2 + r^{-1}|v|^{p+1} + h(t)|v|^{g+1})\,dx\,dt \le c\|v_{0,k}\|^{p+1}_{L_{p+1}(\mathbb{R}^N)}.$$

Thus, due to (5.18)–(5.20), it follows from (5.25) that

$$(5.26)\qquad E_1(r,0) \le cM_k \quad \forall r > 0.$$

Next we return to the inequality (5.22). Due to (5.24) it follows from (5.22) that

$$(5.27)\quad I_1(r) + I_2(r) + I_3(r) \le c(k^{-1} + \chi(r))^{\frac{N(g-p)}{g+1}} h(r)^{-\frac{p+1}{g+1}} (-I_2'(r))^{\frac{p+1}{g+1}} \quad \forall r > 0.$$

We remark that due to (5.26) we have

$$(5.28)\qquad \chi(r) \le c_1 r^{1-\theta_1} M_k^{\frac{(1-\theta_1)(1-p)}{1+p}}.$$

STEP 2. *The first round of computations.* Now we have to define $\tau_k,\ r_k$. First we impose the condition

$$\tau_k \geq c_1 r_k^{1-\theta_1} M_k^{\frac{(1-\theta_1)(1-p)}{1+p}}, \quad c_1 \text{ is from (5.28).} \tag{5.29}$$

Then (5.27) yields

$$I(r) := I_1(r) + I_2(r) + I_3(r) \leq c(k^{-1} + \tau_k)^{\frac{N(g-p)}{g+1}} h(r)^{-\frac{p+1}{g+1}} (-I'(r))^{\frac{p+1}{g+1}} \quad \forall r : 0 < r < r_k. \tag{5.30}$$

Solving this differential inequality we get the estimate

$$I(r) \leq \frac{c(k^{-1} + \tau_k)^N}{(\int_0^r h(s)\,ds)^{\frac{p+1}{g-p}}} \quad \forall r : 0 < r < r_k. \tag{5.31}$$

Remember that the function $h(s)$ has the form $h(s) = s^{(g-1)/(1-p)}\omega(s)^{-1}$, therefore estimate (5.31) yields

$$I(r) \leq \frac{c_2 \omega(r)^{\frac{p+1}{g-p}} (k^{-1} + \tau_k)^N}{r^{\frac{p+1}{1-p}}} \quad \forall r : 0 < r \leq r_k. \tag{5.32}$$

Thus, as a second condition which defines our pair $\tau_k,\ r_k$, we suppose that

$$\frac{c_2 \omega(r_k)^{\frac{p+1}{g-p}} (k^{-1} + \tau_k)^N}{r_k^{\frac{p+1}{1-p}}} \leq c M_{k-1}, \quad c \text{ is from (5.26).} \tag{5.33}$$

Moreover, we will find the pair $\tau_k,\ r_k$ such that

$$k^{-1} + \tau_k \leq 1. \tag{5.34}$$

Then the following inequality is a sufficient condition for validity of (5.33):

$$c_2 \omega(r_k)^{\frac{p+1}{g-p}} r_k^{-\frac{p+1}{1-p}} \leq c M_{k-1}, \quad c \text{ is from (5.26),} \tag{5.35}$$

and we can define r_k by

$$r_k := \left(\frac{c_2}{c}\right)^{\frac{1-p}{p+1}} \omega(r_k)^{\frac{1-p}{g-p}} M_{k-1}^{-\frac{1-p}{p+1}}. \tag{5.36}$$

Now we have to choose the sequence $\{M_k\}$. We set

$$M_k := e^k \qquad \forall k \in \mathbb{N}, \tag{5.37}$$

and we define τ_k, in accordance with assumption (5.29), by

$$\tau_k = c_1 r_k^{1-\theta_1} M_k^{\frac{(1-\theta_1)(1-p)}{1+p}}, \quad c_1 \text{ is from (5.28).} \tag{5.38}$$

Further, due to (5.36) and (5.37), it follows from (5.38) that

$$\tag{5.39} \begin{aligned} \tau_k &= c_1 (r_k^{p+1} M_k^{1-p})^{\frac{1}{N(1-p)+2(p+1)}} \\ &= c_1 \left[\left(\frac{c_2}{c} \right)^{1-p} \omega(r_k)^{\frac{(1-p)(p+1)}{g-p}} M_{k-1}^{-(1-p)} M_k^{1-p} \right]^{\frac{1}{N(1-p)+2(p+1)}} \\ &= c_1 \left(\frac{ec_2}{c} \right)^{\frac{(1-\theta_1)(1-p)}{1+p}} \omega(r_k)^S, \end{aligned}$$

where

$$S = \frac{(1-\theta_1)(1-p)}{g-p} = \frac{(1-p)(p+1)}{(g-p)[N(1-p)+2(p+1)]}.$$

From definition (5.36) and because of (5.37) and (3.43),

$$\tag{5.40} r_k \leq \left(\frac{c_2}{c} \right)^{\frac{1-p}{p+1}} \omega_0^{\frac{1-p}{g-p}} \exp\left(-\frac{1-p}{p+1}(k-1) \right) =: c_3 \exp\left(-\frac{1-p}{p+1} k \right),$$

and $r_k \to 0$ as $k \to \infty$. Therefore, since $\omega(s) \to 0$ as $s \to 0$, it follows from (5.39) that $\tau_k \to 0$ as $k \to \infty$. Consequently, we can suppose k so large that condition (5.34) is satisfied. Thus, we have the pair (τ_k, r_k) for large $k \in \mathbb{N}$.

STEP 3. *The second round of computations*. As a starting global *a priori* estimate of a solution we will now use, instead of (5.25), (5.26), the following estimate:

$$\tag{5.41} I_1(r_k) = \int_{\{t \geq r_k, x \in \mathbb{R}^N\}} |\nabla_x v|^2 \, dx \, dt \leq I(r_k) \leq c M_{k-1},$$

which follows from (5.32), due to definition (5.33), (5.36) of r_k. Using property (5.24), estimate (5.28) and property (5.29), it follows from (5.41) that

$$\tag{5.42} E_1(r, k^{-1} + \tau_k) \leq I_1(r) \leq I_1(r_k) < c M_{k-1} \quad \forall r \geq r_k.$$

Since $v(x, r_k) = 0$ for all x with $|x| \geq k^{-1} + \tau_k$ we deduce similarly to (5.23) that

$$\tag{5.43} \begin{aligned} &\int_{\Omega(\tau)} |v(x, r_k + r)|^{p+1} \, dx + E_1(r_k + r, k^{-1} + \tau_k + \tau) \\ &\leq c r^{\frac{(p+1)(1-\theta_1)}{(p+1)-(1-\theta_1)(1-p)}} \left(-\frac{d}{d\tau} E_1(r_k + r, k^{-1} + \tau_k + \tau) \right)^{\frac{p+1}{p+1-(1-\theta_1)(1-p)}} \qquad \forall r > 0, \ \forall \tau > 0. \end{aligned}$$

Solving this differential inequality, we obtain

$$\tag{5.44} v(x, r_k + r) \equiv 0 \quad \forall x : |x| \geq k^{-1} + \tau_k + c_0 \chi_1(r),$$

where $\chi_1(r) := r^{1-\theta_1} E_1(r_k + r, k^{-1} + \tau_k)^{\frac{(1-\theta_1)(1-p)}{1+p}}$ for $r \geq 0$. But (5.42) implies

$$\tag{5.45} \chi_1(r) \leq c_1 r^{1-\theta_1} M_{k-1}^{\frac{(1-\theta_1)(1-p)}{1+p}}.$$

Now we define $\tau_{k-1},\ r_{k-1}$. In the same way as (5.29) we impose

$$\tau_{k-1} \geq c_1 r_{k-1}^{1-\theta_1} M_{k-1}^{\frac{(1-\theta_1)(1-p)}{1+p}}. \tag{5.46}$$

Similarly to (5.30)–(5.32) we deduce

$$I(r) \leq \frac{c_2 \omega(r)^{\frac{p+1}{g-p}} (k^{-1} + \tau_k + \tau_{k-1})^N}{r^{\frac{p+1}{1-p}}} \quad \forall r : 0 < r \leq r_k + r_{k-1}. \tag{5.47}$$

The second condition defining the pair $\tau_{k-1},\ r_{k-1}$ is analogous to (5.33):

$$\frac{c_2 \omega(r_k + r_{k-1})^{\frac{p+1}{g-p}} (k^{-1} + \tau_k + \tau_{k-1})^N}{(r_k + r_{k-1})^{\frac{p+1}{1-p}}} \leq c M_{k-2}, \quad c \text{ is from (5.26).} \tag{5.48}$$

Supposing that

$$k^{-1} + \tau_k + \tau_{k-1} \leq 1, \tag{5.49}$$

we can define r_{k-1} by the following analogue of (5.36):

$$r_k + r_{k-1} := \left(\frac{c_2}{c}\right)^{\frac{1-p}{p+1}} \omega(r_k + r_{k-1})^{\frac{1-p}{g-p}} M_{k-2}^{-\frac{1-p}{p+1}}. \tag{5.50}$$

And in accordance with (5.46) let us define τ_{k-1} by

$$\tau_{k-1} = c_1 r_{k-1}^{1-\theta_1} M_{k-1}^{\frac{(1-\theta_1)(1-p)}{1+p}}. \tag{5.51}$$

Due to (5.50) we have

$$\begin{aligned}
\tau_{k-1} &\leq c_1 [(r_k + r_{k-1})^{p+1} M_{k-1}^{1-p}]^{\frac{1}{N(1-p)+2(p+1)}} \\
&\leq c_1 \left[\left(\frac{c_2}{c}\right)^{1-p} \omega(r_k + r_{k-1})^{\frac{(1-p)(p+1)}{g-p}} M_{k-2}^{-(1-p)} M_{k-1}^{1-p} \right]^{\frac{1}{N(1-p)+2(p+1)}} \\
&= c_1 \left(\frac{ec_2}{c}\right)^{\frac{(1-\theta_1)(1-p)}{1+p}} \omega(r_k + r_{k-1})^S,
\end{aligned}$$

where S is from (5.39). Notice that, due to (5.47), (5.48), we also have

$$I_1(r_k + r_{k-1}) \leq I(r_k + r_{k-1}) \leq c M_{k-2}, \tag{5.52}$$

and, analogously to (5.42),

$$E_1(r, k^{-1} + \tau_k + \tau_{k-1}) \leq I_1(r) \leq I_1(r_k + r_{k-1}) \leq c M_{k-2} \quad \forall r \geq r_k + r_{k-1}. \tag{5.53}$$

STEP 4. *Completion of the proof.* We can use estimates (5.52), (5.53) instead of (5.41), (5.42) for the third round of computations. After j such rounds we deduce that

$$I_1\Big(\sum_{i=0}^{j} r_{k-i}\Big) \le I\Big(\sum_{i=0}^{j} r_{k-i}\Big) \le cM_{k-j}, \tag{5.54}$$

$$E_1\Big(r, k^{-1} + \sum_{i=0}^{j} \tau_{k-i}\Big) \le I_1(r) \le I_1\Big(\sum_{i=0}^{j} r_{k-i}\Big) \le cM_{k-j} \quad \forall r \ge \sum_{i=0}^{j} r_{k-i}, \tag{5.55}$$

where

$$\tau_{k-i} \le c_1\Big(\frac{ec_2}{c}\Big)^{\frac{(1-\theta_1)(1-p)}{1+p}} \omega\Big(\sum_{l=0}^{i} r_{k-l}\Big)^S, \tag{5.56}$$

with the same S as in (5.39), and

$$\sum_{l=0}^{i} r_{k-l} = \Big(\frac{c_2}{c}\Big)^{\frac{1-p}{p+1}} \omega\Big(\sum_{l=0}^{i} r_{k-l}\Big)^{\frac{1-p}{g-p}} M_{k-i-1}^{-\frac{1-p}{p+1}}. \tag{5.57}$$

Estimates (5.54) will remain true as long as the following analogue of relation (5.49) is valid:

$$k^{-1} + \sum_{i=0}^{j} \tau_{k-i} \le 1.$$

Now we will check this condition. Due to (3.32), it follows from (5.57) that

$$\sum_{l=0}^{i} r_{k-l} \le \Big(\frac{c_2}{c}\Big)^{\frac{1-p}{p+1}} \omega_0^{\frac{1-p}{g-p}} M_{k-i-1}^{-\frac{1-p}{p+1}} =: CM_{k-i-1}^{-\frac{1-p}{p+1}} = C\exp\Big(-\frac{1-p}{p+1}(k-i-1)\Big).$$

Therefore, from (5.56), it follows that

$$\begin{aligned}\tau_{k-i} \ &\le\ c_1\Big(\frac{ec_2}{c}\Big)^{\frac{(1-\theta_1)(1-p)}{1+p}} \Big[\omega\Big(C\exp\Big(-\frac{(1-p)(k-i-1)}{p+1}\Big)\Big)\Big]^S \\ &=: C_1\Big[\omega\Big(C\exp\Big(-\frac{(1-p)(k-i-1)}{p+1}\Big)\Big)\Big]^S.\end{aligned}$$

Thus we have, using in particular the monotonicity of $\omega(s)$,

$$\begin{aligned}\sum_{i=0}^{j} \tau_{k-i} &\le C_1 \sum_{i=0}^{j} \Big[\omega\Big(C\exp\Big(-\frac{(1-p)(k-i-1)}{p+1}\Big)\Big)\Big]^S \\ &\le C_1 \int_{k-j-1}^{k} \Big[\omega\Big(C\exp\Big(-\frac{(1-p)s}{p+1}\Big)\Big)\Big]^S ds \\ &= \frac{C_1(p+1)}{1-p}\int_{A_1}^{A_2} \frac{\omega(s)^S}{s}\, ds,\end{aligned} \tag{5.58}$$

where

$$A_1 = C \exp\left(-\frac{1-p}{p+1}k\right), \qquad A_2 = C \exp\left[-\frac{1-p}{p+1}(k-j-1)\right].$$

Due to condition (5.16) and estimate (5.58) we can find $k_0 \in \mathbb{N}$, which depends on the parameters of problem under consideration, but does not depend on $k \in \mathbb{N}$, such that

$$\sum_{i=0}^{k-k_0} \tau_{k-i} + k^{-1} \le 1 \qquad \forall k \in \mathbb{N}.$$

Finally, our estimates (5.54)–(5.57) are true for all $j \le k - k_0$. Therefore the conclusion of Theorem 5.3 follows from estimates (5.54)–(5.57), in the same way as Theorem 3.1 follows from estimates (3.75)–(3.77). □

6. THE FAST DIFFUSION EQUATION WITH ABSORPTION

When $(1 - 2/N)_+ < m < 1$, it is known that the mere fast diffusion equation

$$\partial_t v - \Delta v^m = 0 \quad \text{in } Q^\infty \tag{6.1}$$

admits a particular fundamental positive solution with initial data $k\delta_0$ ($k > 0$) called the Barenblatt–Zel'dovich–Kompaneets solution, defined by

$$B_k(x,t) = t^{-\ell}\left(C_k + \frac{(1-m)\ell}{2mN}\frac{|x|^2}{t^{2\ell/N}}\right)^{-1/(1-m)}, \tag{6.2}$$

where ℓ and C_k are given in (5.4). The main feature of this expression is that $\lim_{k\to\infty} C_k = 0$, therefore

$$\lim_{k\to\infty} B_k(x,t) = W(x,t) := C_*(t/|x|^2)^{1/(1-m)}, \tag{6.3}$$

where

$$C_* = \left(\frac{(1-m)^3}{2m(mN+2-N)}\right)^{1/(1-m)}.$$

This solution has a persisting singularity and is called a *razor blade* [18]. It also has the property that

$$\lim_{t\to 0} W(x,t) = 0 \qquad \forall x \neq 0.$$

This phenomenon is at the origin of the work of Chasseigne and Vázquez on extended solutions of the fast diffusion equation [3]. Concerning problem (5.1), Proposition 5.1 is still valid provided $m > (1 + 2/N)_+$. We shall denote by $u = u_k$ the solutions of (5.1). Furthermore estimate (5.8) holds. Combining this with the fact that the B_k are supersolutions for the u_k, we derive the following

THEOREM 6.1. *Assume* $(1 - 2/N)_+ < m < 1$ *and* $h \in C(0,\infty)$ *is positive. Assume also that* (5.6) *holds. Then* $u_\infty := \lim_{k\to\infty} u_k$ *has a pointwise singularity at* $(0,0)$ *and the following estimate is satisfied:*

$$u_\infty(x,t) \le \min\left\{C_* t^{-\ell}\left(\frac{|x|^2}{t^{2\ell/N}}\right)^{-1/(1-m)}, \left((q-1)\int_0^t h(s)\,ds\right)^{-1/(q-1)}\right\}. \tag{6.4}$$

REMARK. The profile of u_∞ near $(x,t) = (0,0)$ is completely unknown. In particular a very challenging question could be to give precise estimates of the quantity $\min\{W(x,t), U_h(t)\} - u_\infty(x,t)$.

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