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Partial differential equations. — *Multiple solutions of superlinear elliptic equations*, by PAUL H. RABINOWITZ, JIABAO SU and ZHI-QIANG WANG, communicated on 23 June 2006.

ABSTRACT. — We give some multiplicity results on existence of nontrivial solutions for superlinear elliptic equations with a saddle structure near 0. We make use of a combination of bifurcation theory and minimax methods.

KEY WORDS: Elliptic equation; bifurcation; minimax method.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35J65, 58E05.

1. INTRODUCTION

This paper is concerned with constructing multiple nontrivial solutions of the semilinear elliptic boundary value problem

$$-\Delta u = \lambda u + f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{P}$$

which has received much attention during the last several decades. Here Ω is a bounded smooth domain in $\mathbb{R}^N$. We make the following assumptions on f:

(f_1) $f \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$.
(f_2) $f(x,0) = 0 = f_u(x,0)$.
(f_3) There are $C > 0$ and $2 < p < 2^*$ such that $|f(x,u)| \le C(1 + |u|^{p-1})$ for all $x \in \Omega$ and $u \in \mathbb{R}$, where $2^* = 2N/(N-2)$ for $N \ge 3$ and $2^* = \infty$ for $N = 1, 2$.
(f_4) There are $\mu > 2$ and $M > 0$ such that

$$0 < \mu F(x,u) := \mu \int_0^u f(x,t)\,dt \le u f(x,u)$$

for all $x \in \Omega$ and $|u| \ge M$.

Hypotheses (f_1)–(f_4) are standard conditions used in the paper [1] by Ambrosetti and Rabinowitz and subsequently by many others in the study of superlinear problems. The question of interest here is in giving a lower bound on the number of nontrivial solutions. Denote by $0 < \lambda_1 < \lambda_2 < \cdots$ the distinct eigenvalues of the linear eigenvalue problem

$$-\Delta v = \lambda v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{P$_0$}$$

In [1], for $\lambda < \lambda_1$, one positive and one negative solution were obtained by use of the mountain-pass theorem. When f is also odd in u, infinitely many solutions were obtained for any λ by the symmetric mountain-pass theorem. Without the oddness condition, a third solution for $\lambda < \lambda_1$ was constructed by Wang in [18] by using a two-dimensional linking method and a Morse-theoretic approach. This result has been generalized and proved in other ways by many authors (see [2–5, 7, 12] and the references therein). The question is still open as to whether there exist infinitely many solutions without assuming any symmetry conditions. When $\lambda > \lambda_1$, in general one nontrivial solution is found in [15, 16] under an additional condition: $f(x,u)u \geq 0$. The same conclusion was proved in [10, 11] without this additional condition when $\lambda \neq \lambda_i$ and with a local sign condition on $f(x,u)u$ near zero when $\lambda = \lambda_i$ for some i. Recently, in a paper of Mugnai [13], it is proved that for $\lambda < \lambda_i$ and very close to λ_i, there are three nontrivial solutions. The conditions in [13] seem to be unduly restrictive since it is required that $\mu = p$ with μ in (f_4) and p in (f_3). This requires the nonlinear term to behave exactly like $|u|^{p-2}u$ for $|u|$ large. On the other hand, the bifurcation result ([14, 16]) always gives bifurcation at an eigenvalue λ_i regardless of the behavior of the nonlinearity in the large.

The purpose of the current paper is two-fold. On one hand, we prove the multiplicity result of [13] under more natural conditions. On the other hand, our approach is different in that we make use of a combination of bifurcation analysis and minimax methods, which have been used separately in [14–16] and [3, 18]. Our method also gives some additional information.

Before stating our main results we introduce two additional assumptions.

(f_5) $F(x,u) \geq 0$ for all x and u; and $uf(x,u) > 0$ for $|u| > 0$ small.
(f_6) $uf(x,u) < 0$ for $|u| > 0$ small.

Denote by F^+ and F^- the positive and negative parts of F, respectively, i.e. $F^\pm(x,u) = \max\{\pm F(x,u), 0\}$. The main results in this paper are the following two theorems:

THEOREM 1.1. *Assume* (f_1)–(f_5) *hold and let* $k \geq 1$ *be fixed. Then there is* $\delta > 0$ *such that for* $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1})$, *equation* (P) *has at least three nontrivial solutions.*

THEOREM 1.2. *Assume* (f_1)–(f_4) *and* (f_6) *hold and let* $k \geq 1$ *be fixed. Then there is* $\delta > 0$ *such that when* $\sup_{(x,u)\in\Omega\times\mathbb{R}} F^-(x,u) < \delta$,

(i) *for* $\lambda \in (\lambda_{k+1}, \lambda_{k+1} + \delta)$, *equation* (P) *has at least three nontrivial solutions;*
(ii) *for* $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1}]$, *equation* (P) *has at least two nontrivial solutions.*

REMARK 1.3. The solutions are constructed by a combination of bifurcation arguments, topological linking and Morse theory. In Theorems 1.1 and 1.2(i) two solutions are small while the third one stays away from 0 as $\lambda \to \lambda_{k+1}$. In Theorem 1.2(ii), we have two solutions which are not near 0.

The paper is organized as follows. In Section 2 we recall the classical bifurcation results of [14, 16] and discuss their homological local content. Section 3 gives the existence of a solution by a linking argument which requires λ to be close to λ_{k+1}. We also get information on the critical groups of this solution. Section 4 is devoted to the proof of the main results. We finish Section 4 with a discussion comparing the solutions obtained from the linking structures associated with two adjacent eigenvalues, and prove that for some λ-interval these solutions are different.

2. Bifurcation solutions

In this section we get two small solutions by applying bifurcation theory ([16]) and then discuss their homological local consequences. First let us recall the bifurcation result of [16].

THEOREM 2.1 (Theorem 11.35 in [16]). *Let E be a Hilbert space and $I \in C^2(E, \mathbb{R})$ with*

$$\nabla I(u) = Lu + H(u)$$

where $L \in \mathcal{L}(E, E)$ is symmetric and $H(u) = o(\|u\|)$ as $\|u\| \to 0$. Consider the equation

$$Lu + H(u) = \lambda\, u. \tag{2.2}$$

Let $\mu \in \sigma(L)$ be an isolated eigenvalue of finite multiplicity. Then either

(i) *$(\mu, 0)$ is not an isolated solution of (2.2) in $\{\mu\} \times E$, or*
(ii) *there is a one-sided neighborhood Λ of μ such that for all $\lambda \in \Lambda \setminus \{\mu\}$, (2.2) has at least two distinct nontrivial solutions, or*
(iii) *there is a neighborhood Λ of μ such that for all $\lambda \in \Lambda \setminus \{\mu\}$, (2.2) has at least one nontrivial solution.*

Now we apply Theorem 2.1 to get two small solutions of equation (P). We have

PROPOSITION 2.3. *Let f satisfy (f_1), (f_2) and $k \geq 1$. Then there is a $\delta > 0$ such that equation (P) has at least two nontrivial solutions for*

(i) *every $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1})$ if (f_5) holds,*
(ii) *every $\lambda \in (\lambda_{k+1}, \lambda_{k+1} + \delta)$ if (f_6) holds.*

PROOF. We prove this result by verifying that case (ii) of Theorem 2.1 occurs under the given conditions. First, under (f_1) and (f_2), every eigenvalue λ_j of (P_0) gives rise to a bifurcation point $(\lambda_j, 0)$ of equation (P).

Let $(\lambda, u) \in \mathbb{R} \times E$ be a solution of equation (P) near $(\lambda_{k+1}, 0)$. Consider the linear eigenvalue problem

$$-\Delta v - h(x)v = \mu v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

where $h(x) = f(x, u(x))/u(x)$ for $u(x) \neq 0$ and $h(x) = 0$ for $u(x) = 0$. Its eigenvalues will be denoted by $\mu_1(u) < \mu_2(u) \leq \cdots$.

Suppose (f_5) holds. Then $h(x) \geqslant 0$ and $h(x) > 0$ if $u(x) \neq 0$. Therefore the standard variational characterization of the eigenvalues of (2.4) shows $\mu_i(u)$ is less than the corresponding i-th ordered eigenvalue ν_i of (P_0) for each $i \in \mathbb{N}$ and $\mu_i(u) \to \nu_i$ as $(\lambda, u) \to (\lambda_{k+1}, 0)$. But u is an eigenfunction of (2.4) with eigenvalue λ. It follows that $\lambda < \lambda_{k+1}$ and alternative (i) of Proposition 2.3 holds. Likewise (ii) is valid if (f_6) is satisfied. □

The (weak) solutions of equation (P) correspond to critical points of

$$I(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx - \int_\Omega F(x, u)\, dx, \qquad u \in E := W_0^{1,2}(\Omega).$$

We will use the following notation. For $j \in \mathbb{N}$,

$$E(\lambda_j) = \ker(-\Delta - \lambda_j), \quad E_j = \bigoplus_{i=1}^{j} E(\lambda_i), \quad \nu_j = \dim E(\lambda_j), \quad \ell_j = \dim E_j.$$

Thus $\ell_j = \sum_{i=1}^{j} \nu_i$. For $c \in \mathbb{R}$,

$$I^c = \{u \in E \mid I(u) \leq c\}, \quad \mathcal{K}_c = \{u \in E \mid I'(u) = 0,\ I(u) = c\}.$$

For later use we give information on the critical group of I at 0. We say a functional $I \in C^1(E, \mathbb{R})$ has a *local linking structure* at 0 with respect to a direct sum decomposition $E = Y \oplus Z$ (cf. [10, 11]) if there is an $r > 0$ such that

$$I(u) \leq 0 \text{ for } u \in Y \text{ with } \|u\| \leq r, \quad I(u) > 0 \text{ for } u \in Z \text{ with } 0 < \|u\| \leq r.$$

Recall that the q-th *critical group* of I at its isolated critical point u is defined as

$$C_q(I, u) := H_q(I^c \cap U, I^c \cap U \setminus \{u\}).$$

Here $c = I(u)$ and $H_q(A, B)$ is the q-th relative singular homology group of the topological pair (A, B) with coefficients in a field $\mathbb{F}$. We have

PROPOSITION 2.5. *If (f_5) is satisfied, then $C_q(I, 0) = \delta_{q,\ell_{k+1}}\mathbb{F}$ when $\lambda \in [\lambda_{k+1}, \lambda_{k+2})$. If (f_6) is satisfied, then $C_q(I, 0) = \delta_{q,\ell_k}\mathbb{F}$ when $\lambda \in (\lambda_k, \lambda_{k+1}]$.*

PROOF. The nondegenerate cases are easily seen. At $\lambda = \lambda_{k+1}$, $u = 0$ is an isolated degenerate solution of equation (P) with Morse index ℓ_k and nullity ν_k. When (f_5) is satisfied, I has a local linking at 0 with respect to the decomposition $E = E_{k+1} \oplus E_{k+1}^{\perp}$. When (f_6) is satisfied, we see that $F(x, u) \leq 0$ for $|u|$ small and then I has a local linking at 0 with respect to the decomposition $E = E_k \oplus E_k^{\perp}$. Proposition 2.2 in [17] then gives the conclusions of Proposition 2.5. □

3. MINIMAX SOLUTIONS

In this section we construct a large solution of equation (P) by applying a homological linking argument and give some estimate of its Morse index. This is done for cases (f_5) and (f_6).

LEMMA 3.1. *Let f satisfy (f_1)–(f_3) and $k \geq 1$. Then there exist constants $\beta_1, r_1 > 0$, depending on $\lambda < \lambda_{k+2}$, such that*

$$I(u) \geq \beta_1 \quad \text{for } u \in E_{k+1}^{\perp} \text{ with } \|u\| = r_1. \tag{3.2}$$

PROOF. By (f_2) and (f_3), for $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$F^+(x, t) \leq \frac{\varepsilon}{2} t^2 + C_\varepsilon |t|^p.$$

Thus for $u \in E_{k+1}^{\perp}$,

$$I(u) \geq \frac{1}{2}\left(1 - \frac{\lambda + \varepsilon}{\lambda_{k+2}}\right)\|u\|^2 - C_\varepsilon \int_\Omega |u|^p \, dx.$$

Let $\alpha \in (0, 1)$ be such that

$$\frac{1}{p} = \frac{\alpha}{2^*} + \frac{1-\alpha}{2}.$$

Then by the Gagliardo–Nirenberg inequality, for some $C_1 > 0$ independent of $\lambda < \lambda_{k+2}$,

$$\int_\Omega |u|^p \, dx \leq C_1 \|u\|^{\alpha p} \left(\int_\Omega u^2 \, dx\right)^{(1-\alpha)p/2}.$$

Since

$$\int_\Omega u^2 \, dx \leq \frac{1}{\lambda_{k+2}} \int_\Omega |\nabla u|^2 \, dx, \qquad \forall u \in E_{k+1}^{\perp},$$

one has

$$\int_\Omega |u|^p \, dx \leq C_1 \lambda_{k+2}^{-(1-\alpha)p/2} \|u\|^p, \qquad \forall u \in E_{k+1}^{\perp}.$$

Therefore, setting $\widehat{C} = C_\varepsilon C_1$ gives

$$I(u) \geq \frac{1}{2}\left(1 - \frac{\lambda + \varepsilon}{\lambda_{k+2}}\right)\|u\|^2 - \widehat{C}\lambda_{k+2}^{-(1-\alpha)p/2}\|u\|^p. \tag{3.3}$$

Let $\|u\| = r$ and

$$g(r) = \frac{1}{2}\left(1 - \frac{\lambda + \varepsilon}{\lambda_{k+2}}\right)r^2 - \widehat{C}\lambda_{k+2}^{-(1-\alpha)p/2} \, r^p.$$

It is easy to see that g achieves its maximum on $\mathbb{R}$ at

$$r_1 = r_1(k, \lambda) := \left(\frac{\lambda_{k+2} - (\lambda + \varepsilon)}{p\widehat{C} \, \lambda_{k+2}^{1-(1-\alpha)p/2}}\right)^{1/(p-2)}$$

with the maximum given by

$$g(r_1) = \left(\frac{1}{2} - \frac{1}{p}\right)(p\widehat{C})^{-2/(p-2)}\left(\frac{\lambda_{k+2} - (\lambda + \varepsilon)}{\lambda_{k+2}^{\alpha}}\right)^{p/(p-2)} =: \beta_1 = \beta_1(k, \lambda). \tag{3.4}$$

Hence (3.3) and (3.4) show (3.2) holds. The proof is complete. □

Next take an eigenfunction φ_{k+2} corresponding to λ_{k+2}. Set $V_{k+1} = E_{k+1} \oplus \operatorname{span}\{\varphi_{k+2}\}$ and let

$$Q_1 = \{u \in V_{k+1} \mid \|u\| \leq R_1, \; u = v + t\varphi_{k+2}, \; v \in E_{k+1}, \; t \geq 0\},$$

where $R_1 > 0$ will be given below. We have

LEMMA 3.5. *Let f satisfy (f_4), (f_5) and $k \geq 1$. Then there exists $R_1 > 0$ independent of $\lambda < \lambda_{k+2}$, $\delta_1 > 0$ and $\sigma_1 \in \mathbb{R}$ such that*

$$I(u) \leq \sigma_1 < \beta_1 \quad \textit{for } u \in \partial Q_1, \quad \forall \lambda \in (\lambda_{k+1} - \delta_1, \lambda_{k+1}).$$

PROOF. It follows from (f_4) that

$$F(x,t) \geq C|t|^\mu, \quad \forall |t| \geq M,$$

for some positive constant C independent of λ. For $u \in V_{k+1}$, write $u = y + z$, where $y \in E_k$ and $z \in E(\lambda_{k+1}) \oplus \operatorname{span}\{\varphi_{k+2}\}$. Then

$$I(u) \leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|y\|^2 + \frac{1}{2}\left(1 - \frac{\lambda_k}{\lambda_{k+2}}\right)\|z\|^2 - C\|u\|^\mu_{L^\mu} + C. \tag{3.6}$$

Since $\mu > 2$ and V_{k+1} is finite-dimensional, (3.6) shows there exists $R_1 > 0$ independent of λ such that

$$I(u) \leq 0 \quad \text{for } u \in V_{k+1} \text{ with } \|u\| = R_1.$$

Now fixing such $R_1 > 0$, notice that

$$\partial Q_1 = \{u = v + t\varphi_{k+2} \mid v \in E_{k+1},\ (\|v\| \leq R_1,\ t = 0) \text{ or } (\|u\| = R_1,\ t \geq 0)\}.$$

For $v \in E_{k+1}$ with $\|v\| \leq R_1$, write $v = w + z$, where $w \in E_k$ and $z \in E(\lambda_{k+1})$. Then

$$\begin{aligned} I(v) &= \frac{1}{2}\int_\Omega (|\nabla w|^2 - \lambda w^2)\,dx + \frac{1}{2}\int_\Omega (|\nabla z|^2 - \lambda z^2)\,dx - \int_\Omega F(x,v)\,dx \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\|z\|^2 \leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)R_1^2. \end{aligned} \tag{3.7}$$

Here we only used the assumption $F(x,t) \geq 0$ in (f_5). If we take $\delta_1 = \beta_1\lambda_{k+1}/R_1^2$ and $\sigma_1 = \beta_1/2$, the conclusion of Lemma 3.5 follows from (3.7). □

REMARK 3.8. If (f_4) is strengthened to

(f_4') There is $\mu > 2$ such that

$$0 < \mu F(x,u) := \mu \int_0^u f(x,t)\,dt \leq u f(x,u), \quad \forall x \in \Omega, u \neq 0,$$

we can get a sharper estimate for σ_1 in the last lemma. In fact, using similar arguments to the above we have

$$I(u) \leq \sigma_1', \quad \forall u \in \partial Q,$$

where

$$\sigma_1' = \sigma_1'(k,\lambda) = \frac{\mu - 2}{2\mu}(C\mu)^{-2/(\mu-2)}\,|\Omega|(\lambda_{k+1} - \lambda)^{\mu/(\mu-2)}$$

in which C is such that $F(x,t) \geq C|t|^\mu$, following from (f_4').

Set

$$S_1 := \{u \in E_{k+1}^{\perp} \mid \|u\| = r_1\}.$$

It follows from Lemmas 3.1 and 3.5 that ∂Q_1 and S_1 link homologically ([6]) since we can choose $R_1 > r_1$. Define

$$c_1 := \inf_{\tau \in \Gamma_1} \sup_{u \in |\tau|} I(u)$$

where

$$\Gamma_1 = \{\tau \mid \tau \text{ is a singular } \ell_{k+1} + 1\text{-chain with } \partial\tau = \partial Q_1\}.$$

It is well known ([16]) that the functional I satisfies the Palais–Smale condition. By Theorem 1.5 of Chapter II in [6] we have

LEMMA 3.9. *Assume (f_1)–(f_5) hold. Then $c_1 \geq \beta_1 > 0$ is a critical value of I and there is a $u_2 \in \mathcal{K}_{c_1}$ such that*

$$C_{\ell_{k+1}+1}(I, u_2) \neq 0. \tag{3.10}$$

Next we consider (f_6) instead of (f_5) for f near 0. Set

$$Q_2 = \{u \in V_{k+1} \mid \|u\| \leq R_2,\ u = v + t\varphi_{k+2},\ v \in E_{k+1},\ t \geq 0\},$$

where $R_2 > 0$ will be given below. We have

LEMMA 3.11. *Let f satisfy (f_3), (f_4) and (f_6), and $k \geq 1$. There exists $R_2 > 0$ independent of λ, $\delta_2 > 0$ and $\sigma_2 \in \mathbb{R}$ such that when $\sup_{(x,t)\in\Omega\times\mathbb{R}} F^-(x,t) < \delta_2$,*

$$I(u) \leq \sigma_2 < \beta_1 \quad \textit{for } u \in \partial Q_2, \quad \forall\lambda \in (\lambda_{k+1} - \delta_2, \lambda_{k+1} + \delta_2). \tag{3.12}$$

PROOF. With the same $R_1 > 0$ as given in Lemma 3.5, we have

$$I(u) \leq 0 \quad \text{for } u \in V_{k+1} \text{ with } \|u\| = R_1.$$

Checking the proof there, we see $R_1 > 0$ can be chosen to be the same if we make $\sup_{(x,u)\in\Omega\times\mathbb{R}} F^-(x,u)$ smaller. Now set $R_2 := R_1$. Notice that

$$\partial Q_2 = \{u = v + t\varphi_{k+2} \mid v \in E_{k+1},\ (\|v\| \leq R_2,\ t = 0) \text{ or } (\|u\| = R_2,\ t \geq 0)\}.$$

Let $\hat{M} = \sup_{(x,u)\in\Omega\times\mathbb{R}} F^-(x,u)$. For $v \in E_{k+1}$ with $\|v\| \leq R_2$, writing $v = w + z$, where $w \in E_k$ and $z \in E(\lambda_{k+1})$, we have

$$\begin{aligned} I(v) &= \frac{1}{2}\int_\Omega (|\nabla w|^2 - \lambda w^2)\,dx + \frac{1}{2}\int_\Omega (|\nabla z|^2 - \lambda z^2)\,dx - \int_\Omega F(x,v)\,dx \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|w\|^2 + \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\|z\|^2 + \hat{M}|\Omega| \\ &\leq \frac{1}{2}\left(\frac{\lambda - \lambda_{k+1}}{\lambda_{k+1}}\right)R_2^2 + \hat{M}|\Omega|. \end{aligned} \tag{3.13}$$

Take

$$\delta_2 = \frac{\beta_1\lambda_{k+1}}{R_2^2 + |\Omega|\lambda_{k+1}} \quad \text{and} \quad \sigma_2 = \frac{\beta_1}{2}.$$

If $\hat{M} < \delta_2$, then (3.13) shows that (3.12) holds. □

It follows from Lemmas 3.1 and 3.11 that ∂Q_2 and S_1 link homologically ([6]) since we can choose $R_2 > r_2$. Define

$$c_2 := \inf_{\tau \in \Gamma_2} \sup_{u \in |\tau|} I(u)$$

where

$$\Gamma_2 = \{\tau \mid \tau \text{ is a singular } \ell_{k+1} + 1\text{-chain with } \partial\tau = \partial Q_2\}.$$

Now applying Theorem 1.5 of Chapter II in [6] again, we have

LEMMA 3.14. *Assume* (f_1)–(f_4) *and* (f_6) *hold. Then* $c_2 \geq \beta_1 > 0$ *is a critical value of* I *and there is a* $u_2 \in \mathcal{K}_{c_2}$ *such that*

$$C_{\ell_{k+1}+1}(I, u_2) \neq 0. \tag{3.15}$$

4. PROOFS OF THE MAIN RESULTS AND FURTHER REMARKS

We begin by giving the proofs of our main results, using the partial results of the previous sections.

PROOF OF THEOREM 1.1. By Proposition 2.3(i), equation (P) has two nontrivial solutions which are small. By Lemma 3.9, equation (P) has a solution with positive energy bounded away from 0 for λ near λ_{k+1}. Hence these three solutions are different. □

PROOF OF THEOREM 1.2. For case (i), the proof is similar to that of Theorem 1.1. By Proposition 2.3(ii) and Lemma 3.14, we obtain two small solutions from the bifurcation result and one large one from the linking argument. As above, these three solutions are different.

We prove case (ii) next. It follows from Lemma 3.14 that I has a critical point u_2 with $I(u_2) \geq \beta_1 > 0$ and $C_{\ell_{k+1}+1}(I, u_2) \neq 0$. Assume I has only two critical points 0 and u_2. Denote by S^∞ the unit sphere in E. Choose $a_0 < 0$. Following the same arguments as in [18], we have

$$H_q(I^{a_0}) \cong H_q(S^\infty), \quad H_q(E, I^{a_0}) = 0, \quad \forall q = 0, 1, 2, \ldots. \tag{4.1}$$

Then it is easy to see that $C_{q+1}(I, u_2) \cong C_q(I, 0)$ for all q. But this is impossible since by Proposition 2.5, $C_q(I, 0) \cong \delta_{q,\ell_k}\mathbb{F}$ for any $\lambda \in (\lambda_k, \lambda_{k+1}]$. The proof is complete. □

We conclude this section with further discussion on the linking structure used to construct the solutions in Theorems 1.1 and 1.2 which stay away from zero. Under (f_1)–(f_4), when $\lambda \neq \lambda_i$, it is well known that there is a solution given by the linking method. Of course, the linking structure used depends on where λ is located. For $\lambda \in (\lambda_i, \lambda_{i+1})$ the same linking structure is used. In Section 3, we proved that the solution constructed by using the linking associated with $(\lambda_{k+1}, \lambda_{k+2})$ is still valid for $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1}]$ producing one of the larger solutions in the main theorems. Next, we examine the difference between the solutions constructed by using the linking associated with $(\lambda_k, \lambda_{k+1})$ and $(\lambda_{k+1}, \lambda_{k+2})$. By showing they are different and by getting information on their local critical groups, we can give a different proof of Theorem 1.1. This proof provides some different information on the solutions although it requires a slightly stronger condition.

Take an eigenfunction φ_{k+1} corresponding to λ_{k+1} and let $V_k = E_k \oplus \operatorname{span}\{\varphi_{k+1}\}$. By arguments as in Section 3, we have

LEMMA 4.2. *Let f satisfy (f_1)–(f_4). There exists $R > 0$ independent of $\lambda \in (\lambda_k, \lambda_{k+1})$ such that*

$$I(u) \le 0 \quad \text{for } u \in V_k \text{ with } \|u\| = R,$$

and there exist $\beta_\lambda = \beta(\lambda) > 0$ and $r_\lambda = r(\lambda) > 0$, dependent on $\lambda \in (\lambda_k, \lambda_{k+1})$, such that

$$I(u) \ge \beta_\lambda \quad \text{for } u \in E_k^\perp \text{ with } \|u\| = r_\lambda.$$

Now for fixed $R > 0$ given in Lemma 4.2, define

$$Q_\lambda := \{u \in V_k \mid u = v + t\varphi_{k+1},\ v \in E_k,\ t \ge 0,\ \|u\| \le R\},$$
$$S_\lambda := \{u \in E_k^\perp \mid \|u\| = r_\lambda\}.$$

It follows from Lemma 4.2 that ∂Q_λ and S_λ link since we can choose $R > r_\lambda$ for any $\lambda \in (\lambda_k, \lambda_{k+1})$. By Theorem 1.2 of Chapter II in [6], ∂Q_λ and S_λ also link homologically. Therefore we can define

$$c_\lambda := \inf_{\tau \in \Gamma} \sup_{u \in |\tau|} I(u)$$

where

$$\Gamma = \{\tau \mid \tau \text{ is a singular } \ell_k + 1\text{-chain with } \partial\tau = \partial Q_\lambda\}.$$

Then $c_\lambda \ge \beta_\lambda > 0$ is a critical value of I for $\lambda \in (\lambda_k, \lambda_{k+1})$ and there is $u_\lambda \in \mathcal{K}_{c_\lambda}$ such that

$$C_{\ell_k+1}(I, u_\lambda) \ne 0. \tag{4.3}$$

A similar argument to that in [1] shows that u_λ is bounded in E uniformly in $\lambda \in (\lambda_k, \lambda_{k+1})$. By standard elliptic regularity arguments, u_λ is bounded in $C^1(\Omega)$ uniformly in $\lambda \in (\lambda_k, \lambda_{k+1})$. We give the asymptotic behavior of the solution u_λ as $\lambda \to \lambda_{k+1}^-$. For this purpose we assume

(f_7) $uf(x,u) \ge 2F(x,u) \ge 0$ for all x and u and the first inequality is strict for $|u| > 0$ small.

Note that (f_7) is slightly stronger than (f_5). Under (f_7) we have

LEMMA 4.4. *$c_\lambda \le \sup_{u \in Q_\lambda} I(u) \to 0$, and $u_\lambda \to 0$ in $C^0(\bar{\Omega})$ as $\lambda \to \lambda_{k+1}^-$.*

PROOF. It is easy to see $c_\lambda \to 0$ as $\lambda \to \lambda_{k+1}^-$. By regularity for a subsequence $\tau_n \to \lambda_{k+1}^-$, we may assume $u_{\tau_n} \to u$ in $C^0(\bar{\Omega})$ as $n \to \infty$. We only need to show $u = 0$. Since

$$2c_{\tau_n} = 2I(u_{\tau_n}) - \langle I'(u_{\tau_n}), u_{\tau_n}\rangle = \int_\Omega (f(x, u_{\tau_n})u_{\tau_n} - 2F(x, u_{\tau_n}))\,dx,$$

letting $n \to \infty$, the conclusion follows from (f_7). $\square$

Using the fact that u_λ is small, we get an estimate for the Morse index of u_λ for λ near λ_{k+1}. Denote by $m(u_\lambda)$ and $n(u_\lambda)$ the Morse index and nullity of u_λ, respectively. Then

LEMMA 4.5. *There is $\delta_1 > 0$ such that*

$$m(u_\lambda) \geq \ell_k, \quad n(u_\lambda) \leq \nu_{k+1}, \quad \forall \lambda \in (\lambda_{k+1} - \delta_1, \lambda_{k+1}).$$

We summarize the above results:

PROPOSITION 4.6. *Let f satisfy (f_1)–(f_4) and (f_7). For $\lambda \in (\lambda_k, \lambda_{k+1})$, there is a solution u_λ of equation* (P) *satisfying $I(u_\lambda) > 0$, $C_{\ell_k+1}(I, u_\lambda) \neq 0$, $I(u_\lambda) \to 0$ and $u_\lambda \to 0$ (in $C^0(\Omega)$) as $\lambda \to \lambda_{k+1}^-$. Furthermore, there is $\delta_1 > 0$ such that*

$$C_q(I, u_\lambda) = 0, \quad \forall q \notin [\ell_k, \ell_{k+1}], \ \forall \lambda \in (\lambda_{k+1} - \delta_1, \lambda_{k+1}).$$

Now for λ close to λ_{k+1} from the left, we can construct two solutions of equation (P) by using the linking associated with $(\lambda_k, \lambda_{k+1})$ and $(\lambda_{k+1}, \lambda_{k+2})$. As $\lambda \to \lambda_{k+1}^-$, one solution tends to 0 and the other stays away from 0. Hence we have the following result.

THEOREM 4.7. *Let f satisfy (f_1)–(f_4) and (f_7). There is $\delta > 0$ such that for $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1})$, the solutions of equation* (P) *constructed by using linking associated with $(\lambda_k, \lambda_{k+1})$ and $(\lambda_{k+1}, \lambda_{k+2})$ both exist and are different.*

Finally, we give a different proof of Theorem 1.1 (under (f_7)) by showing the existence of a third nontrivial solution via a Morse-theoretic approach.

Let $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1})$ and let u_λ, u_2 be the solutions constructed above with $0 < I(u_\lambda) < I(u_2)$. Assume that I has only three critical points $\{0, u_\lambda, u_2\}$. Choose $a_0, a_1, a_2 \in \mathbb{R}$ such that $a_0 < 0 < a_1 < I(u_\lambda) < a_2 < I(u_2)$. Then by the deformation and excision properties of homology (see e.g. [6]), we have

$$C_q(I, 0) \cong H_q(I^{a_1}, I^{a_0}), \quad C_q(I, u_\lambda) \cong H_q(I^{a_2}, I^{a_1}), \quad C_q(I, u_2) \cong H_q(E, I^{a_2}).$$

LEMMA 4.8. *For all $q = 0, 1, 2, \ldots,$*

$$C_q(I, 0) \cong \delta_{q,\ell_k}\mathbb{F}, \quad H_q(I^{a_1}) \cong \delta_{q,\ell_k}\mathbb{F}, \quad H_q(E, I^{a_1}) \cong \delta_{q,\ell_k+1}\mathbb{F}.$$

PROOF. The first result follows from 0 being a nondegenerate critical point of I with Morse index ℓ_k. The others follow from (4.1) and the exact sequence of the triple (E, I^{a_1}, I^{a_0}). □

LEMMA 4.9.

$$C_{q+1}(I, u_2) \cong C_q(I, u_\lambda) \qquad \text{for } q \geq \ell_k + 2,$$
$$C_q(I, u_2) \cong C_{q-1}(I, u_\lambda) \qquad \text{for } q \leq \ell_k,$$

and we have an exact sequence

$$0 \to C_{\ell_k+2}(I, u_2) \to C_{\ell_k+1}(I, u_\lambda) \to H_{\ell_k+1}(E, I^{a_1}) \to C_{\ell_k+1}(I, u_2) \to C_{\ell_k}(I, u_\lambda) \to 0.$$

PROOF. This lemma is obtained by using the exact sequence of the triple (E, I^{a_2}, I^{a_1}) and Lemma 4.8. □

NEW PROOF OF THEOREM 1.1. In the case $\nu_{k+1} \geq 2$, $\ell_{k+1} = \ell_k + \nu_{k+1} \geq \ell_k + 2$, by Lemma 4.9 and (3.10), we get

$$C_{\ell_{k+1}}(I, u_\lambda) \cong C_{\ell_{k+1}+1}(I, u_2) \neq 0.$$

However, by Lemma 4.5, we have $C_q(I, u_\lambda) = 0$ for $q \notin [\ell_k, \ell_{k+1}]$ and the Morse index of u_λ is either ℓ_k or $\ell_k + 1$. If it is ℓ_k, then by the shifting theorem ([6]) we have

$$C_{\ell_{k+1}}(I, u_\lambda) \cong C_{\ell_k+\nu_{k+1}}(I, u_\lambda) \cong C_{\nu_{k+1}}(\tilde{I}, u_\lambda) \neq 0$$

where $\tilde{I}$ is the restriction of I to the kernel of $I''(u_\lambda)$. Therefore u_λ is a local maximum point of $\tilde{I}$ and we get $C_q(I, u_\lambda) \cong \delta_{q,\ell_{k+1}}\mathbb{F}$. This contradicts (4.3). If the Morse index is $\ell_k + 1$, we can use the shifting theorem to get $C_q(I, u_\lambda) \cong \delta_{q,\ell_k+1}\mathbb{F}$, still a contradiction for $\ell_{k+1} > \ell_k + 1$.

Next assume $\nu_{k+1} = 1$; then $\ell_{k+1} = \ell_k + 1$. Since $C_{\ell_k+1}(I, u_\lambda) \neq 0$ we have $C_q(I, u_\lambda) \cong \delta_{q,\ell_k+1}\mathbb{F}$. By Lemma 4.9,

$$0 \to C_{\ell_k+2}(I, u_2) \to C_{\ell_k+1}(I, u_\lambda) \to H_{\ell_k+1}(E, I^{a_1}) \to C_{\ell_k+1}(I, u_2) \to 0.$$

Since the map from $C_{\ell_k+1}(I, u_\lambda)$ to $H_{\ell_k+1}(E, I^{a_1})$ is injective (cf. [3] for a proof), we get $C_{\ell_k+2}(I, u_2) = 0$, a contradiction with Lemma 3.9. The proof is complete. □

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