*Rend. Lincei Mat. Appl. 18 (2007), 125–138*

**Ordinary differential equations.** — *Hardy inequality and properties of the quasilinear Sturm–Liouville problem*, by PAVEL DRÁBEK and ALOIS KUFNER.

ABSTRACT. — We present a necessary and sufficient condition for discreteness of the set of all eigenvalues (having the usual Sturm–Liouville properties) of a quasilinear Sturm–Liouville–type problem with weights on an infinite interval. We point out that the same condition is necessary and sufficient for the compact embedding of certain weighted Sobolev and Lebesgue spaces. Our result completes those from the linear theory.

KEY WORDS: Hardy inequality; weighted spaces; Sturm–Liouville problem; infinite interval.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 34L05, 47E05, 34B40.

## 1. INTRODUCTION

Let $p > 1$ be a real number and let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined as $\varphi(s) = |s|^{p-2}s$ for $s \neq 0$, $\varphi(0) = 0$. Let $r = r(t)$, $c = c(t)$ be continuous and positive functions on $[0, \infty)$. For $x = x(t)$ defined on $[0, \infty)$ set $x(\infty) = \lim_{t\to\infty} x(t)$. We study the eigenvalue problem

$$\begin{cases} (r(t)\varphi(x'(t)))' + \lambda c(t)\varphi(x(t)) = 0, & t \geq 0, \\ x'(0) = 0, \quad x(\infty) = 0, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter.

Let $W^{1,p}_{\infty}(r)$ be the set of all absolutely continuous functions $x = x(t)$ defined on $[0, \infty)$ such that $x(\infty) = 0$ and

$$\|x\|_{1,p;r} := \left(\int_0^\infty r(t)|x'(t)|^p\,dt\right)^{1/p} < \infty. \tag{1.2}$$

Then $W^{1,p}_{\infty}(r)$ equipped with the norm $\|\cdot\|_{1,p;r}$ is a uniformly convex Banach space.

A function $x \in W^{1,p}_{\infty}(r)$ is called a *weak solution* of (1.1) if the integral identity

$$\int_0^\infty r(t)\varphi(x'(t))y'(t)\,dt = \lambda\int_0^\infty c(t)\varphi(x(t))y(t)\,dt \tag{1.3}$$

holds for all $y \in W^{1,p}_{\infty}(r)$ (with both integrals being finite).

The parameter $\lambda$ is called an *eigenvalue* of (1.1) if this problem has a nontrivial (i.e. nonzero) weak solution (called an *eigenfunction* of (1.1) associated with $\lambda$).

We say that the *(S.L.) Property for* (1.1) is satisfied if

- the set of all eigenvalues of (1.1) forms an increasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lambda_1 > 0$ and $\lim_{n\to\infty}\lambda_n = \infty$. Every eigenvalue $\lambda_n$, $n = 1, 2, \ldots$, is simple in the sense that

there exists a unique normalized eigenfunction $x_{\lambda_n}$ associated with $\lambda_n$. Moreover, the eigenfunction $x_{\lambda_n}$ has precisely $n-1$ zeros in $(0,\infty)$. In particular, $x_{\lambda_1}$ does not change sign in $(0,\infty)$. For $n \geq 3$, between two consecutive zeros of $x_{\lambda_{n-1}}$ in $(0,\infty)$, there is exactly one zero of $x_{\lambda_n}$.

Our main result depends on the following condition on the weight functions (coefficients in the equation) $r$ and $c$:

$$\lim_{t\to\infty}\left(\int_0^t c(\tau)\,d\tau\right)^{1/p}\left(\int_t^\infty r^{1-p'}(\tau)\,d\tau\right)^{1/p'} = 0, \tag{1.4}$$

and we state it in the next theorem.

THEOREM 1.1. *The (S.L.) Property for* (1.1) *is satisfied if and only if* (1.4) *holds.*

It is an interesting fact that condition (1.4) is closely connected with the *compact embedding*

$$W^{1,p}_\infty(r) \hookrightarrow\hookrightarrow L^p(c), \tag{1.5}$$

where $L^p(c)$ is the weighted Lebesgue space of all functions $x = x(t)$ defined on $(0,\infty)$ for which

$$\|x\|_{p;c} := \left(\int_0^\infty c(t)|x(t)|^p\,dt\right)^{1/p} < \infty.$$

(Note that $L^p(c)$ equipped with the norm $\|\cdot\|_{p;c}$ is a uniformly convex Banach space.) Indeed, it follows from the results of Opic and Kufner [10, Theorem 7.13 and Remark 7.14] that (1.5) holds if and only if (1.4) and also

$$\lim_{t\to 0+}\left(\int_0^t c(\tau)\,d\tau\right)^{1/p}\left(\int_t^\infty r^{1-p'}(\tau)\,d\tau\right)^{1/p'} = 0 \tag{1.6}$$

are satisfied. However, the weight functions $r$ and $c$ are much more general in [10] than in our situation. Namely, the continuity and positivity of $r$ and $c$ in $[0,\infty)$ imply that (1.6) is always satisfied provided (1.4) holds. So, due to [10], we also have the following assertion.

THEOREM 1.2. *The compact embedding* (1.5) *holds if and only if* (1.4) *is satisfied.*

As a consequence of Theorems 1.1 and 1.2 we then immediately obtain the following corollary.

COROLLARY 1.3. *The (S.L.) Property for* (1.1) *is satisfied if and only if the compact embedding* (1.5) *holds.*

REMARK 1.4. It is interesting to point out that if (1.4) is violated, but

$$\sup_{t\in(0,\infty)}\left(\int_0^t c(\tau)\,d\tau\right)^{1/p}\left(\int_t^\infty r^{1-p'}(\tau)\,d\tau\right)^{1/p'} < \infty, \tag{1.7}$$

then only a *continuous embedding*

$$W^{1,p}_{\infty}(r) \hookrightarrow L^p(c)$$

holds instead of a compact one (see [10, Theorem 1.14]). Condition (1.7) clearly guarantees the boundedness of possible eigenvalues from below but, as the following example shows, it is not sufficient to make our approach work.

EXAMPLE 1.5. Let $p = 2$, $r(t) = (t+1)^2$, $c(t) \equiv 1$. Then

$$\left(\int_0^t c(\tau)\,d\tau\right)^{1/p}\left(\int_t^\infty r^{1-p'}(\tau)\,d\tau\right)^{1/p'} = \left(\int_0^t d\tau\right)^{1/2}\left(\int_t^\infty \frac{d\tau}{(1+\tau)^2}\right)^{1/2} = \left(\frac{t}{1+t}\right)^{1/2},$$

i.e. condition (1.7) holds but (1.4) is violated. The initial value problem (IVP)

$$((t+1)^2x'(t))' + \lambda x(t) = 0, \quad x(0) = 1, \quad x'(0) = 0,$$

has the following solutions: for $\lambda = \frac{1}{4}$,

$$x(t) = (t+1)^{-1/2}\left(1 + \frac{1}{2}\ln(t+1)\right);$$

for $\lambda < \frac{1}{4}$,

$$x(t) = (t+1)^{-1/2}\left[\left(\frac{1}{2} - \frac{1}{2\sqrt{1-4\lambda}}\right)(t+1)^{\frac{1}{2}\sqrt{1-4\lambda}} + \left(\frac{1}{2} - \frac{1}{2\sqrt{1-4\lambda}}\right)(t+1)^{-\frac{1}{2}\sqrt{1-4\lambda}}\right];$$

for $\lambda > \frac{1}{4}$,

$$x(t) = (t+1)^{-1/2}\left[\cos\left(\frac{1}{2}\sqrt{4\lambda-1}\ln(t+1)\right) - \frac{1}{\sqrt{4\lambda-1}}\sin\left(\frac{1}{2}\sqrt{4\lambda-1}\ln(t+1)\right)\right].$$

It follows that the boundary value problem (BVP)

$$((t+1)^2x'(t))' + \lambda x(t) = 0, \quad x'(0) = x(\infty) = 0, \tag{1.8}$$

has no solution $x \in W^{1,2}_{\infty}(r)$, i.e. there is no eigenvalue of (1.8).

REMARK 1.6 (Regularity of the weak solution). It can be shown that for any weak solution $x = x(t)$ of (1.1) we have $r\varphi(x') \in C^1[0,\infty)$, the equation holds at every point and $x'(0) = 0$. Indeed, take an arbitrary $y \in C_0^\infty(0,\infty)$ in (1.3) and integrate by parts to get

$$\int_0^\infty \left[r(t)\varphi(x'(t)) - \int_0^t \lambda c(\tau)\varphi(x(\tau))\,d\tau\right]y'(t)\,dt = 0,$$

i.e. the distributional derivative of the expression in brackets is equal to zero. Hence there is a constant $k \in \mathbb{R}$ such that

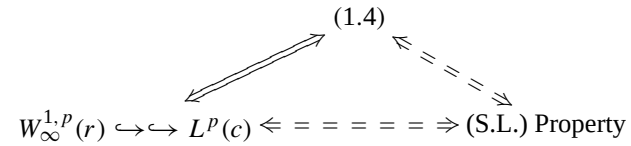
$$r(t)\varphi(x'(t)) - \int_0^t \lambda c(\tau)\varphi(x(\tau))\,d\tau = k \tag{1.9}$$

a.e. in $(0, \infty)$. However, continuity of $c\varphi(x)$ implies that $r\varphi(x') \in C^1[0, \infty)$ and (1.9) holds at every point of $[0, \infty)$. It then follows that the equation in (1.1) also holds at every point of $[0, \infty)$. Now, taking into account this fact and a test function $y \in W^{1,p}_\infty(r)$, $y(0) \neq 0$, in (1.3), integrating by parts we arrive at $x'(0) = 0$.

REMARK 1.7 (The linear case $p = 2$). Let us note that if $p = 2$, it has been proved by Lewis [8] that condition (1.4) is necessary and sufficient for the discreteness and boundedness from below of the spectrum of the maximal self-adjoint extension of the abstract linear operator generated by the equation in BVP (1.1) (the so-called *BD property*). However, it is not completely clear what is the analogue of such spectrum in the nonlinear case $p \neq 2$ (see e.g. Appell, DePascale and Vignoli [2]).

On the other hand, if $p = 2$ then most of our results follow directly from Theorem 1.2 and the linear theory (see e.g. Yosida [12]). Indeed, the properties of compact linear self-adjoint operators could be used to study the eigenvalue problem (1.1).

REMARK 1.8. The properties of the eigenvalues and eigenfunctions of (1.1) have been studied already in Drábek and Kufner [5]. However, the necessity of (1.4) was not discussed there at all and the proofs of some key assertions showing the sufficiency of (1.4) were not included. Here, we include the proofs of all key assertions and complete the following picture:




REMARK 1.9. The reader will easily figure out that similar results also hold for other boundary conditions in (1.1). In particular, one can prove these results for "Dirichlet boundary conditions" $x(0) = x(\infty) = 0$.

This paper is organized as follows. In Section 2 we apply a variational argument to construct a sequence of eigenvalues of (1.1) approaching infinity. From this construction it is not clear if this sequence exhausts the entire set of eigenvalues. That is why we state some comparison and oscillation results for quasilinear equations in Section 3 in order to get more information about the zeros of eigenfunctions. The proof of the main result (Theorem 1.1) is given in Section 4. In order not to interrupt the continuous flow of ideas of the proof, we postpone some technical assertions and their proofs to the Appendix.

## 2. VARIATIONAL EIGENVALUES

The following assertion is a standard consequence of the Lagrange multiplier method and compactness of the embedding (1.5).

LEMMA 2.1. *Assume that* (1.4) *holds. Then* (1.1) *has the least (principal) eigenvalue* $\lambda_1 > 0$ *which can be characterized as follows:*

$$\lambda_1 = \min \frac{\int_0^\infty r(t)|x'(t)|^p\,dt}{\int_0^\infty c(t)|x(t)|^p\,dt}, \tag{2.1}$$

*where the minimum is taken over all* $x \in W_\infty^{1,p}(r)$, $x \neq 0$.

Further in this section, we assume that (1.4) holds. In order to get the higher eigenvalues we employ a variational argument of Ljusternik–Schnirelmann type. Let

$$\mathcal{S} := \{x \in W_\infty^{1,p}(r) : \|x\|_{p;c} = 1\}$$

and let $\mathcal{S}^{k-1}$ be the unit sphere in $\mathbb{R}^k$. For $k \in \mathbb{N}$, let

$\mathcal{F}_k := \{\mathcal{A} \subset \mathcal{S} : \mathcal{A} = h(\mathcal{S}^{k-1})$, where $h$ is a continuous odd function from $\mathcal{S}^{k-1}$ into $\mathcal{S}\}$.

Define

$$\lambda_k := \inf_{\mathcal{A}\in\mathcal{F}_k} \sup_{x\in\mathcal{A}} \|x\|_{1,p;r}^p. \tag{2.2}$$

Following literally the proofs from Drábek and Robinson [6, Section 3], one can show (due to the compactness of the embedding (1.5)) that $\lambda_k$, $k = 1, 2, \ldots$, are the eigenvalues of (1.1) and $\lim_{k\to\infty} \lambda_k = \infty$. Note also that every $\mathcal{A} \in \mathcal{F}_1$ is formed by two antipodal points from $\mathcal{S}$ and so, for $k = 1$, the two characterizations (2.1) and (2.2) coincide. Following verbatim Drábek and Robinson [7, Section 3] one can also show that the eigenfunction $x_{\lambda_n}$ associated with the eigenvalue $\lambda_n$ has at most $n - 1$ zeros in $(0, \infty)$. We thus have

PROPOSITION 2.2. *Assume that* (1.4) *holds. Then* $\{\lambda_k\}_{k=1}^\infty$ *defined by* (2.2) *is a sequence of eigenvalues of* (1.1), $\lim_{k\to\infty} \lambda_k = \infty$, *and any eigenfunction associated with* $\lambda_n$, $n = 1, 2, \ldots$, *has at most* $n - 1$ *zeros in* $(0, \infty)$.

REMARK 2.3. We point out that at the moment it is not clear if the sequence of *variational eigenvalues* constructed by (2.2) exhausts the set of all eigenvalues of (1.1). To prove this fact requires more effort. See the following two sections.

Let $\mathcal{A} \subset \mathcal{S}$ be a compact symmetric set and let

$$\gamma(\mathcal{A}) := \inf\{m \in \mathbb{N} : \exists \text{ continuous and odd mapping of } \mathcal{A} \text{ into } \mathbb{R}^m \setminus \{0\}\},$$
$$\gamma(\mathcal{A}) := \infty \quad \text{if no such } m \text{ exists},$$

be its Krasnosel'skiĭ genus. Define the family of sets

$$\mathcal{F}_k^* := \{\mathcal{A} \subset \mathcal{S} : \mathcal{A} \text{ compact and symmetric, } \gamma(\mathcal{A}) = k\}, \qquad k \in \mathbb{N},$$

and

$$\lambda_k^* := \inf_{\mathcal{A}\in\mathcal{F}_k^*} \sup_{x\in\mathcal{A}} \|x\|_{1,p;r}^p. \tag{2.3}$$

Then $\lambda_k^*$ is also a sequence of variational eigenvalues of (1.1) (cf. Drábek and Robinson [6, Section 3]), and, since $\mathcal{F}_k \subset \mathcal{F}_k^*$, we have $\lambda_k^* \le \lambda_k,\ k \in \mathbb{N}$. As in Drábek and Robinson [6, Section 3], one can easily show that $\lambda_1^* = \lambda_1$ and $\lambda_2^* = \lambda_2$.

The following assertion which follows directly from the result of Szulkin [11] will be important for us.

PROPOSITION 2.4. *Assume that for some $k \ge 2,\ j \ge 1$, we have*

$$\lambda_k^* = \lambda_{k+1}^* = \cdots = \lambda_{k+j}^*. \tag{2.4}$$

*Then the set of eigenfunctions associated with $\lambda_k^*$ and normalized by $\|x_{\lambda_k^*}\|_{p;c} = 1$ consists of more than two antipodal points.*

## 3. COMPARISON AND OSCILLATION RESULTS

Let $t_0 \in [0, \infty),\ A, B \in \mathbb{R}$. Consider the IVP

$$\begin{cases} (r(t)\varphi(x'(t)))' + \lambda c(t)\varphi(x(t)) = 0, \\ x(t_0) = A, \quad x'(t_0) = B. \end{cases} \tag{3.1}$$

By a *solution to IVP* (3.1) we understand an absolutely continuous function $x = x(t)$ defined on $[0, \infty)$ such that $r\varphi(x') \in C^1[0, \infty)$, the equation in (3.1) is satisfied at every point and the initial conditions hold. According to Došlý [4, Theorem 1.1], IVP (3.1) has a unique solution which is extensible to the entire interval $[0, \infty)$.

Recall that the equation in (3.1) is called *disconjugate* on an interval $[a, b] \subset [0, \infty)$ if any nontrivial solution to this equation has at most one zero in $[a, b]$.

Let $W_0^{1,p}(a, b)$ denote the usual Sobolev space of functions $y$ on $(a, b)$ with $y(a) = y(b) = 0$. The following separation and comparison results can also be found in [4].

PROPOSITION 3.1 ([4, Theorem 2.2]). *Let $[a, b] \subset [0, \infty)$. Then the following statements are equivalent:*

(i) *The equation in* (3.1) *is disconjugate on* $[a, b]$.
(ii) *There exists a solution of the equation in* (3.1) *having no zero in* $[a, b]$.
(iii) *The functional*

$$\mathcal{F}(y; a, b) := \int_a^b [r(t)|y'(t)|^p - c(t)|y(t)|^p]\,dt$$

*is positive for every $y \in W_0^{1,p}(a, b),\ y \ne 0$.*

In particular, the following two assertions follow immediately from Proposition 3.1.

COROLLARY 3.2. *Let $x_i = x_i(t),\ t \in [0, \infty),\ i = \alpha, \beta$, be two solutions of IVP* (3.1) *with $t_0 = 0,\ A = 1,\ B = 0,\ \lambda = \mu_i,\ i = \alpha, \beta,\ 0 < \mu_\alpha < \mu_\beta$, and assume that both $x_i$, $i = \alpha, \beta$, have at least one zero in $[0, \infty)$. Let $t_i$ be the first zero of $x_i,\ i = \alpha, \beta$, in $(0, \infty)$. Then $t_\beta < t_\alpha$.*

COROLLARY 3.3. *Let $x_i = x_i(t)$, $i = \alpha, \beta$, be as above and assume that they both have more than one zero in $(0, \infty)$. Then between two consecutive zeros of $x_\alpha$ there is at least one zero of $x_\beta$.*

Recall that the equation (3.1) is called *nonoscillatory* if for any nontrivial solution to this equation $x = x(t)$ there exists $T = T(x) > 0$ such that $x(t) \neq 0$ for all $t \in [T, \infty)$. If this is not true, the equation in (3.1) is called *oscillatory*.

The following oscillation and nonoscillation criteria from Došlý [3] will be used.

PROPOSITION 3.4 ([3, Theorems 6 and 4]). *If*

$$\limsup_{t\to\infty}\left(\int_t^\infty r^{1-p'}(\tau)\,d\tau\right)^{p-1}\left(\int_0^t c(\tau)\,d\tau\right)^{1/p} < \frac{(p-1)^{p-1}}{\lambda p^p},$$

*then the equation in* (3.1) *is nonoscillatory, while if*

$$\limsup_{t\to\infty}\left(\int_t^\infty r^{1-p'}(\tau)\,d\tau\right)^{p-1}\left(\int_0^t c(\tau)\,d\tau\right)^{1/p} > \frac{1}{\lambda},$$

*then it is oscillatory.*

We immediately get

COROLLARY 3.5. *Assume that* (1.4) *holds. Then the equation in* (3.1) *is nonoscillatory for all $\lambda \in \mathbb{R}$. On the other hand, if* (1.4) *is violated, i.e.*

$$\limsup_{t\to\infty}\left(\int_t^\infty r^{1-p'}(\tau)\,d\tau\right)^{1/p'}\left(\int_0^t c(\tau)\,d\tau\right)^{1/p} > 0,$$

*then there exists $\lambda_0 > 0$ such that the equation in* (3.1) *is oscillatory provided $\lambda \geq \lambda_0$.*

We also have

COROLLARY 3.6. *Assume that* (1.4) *holds and let $x = x(t)$ be a nontrivial solution of* (3.1). *Then $x$ has at most a finite number of zeros in $(0, \infty)$.*

PROOF. There is the largest zero $\xi$ of $x$ by Corollary 3.5. The assertion now follows from the compactness of the interval $[0, \xi]$ and from the uniqueness of the solution of IVP (3.1) with a $t_0 \in [0, \xi]$. □

The following result on comparison of the largest zeros of eigenfunctions corresponding to two different eigenvalues requires more effort.

LEMMA 3.7. *Assume that* (1.4) *holds and let $x_i = x_i(t)$, $t \in [0, \infty)$, $i = \alpha, \beta$, be two eigenfunctions corresponding to eigenvalues $\lambda_\alpha$, $\lambda_\beta$ such that $0 < \lambda_\alpha < \lambda_\beta$. Assume that both $x_i$, $i = \alpha, \beta$, have at least one zero in $(0, \infty)$. Denote by $\xi_i \in (0, \infty)$ the largest zero of $x_i$, $i = \alpha, \beta$. Then $0 < \xi_\alpha < \xi_\beta$.*

PROOF. The existence of the largest zeros $\xi_i$, $i = \alpha, \beta$, follows from Remark 1.6 and Corollary 3.5. The proof that $\lambda_\alpha < \lambda_\beta$ implies $0 < \xi_\alpha < \xi_\beta$ is postponed to the Appendix (see Corollary A.4). □

## 4. Proof of Theorem 1.1

*Necessity of (1.4).* We proceed via contradiction. Assume that (1.4) is violated. It then follows from Corollary 3.5 that there exists $\lambda_0 > 0$ such that the equation in (3.1) is oscillatory for all $\lambda \geq \lambda_0$. In particular, taking $n$ large enough, the (S.L.) Property implies that it is oscillatory with $\lambda = \lambda_n$. Hence there exists a nontrivial solution $x$ of this equation with infinitely many zeros approaching infinity. The (S.L.) Property also entails that the eigenfunction $x_{\lambda_n}$ associated with $\lambda_n$ has the largest zero $\xi_n \in (0, \infty)$. Now, choose $\xi_n < a < b$ such that the interval $(a, b)$ contains at least two zeros of $x$. Then we have a contradiction with Proposition 3.1(i), (ii).

*Sufficiency of (1.4).* Some auxiliary assertions are needed.

Lemma 4.1. *Assume that* (1.4) *holds. Then every eigenvalue of* (1.1) *is simple, i.e., for any eigenvalue* $\lambda_e$ *of* (1.1)*, all associated eigenfunctions* $x_{\lambda_e}$ *are mutually proportional.*

This lemma follows immediately from the homogeneity of (1.1) and from the uniqueness of the solution of IVP (3.1) with $t_0 = 0$, $A = 1$ and $B = 0$.

Lemma 4.2. *Assume that* (1.4) *holds. If* $x_{\lambda_e}$ *is an eigenfunction of* (1.1) *associated with an eigenvalue* $\lambda_e > \lambda_1$*, then* $x_{\lambda_e}$ *has at least one zero in* $(0, \infty)$.

The proof of this lemma relies on the method invented by Anane [1] and Lindqvist [9] and follows the lines of the proof of Proposition A.2.

Lemma 4.3. *Assume that* (1.4) *holds. Let* $\lambda_\alpha$, $\lambda_\beta$ *be two eigenvalues of* (1.1) *such that* $\lambda_\alpha < \lambda_\beta$ *and* $x_{\lambda_\alpha}$, $x_{\lambda_\beta}$ *be corresponding eigenfunctions having at least one zero in* $(0, \infty)$*, respectively. Then the number of zeros of* $x_{\lambda_\beta}$ *in* $(0, \infty)$ *is strictly larger than the number of zeros of* $x_{\lambda_\alpha}$ *in* $(0, \infty)$.

This assertion follows directly from Corollaries 3.2, 3.3, 3.6 and from Lemma 3.7.

Lemma 4.4. *Assume that* (1.4) *holds. Let* $\{\lambda_m\}_{m=1}^{\infty}$ *be a sequence of eigenvalues of* (1.1) *such that* $\lim_{m\to\infty} \lambda_m = \lambda_0$*. Then* $\lambda_0 > 0$ *is also an eigenvalue of* (1.1).

Proof. Since $\lambda_m \geq \lambda_1$, we have $\lambda_0 \geq \lambda_1 > 0$. Let us denote by $\{x_{\lambda_m}\}_{m=1}^{\infty}$ the set of eigenfunctions associated with $\{\lambda_m\}_{m=1}^{\infty}$ and normalized by $\|x_{\lambda_m}\|_{1,p;r} = 1$, $x_{\lambda_m}(0) > 0$. Due to the reflexivity of $W_\infty^{1,p}(r)$ and compactness of the embedding (1.5) we can pass to a subsequence such that $x_n \rightharpoonup x_0$ (weakly) in $W_\infty^{1,p}(r)$ and $x_{\lambda_m} \to x_0$ (strongly) in $L^p(c)$ for some $x_0 \in W_\infty^{1,p}(r)$. Taking $y = x_{\lambda_m}$ in

$$\int_0^\infty r(t)\varphi(x'_{\lambda_m}(t))y'(t)\,dt = \lambda_m \int_0^\infty c(t)\varphi(x_{\lambda_m}(t))y(t)\,dt, \tag{4.1}$$

we arrive at

$$1 = \|x_{\lambda_m}\|_{1,p;r}^p = \lambda_m \|x_{\lambda_m}\|_{p;c}^p,$$

i.e.

$$\|x_{\lambda_m}\|_{p;c} = 1/\lambda_m^{1/p}.$$

In particular, $\|x_0\|_{p;c} = 1/\lambda_0^{1/p}$, i.e. $x_0 \neq 0$. The integral identity (4.1) is equivalent to the operator equation

$$J(x_{\lambda_m}) = f_m \tag{4.2}$$

with $f_m = \lambda_m c\varphi(x_{\lambda_m})$, where $J : W_\infty^{1,p}(r) \to (W_\infty^{1,p}(r))^*$ is defined in the Appendix. It follows from $x_m \to x_0$ in $L^p(c)$ that $f_m \to \lambda_0 c\varphi(x_0)$ in $(W_\infty^{1,p}(r))^*$. By Lemma A.1 we infer that $x_{\lambda_m}$ is strongly convergent in $W_\infty^{1,p}(r)$, i.e. $x_{\lambda_m} \to x_0$ in $W_\infty^{1,p}(r)$. Now, it follows from continuity of $J$ and (4.2) that

$$J(x_0) = \lambda_0 c\varphi(x_0),$$

i.e. $x_0 = x_{\lambda_0}$ is an eigenfunction of (1.1) associated with the eigenvalue $\lambda_0$. □

PROPOSITION 4.5. *Assume that* (1.4) *holds. Then the set of all eigenvalues of* (1.1) *consists of isolated points from the interval* $(0, \infty)$.

PROOF. Let $\lambda_0$ be a limit point of some sequence $\{\lambda_m\}_{m=1}^\infty$ of eigenvalues of (1.1). Then $\lambda_0 > 0$ is an eigenvalue of (1.1) with associated eigenfunction $x_{\lambda_0}$ (see Lemma 4.4). We can extract an increasing or a decreasing subsequence converging to $\lambda_0$. We denote it again $\{\lambda_m\}_{m=1}^\infty$. If $\{\lambda_m\}_{m=1}^\infty$ is increasing and $\lambda_m \to \lambda_0$, then the number of zeros of $x_{\lambda_m}$ strictly increases with $m$ (due to Lemmas 4.2 and 4.3), so $x_{\lambda_0}$ has more zeros than any $x_{\lambda_m}$, a contradiction with Corollary 3.6. In the case that $\{\lambda_m\}_{m=1}^\infty$ is decreasing, by Lemmas 4.2 and 4.3 the number of zeros of $x_{\lambda_m}$ is finite for any $m \in \mathbb{N}$ and strictly decreasing as $m \to \infty$, a contradiction. □

We now finish the proof of sufficiency of (1.4). Let us consider the sequence of variational eigenvalues $\{\lambda_k\}_{k=1}^\infty$ and $\{\lambda_k^*\}_{k=1}^\infty$ given in (2.2) and (2.3), respectively. Our aim is to show that $\lambda_k = \lambda_k^*$, $k \in \mathbb{N}$, and that this is the entire set of all eigenvalues of (1.1) with all properties stated in Section 1. Indeed, every $\lambda_k$, $k = 1, 2, \ldots$, is a simple eigenvalue according to Lemma 4.1, and $x_{\lambda_1}$ has no zero in $(0, \infty)$ due to Proposition 2.2. The eigenfunction $x_{\lambda_2}$ has exactly one zero in $(0, \infty)$ according to Lemma 4.2 and Proposition 2.2, and there is no eigenvalue of (1.1) strictly between $\lambda_1$ and $\lambda_2$ thanks to Lemmas 4.2 and 4.3. If $\lambda_3 = \lambda_2$ $(= \lambda_2^*)$ then $\lambda_3^* = \lambda_2^*$ and Proposition 2.4 would contradict Lemma 4.1. Hence $\lambda_3 > \lambda_2$. It follows from Lemmas 4.2, 4.3 and Proposition 2.2 that the eigenfunction $x_{\lambda_3}$ has exactly two zeros in $(0, \infty)$ and that there is no eigenvalue of (1.1) lying strictly between $\lambda_2$ and $\lambda_3$. In particular, $\lambda_3^* = \lambda_3$. We can continue by induction. The interlacing property of the zeros of $x_{\lambda_{k-1}}$ and $x_{\lambda_k}$ then follows from Corollary 3.2 and Lemma 3.7.

This completes the proof of Theorem 1.1.

## A. APPENDIX

In this section we assume that (1.4) holds and present some technical assertions which are used in the proofs of the main results of this paper. Let us define the operator

$$J : W_\infty^{1,p}(r) \to (W_\infty^{1,p}(r))^*$$

by

$$\langle J(x), y\rangle = \int_0^\infty r(t)\varphi(x'(t))y'(t)\,dt$$

for $x, y \in W_\infty^{1,p}(r)$. Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(W_\infty^{1,p}(r))^*$ and $W_\infty^{1,p}(r)$. The operator $J$ is continuous and $(p-1)$-homogeneous. It also has the following properties.

LEMMA A.1. *There exists an inverse operator*

$$J^{-1} : (W_\infty^{1,p}(r))^* \to W_\infty^{1,p}(r),$$

*it is bounded and continuous.*

PROOF. The strict monotonicity of $\varphi$ implies that

$$\langle J(x) - J(y), x - y\rangle > 0 \quad \text{for } x \neq y.$$

Hence $J$ is injective. Using the Hölder inequality we get

$$\langle J(x) - J(y), x - y\rangle \geq (\|x\|_{1,p;r}^{p-1} - \|y\|_{1,p;r}^{p-1})(\|x\|_{1,p;r} - \|y\|_{1,p;r}) \tag{A.1}$$

and the boundedness of $J$ follows.

To prove that $J^{-1}$ is continuous we proceed via contradiction. Suppose it is not, i.e., there exists a sequence $\{f_n\}_{n=1}^\infty$, $f_n \to f$ (strongly) in $(W_\infty^{1,p}(r))^*$ and

$$\|J^{-1}(f_n) - J^{-1}(f)\|_{1,p;r} \geq \delta \quad \text{for a } \delta > 0.$$

Let $x_n := J^{-1}(f_n)$ and $x := J^{-1}(f)$. It follows that

$$\|f_n\|_*\|x_n\|_{1,p;r} \geq \langle f_n, x_n\rangle = \langle J(x_n), x_n\rangle = \|x_n\|_{1,p;r}^p,$$

i.e.

$$\|x_n\|_{1,p;r}^{p-1} \leq \|f_n\|_*,$$

where $\|\cdot\|_*$ is the norm on $(W_\infty^{1,p}(r))^*$. Due to the reflexivity of $W_\infty^{1,p}(r)$ we may then assume that $x_n \rightharpoonup \tilde{x}$ in $W_\infty^{1,p}(r)$. Hence

$$\begin{aligned}\langle J(x_n) - J(\tilde{x}), x_n - \tilde{x}\rangle &= \langle J(x_n) - J(x), x_n - \tilde{x}\rangle \\ &\quad + \langle J(x) - J(\tilde{x}), x_n - \tilde{x}\rangle \to 0\end{aligned} \tag{A.2}$$

since $J(x_n) \to J(x)$ in $(W_\infty^{1,p}(r))^*$. It follows from (A.1) (with $x := x_n$, $y := \tilde{x}$) and (A.2) that $\|x_n\|_{1,p;r} \to \|\tilde{x}_n\|_{1,p;r}$. The uniform convexity of $W_\infty^{1,p}(r)$ then implies that $x_n \to \tilde{x}$ in $W_\infty^{1,p}(r)$. Since $J$ is injective, we obtain $\tilde{x} = x$, a contradiction. □

For $a \in [0, \infty)$ let $W_{a,\infty}^{1,p}(r)$ be the set of all absolutely continuous functions $x = x(t)$ defined on $[a, \infty)$ such that $x(a) = x(\infty) = 0$ and

$$|||x||| := \left(\int_a^\infty r(t)|x'(t)|^p\,dt\right)^{1/p} < \infty$$

Then $W^{1,p}_{a,\infty}(r)$ equipped with the norm $|||\cdot|||$ is a uniformly convex Banach space. Since (1.4) implies

$$\lim_{t\to\infty}\left(\int_a^t c(\tau)\,d\tau\right)^{1/p}\left(\int_t^\infty r^{1-p'}(\tau)\,d\tau\right)^{1/p'}=0, \tag{A.3}$$

we also have a compact embedding

$$W^{1,p}_{a,\infty}(r)\hookrightarrow\hookrightarrow L^p_a(c), \tag{A.4}$$

where $L^p_a(c)$ is the weighted Lebesgue space of all functions $x=x(t)$ defined on $(0,\infty)$ for which

$$\|x\|=\left(\int_a^\infty c(t)|x(t)|^p\,dt\right)^{1/p}<\infty$$

(cf. [1, Theorem 7.13 and Remark 7.14]).

It follows from the Lagrange multiplier method and the compactness of the embedding (A.4) that the infimum

$$\lambda_a=\inf\frac{\int_a^\infty r(t)|x'(t)|^p\,dt}{\int_a^\infty c(t)|x(t)|^p\,dt} \tag{A.5}$$

(taken over all $x\in W^{1,p}_{a,\infty}(r)$, $x\neq 0$) is achieved at some $x_a\in W^{1,p}_{a,\infty}(r)$, $\lambda_a$ is the principal eigenvalue of

$$\begin{cases}(r(t)\varphi(x'(t)))'+\lambda c(t)\varphi(x(t))=0,\\ x(a)=x(\infty)=0,\end{cases} \tag{A.6}$$

and $x_a$ is the corresponding principal eigenfunction.

The value $\lambda\in\mathbb{R}$, $\lambda\neq\lambda_a$, for which there is a nonzero solution $x\in W^{1,p}_{a,\infty}(r)$ of (A.6) is called a *higher eigenvalue* of (A.6) and $x$ is a corresponding *higher eigenfunction*. The variational characterization (A.5) implies that $\lambda>\lambda_a$ for any higher eigenvalue $\lambda$. It also follows easily from (A.5) that for any $x\in W^{1,p}_{a,\infty}(r)$ we have

$$\int_a^\infty r(t)|x'(t)|^p\,dt-\lambda_a\int_a^\infty c(t)|x(t)|^p\,dt\geq 0. \tag{A.7}$$

Another consequence of (A.5) is that if the infimum in (A.5) is achieved at some $x\in W^{1,p}_{a,\infty}(r)$ then it must also be achieved at $|x|\in W^{1,p}_{a,\infty}(r)$. But the regularity argument similar to that from Remark 1.6 combined with the uniqueness of the solution of IVP (3.1) shows that no minimizer in (A.5) has a zero in $(a,\infty)$. Further, we normalize $x_a$ in such a way that $x_a>0$ in $(a,\infty)$. The situation with higher eigenfunctions is different.

PROPOSITION A.2. *Let $x\in W^{1,p}_{a,\infty}(r)$ be an eigenfunction associated with a higher eigenvalue $\lambda$ of* (A.6)*. Then $x$ has at least one zero in $(a,\infty)$.*

PROOF. Assume that there exists a higher eigenfunction $x \in W^{1,p}_{a,\infty}(r)$ such that $x > 0$ in $(a, \infty)$. Then

$$\int_a^\infty r(t)\varphi(x'(t))z'(t)\,dt = \lambda \int_a^\infty c(t)\varphi(x(t))z(t)\,dt \tag{A.8}$$

for any $z \in W^{1,p}_{a,\infty}(r)$. But we also have

$$\int_a^\infty r(t)\varphi(x_a'(t))y'(t)\,dt = \lambda_a \int_a^\infty c(t)\varphi(x_a(t))y(t)\,dt \tag{A.9}$$

for any $y \in W^{1,p}_{a,\infty}(r)$. For $\varepsilon > 0$ and $\mathcal{X}(t) = \max_{t\in(a,\infty)}\{x_a(t), x(t)\}$ set

$$x_{a,\varepsilon}(t) = x_a(t) + \varepsilon\mathcal{X}(t), \qquad x_\varepsilon(t) = x(t) + \varepsilon\mathcal{X}(t)$$

and

$$y(t) = \frac{x^p_{a,\varepsilon}(t) - x^p_\varepsilon(t)}{x^{p-1}_{a,\varepsilon}(t)}, \quad z(t) = \frac{x^p_\varepsilon(t) - x^p_{a,\varepsilon}(t)}{x^{p-1}_\varepsilon(t)}.$$

Then $x_{a,\varepsilon}/x_\varepsilon, x_\varepsilon/x_{a,\varepsilon} \in L^\infty(a,\infty)$ and hence $y, z \in W^{1,p}_{a,\infty}(r)$. Adding (A.8) and (A.9) with $y$ and $z$ chosen as above we obtain

$$\begin{aligned}\int_a^\infty r(t)\left\{\left[1 + (p-1)\left(\frac{x_\varepsilon}{x_{a,\varepsilon}}\right)^p\right]|x'_{a,\varepsilon}|^p + \left[1 + (p-1)\left(\frac{x_{a,\varepsilon}}{x_\varepsilon}\right)^p\right]|x'_\varepsilon|^p\right\}dt \\ - \int_a^\infty r(t)\left[p\left(\frac{x_\varepsilon}{x_{a,\varepsilon}}\right)^{p-1}|x'_{a,\varepsilon}|^{p-2}x'_{a,\varepsilon}x'_\varepsilon + p\left(\frac{x_{a,\varepsilon}}{x_\varepsilon}\right)^{p-1}|x'_\varepsilon|^{p-2}x'_\varepsilon x'_{a,\varepsilon}\right]dt \\ = \int_a^\infty c(t)\left[\lambda\left(\frac{x}{x_\varepsilon}\right)^{p-1} - \lambda_a\left(\frac{x_a}{x_{a,\varepsilon}}\right)^{p-1}\right](x^p_\varepsilon - x^p_{a,\varepsilon})\,dt.\end{aligned} \tag{A.10}$$

Since for a function $u = u(t)$, $u > 0$ in $(a,\infty)$, we have $|(\log u)'| = |u'|/u$, we can rewrite (A.10) as follows:

$$\begin{aligned}\int_a^\infty r(t)(x^p_{a,\varepsilon} - x^p_\varepsilon)[|(\log x_{a,\varepsilon})'|^p - |(\log x_\varepsilon)'|^p]\,dt \\ - \int_a^\infty r(t)px^p_\varepsilon|(\log x_{a,\varepsilon})'|^{p-2}(\log x_{a,\varepsilon})'[(\log x_\varepsilon)' - (\log x_{a,\varepsilon})']\,dt \\ - \int_a^\infty r(t)px^p_{a,\varepsilon}|(\log x_\varepsilon)'|^{p-2}(\log x_\varepsilon)'[(\log x_{a,\varepsilon})' - (\log x_\varepsilon)']\,dt \\ = \int_a^\infty c(t)\left[\lambda\left(\frac{x}{x_\varepsilon}\right)^{p-1} - \lambda_a\left(\frac{x_a}{x_{a,\varepsilon}}\right)^{p-1}\right](x^p_\varepsilon - x^p_{a,\varepsilon})\,dt.\end{aligned} \tag{A.11}$$

It follows from the inequality

$$|a|^p - |b|^p \geq p|b|^{p-2}b(a-b),$$

which holds for any $a, b \in \mathbb{R}$, that the left hand side in (A.11) is nonnegative, i.e. we have

$$\int_a^\infty c(t)\left[\lambda\left(\frac{x}{x_\varepsilon}\right)^{p-1} - \lambda_a\left(\frac{x_a}{x_{a,\varepsilon}}\right)^{p-1}\right](x_\varepsilon^p - x_{a,\varepsilon}^p)\,dt \geq 0. \tag{A.12}$$

Note that for all $\varepsilon \leq \varepsilon_0$ with $\varepsilon_0$ small enough, the integrand in (A.12) is bounded by a function from $L^1(a, \infty)$. Letting $\varepsilon \to 0$ in (A.12), it follows from the Lebesgue theorem that

$$(\lambda - \lambda_a)\int_a^\infty c(t)(x^p - x_a^p)\,dt \geq 0.$$

However, renormalizing $x$ so that the last integral is negative, we arrive at a contradiction. □

REMARK A.3. The method from the previous proof is taken from Anane [1] and Lindqvist [9]. A similar approach proves that $\lambda_a$ is a simple eigenvalue (cf. [1], [9]). In our case the simplicity of $\lambda_a$ is a consequence of the uniqueness of the solution of IVP (3.1).

COROLLARY A.4. *Let $x_i = x_i(t)$, $t \in [0, \infty)$, $i = \alpha, \beta$, be two eigenfunctions of* (1.1) *corresponding to the eigenvalues $0 < \lambda_\alpha < \lambda_\beta$ and assume that $x_\alpha, x_\beta$ have the largest zeros $\xi_\alpha, \xi_\beta \in (0, \infty)$. Assume that* (1.4) *holds. Then $\xi_\alpha < \xi_\beta$.*

PROOF. Assume the contrary: $\xi_\alpha \geq \xi_\beta$. If $\xi_\alpha > \xi_\beta$, we define a function $\bar{x}_\alpha = \bar{x}_\alpha(t)$, $t \in [0, \infty)$, as follows:

$$\bar{x}_\alpha(t) = \begin{cases} 0, & t \in [0, \xi_\alpha), \\ x_\alpha(t), & t \in [\xi_\alpha, \infty), \end{cases}$$

and let be $\hat{x}_\alpha$ the restriction of $\bar{x}_\alpha$ to the interval $[\xi_\beta, \infty)$. Since $x_\alpha \in W_\infty^{1,p}(r)$, we have $\hat{x}_\alpha \in W_{\xi_{\beta,},\infty}^{1,p}(r)$. Moreover,

$$\int_{\xi_\beta}^\infty r(t)|\hat{x}_\alpha'(t)|^p\,dt - \lambda_\alpha\int_{\xi_\beta}^\infty c(t)|\hat{x}_\alpha(t)|^p\,dt = 0. \tag{A.13}$$

Now, we define a function $\hat{x}_\beta \in W_{\xi_{\beta,},\infty}^{1,p}(r)$ as $\hat{x}_\beta(t) = x_\beta(t)$, $t \in [\xi_\beta, \infty)$. Then

$$\int_{\xi_\beta}^\infty r(t)\varphi(\hat{x}_\beta'(t))y'(t)\,dt - \lambda_\beta\int_{\xi_\beta}^\infty c(t)\varphi(\hat{x}_\beta(t))y(t)\,dt = 0 \tag{A.14}$$

for any $y \in W_{\xi_{\beta,},\infty}^{1,p}(r)$. Since $\hat{x}_\beta$ does not change sign in $(\xi_\beta, \infty)$ (note that $\xi_\beta$ is the last zero of $x_\beta$ in $(0, \infty)$), $\lambda_\beta$ is the principal eigenvalue and $\hat{x}_\beta$ is the corresponding eigenfunction of (A.6) with $a = \xi_\beta$ (cf. Proposition A.2). It then follows from (A.7) that

$$\int_{\xi_\beta}^\infty r(t)|\hat{x}_\alpha'(t)|^p\,dt - \lambda_\beta\int_{\xi_\beta}^\infty c(t)|\hat{x}_\alpha(t)|^p\,dt \geq 0,$$

which contradicts $\lambda_\beta > \lambda_\alpha$ and (A.13). If $\xi_\alpha = \xi_\beta$ we proceed similarly to conclude that $\lambda_\alpha < \lambda_\beta$ are both eigenvalues of (A.6) with $a = \xi_\alpha = \xi_\beta$ having eigenfunctions $\hat{x}_\alpha$, $\hat{x}_\beta$ which do not have a zero in $(\xi_\alpha, \infty)$. This contradicts Proposition A.2. □

ACKNOWLEDGEMENTS. The first author was supported by the Research Plan MSN 4977751301 of the Ministry of Education, Youth and Sports of the Czech Republic.

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Received 28 August 2006,
and in revised form 12 September 2006.

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