Rend. Lincei Mat. Appl. 18 (2007), 117–124

Mathematical analysis. — *Transversality of covariant mappings admitting a potential*, by MARC LESIMPLE and TULLIO VALENT.

ABSTRACT. — It is shown that for any mapping between Banach spaces which is covariant under a nonlinear action of a Lie group and has a symmetric first derivative with respect to some duality, the image of any point close to the origin belongs to the polar of the orbit of that point under the commutator subgroup of the symmetry group.

KEY WORDS: Perturbation problems; nonlinear symmetry; covariant mappings; potential.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 22E45, 81C40, 47H60, 47J05, 58C25, 58E99, 57R99.

INTRODUCTION

The problem of existence of local solutions for a perturbation equation of the type

$$A(x) + \epsilon B(x) = 0, \tag{0.1}$$

where A and B are mappings defined on a topological vector space M with values in a topological vector space N and $\epsilon \in \mathbb{R}$, is related to the covariance of the operator A under the actions of a symmetry group $\boldsymbol{G}$ on M and N respectively. The mapping A is supposed to be "covariant" in the sense that it commutes with the two actions of $\boldsymbol{G}$ (see (1.1)).

Perturbation problems in the presence of linear or affine symmetries have been studied in [5–7] (where the actions on M and N, assumed to be Banach spaces, are given by linear or affine representations of $\boldsymbol{G}$). But nonlinear actions of groups arise in boundary value problems with symmetries (see [7]). So it is important to extend the results previously obtained in [5] to the case of nonlinear actions of groups.

In the linear case, the existence of local solutions for equation (0.1) is linked to the "transversality" of the mapping A at any point x close to the origin in M. We shall seek conditions guaranteeing the transversality of A in the nonlinear case.

Having in mind existence results depending on the implicit function theorem, we shall suppose that M and N are Banach spaces. Moreover the action of the symmetry group $\boldsymbol{G}$, supposed to be a Lie group, is realized as nonlinear representations S and T obtained from linear representations of $\boldsymbol{G}$ on M and N, as in the theory of nonlinear group representations elaborated in [1, 3]. If a duality pairing $\langle\ ,\ \rangle$ exists between M and N, then there exist a linear operator $\kappa : M \to N$ and an inner product $(\ |\ )$ on N such that $\langle x, y\rangle = (\kappa(x) \,|\, y)$ for every $(x, y) \in M \times N$ (see [8]).

If the linear parts, say S^1 and T^1, of the two representations S and T are contragredient with respect to the bilinear form $\langle\ ,\ \rangle$ and if the product $(\ |\ )$ on N is T^1-invariant, then the operator κ intertwines S^1 and T^1.

Once a duality is given on $M \times N$, a differential form ω_A can be associated to A and we show that if ω_A is closed (which can be compared to a compatibility condition as in perturbation theories), then A is transversal with respect to the orbit of any point sufficiently close to the origin, under the commutator subgroup of $\boldsymbol{G}$. Here, we say that A is transversal at a point x if $A(x)$ belongs to the polar of the orbit of x (with respect to the duality), which is equivalent to saying that $A(x)$ is orthogonal to the image under κ of the orbit of x (with respect to the inner product $(\ |\ )$ on N). The difficulties encountered in attempting to prove the local existence of solutions for a perturbation problem of the form (0.1) rely on the covariance of the mapping A; and this is a point that we aim to address by showing, in some sense, the transversality of A, in order to deduce local existence results from the implicit function theorem [2].

We first show that A is transversal for the linear action of $\boldsymbol{G}$ given by S^1 and T^1.

To the nonlinear representation S on M (resp. T on N) there corresponds a linear representation $\widetilde{S}$ (resp. $\widetilde{T}$) on the space of polynomials on M (resp. on N) [1, 3]. For instance, the restriction of $\widetilde{S}$ to the space M_{n+1} of polynomials of degree $n+1$ is realized as an extension of the restriction of $\widetilde{S}$ to the space M_n of polynomials of degree n by the $n+1$-tensorial representation $\otimes^{n+1} S^1$ on homogeneous polynomials of degree $n+1$ (as defined in Section 2; see [3] for more details). Turning to that associated linear action, we proceed further by supposing the mapping A to be analytic and by assuming that the linear representations associated to S and T are contragredient. Therefore A becomes an intertwining operator for S and T, which leads to an intertwining operator $\widetilde{A}$ for $\widetilde{S}$ and $\widetilde{T}$, which in turn is shown to be transversal with respect to the orbit (under the associated linear action) of any point belonging to a neighborhood of the origin in the strict inductive limit of the spaces M_n, $n \geq 1$. Next, by induction on the degree n, referring to the construction of $\widetilde{S}$, we conclude that A is transversal. Then the case of C^∞ mappings is considered.

Definitions and some results pertaining to nonlinear representations of Lie groups will be recalled when needed, referring to [1, 3].

1. TRANSVERSALITY OF MAPPINGS ADMITTING A POTENTIAL WITH RESPECT TO A DUALITY

Let $\boldsymbol{G}$ be a Lie group and $\mathfrak{g}$ its Lie algebra. The mapping A is defined on a Banach space M with values in a Banach space N. We consider two analytic representations, (S, M) and (T, N), of $\boldsymbol{G}$ on M and N respectively [1]. Namely, S (resp. T) is a morphism from $\boldsymbol{G}$ to the group of invertible elements in the space of formal power series of the form (we use the notations of [1, 3])

$$S_g = S_g^1 + \sum_{n \geq 2} S_g^n \quad \text{(resp. } T_g = T_g^1 + \sum_{n \geq 2} T_g^n\text{)}$$

where $g \in \boldsymbol{G}$ and $S_g^n \in \mathcal{L}_n(M)$ (resp. $T_g^n \in \mathcal{L}_n(N)$), the space of symmetric n-linear continuous mappings from $M \times \cdots \times M$ to M (resp. from $N \times \cdots \times N$ to N); and for every g in a neighborhood of the identity e in $\boldsymbol{G}$, the map $x \mapsto S_g(x) = S_g^1(x) + \sum_{n \geq 2} S_g^n(x)$ is analytic in some ball $B(0, r)$ in M (resp. $y \mapsto T_g(y)$ is analytic in some ball $B(0, r')$

in N). We denote by $\hat{\bigotimes}_n M$ the image under the symmetrization operator σ_n, defined by

$$\sigma_n(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}, \qquad x_1, \ldots, x_n \in M,$$

of the completed projective tensor product of M, and identify $\mathcal{L}_n(M, N)$ with $\mathcal{L}(\hat{\bigotimes}_n M, N)$. The free part S^1 (resp. T^1) of S (resp. T) is a continuous linear representation of $\boldsymbol{G}$. Since we shall be concerned with the restriction of S^1 to $C^\infty(S^1)$ (resp. of T^1 to $C^\infty(T^1)$), we assume that S^1 is a C^∞ representation in M (resp. T^1 is a C^∞ representation in N).

The second degree term S^2 in the series $S^1_g + \sum_{n\geq 2} S^n_g$ represents the "infinitesimal perturbation" of the linear representation S^1, in the sense that the mapping $F^2 \colon \boldsymbol{G} \to \mathcal{L}_2(M, M)$ defined by $F^2_g = S^2_g \circ (S^1_{g^{-1}} \otimes S^1_{g^{-1}})$ (for $g \in \boldsymbol{G}$) belongs to the space $Z^1(\boldsymbol{G}, \mathcal{L}(M \hat{\otimes} M, M))$ of 1-cocycles on $\boldsymbol{G}$ with values in the $\boldsymbol{G}$-module $\mathcal{L}(M \hat{\otimes} M, M)$ (endowed with the following structure: if $g \in \boldsymbol{G}$ and $u \in \mathcal{L}(M \hat{\otimes} M, M)$, then $g.u = S^1_g \circ u \circ S^1_{g^{-1}} \otimes S^1_{g^{-1}}$; see e.g. [4]). Affine symmetries considered in [5] are expressed by affine representations: $S_g = S^0_g + S^1_g$ with $S^0 \in Z^1(\boldsymbol{G}, M)$, the 1-cocycle space on $\boldsymbol{G}$ with values in M [4].

We keep the same notations for an n-linear operator and the associated homogeneous polynomial of degree n. There exists a neighborhood ω_e of e in $\boldsymbol{G}$ such that the map $g \mapsto S^n_g$ from ω_e to $\mathcal{L}_n(M)$ is C^∞ for every n (see [1]). So we can define the differential dS of S as an analytic representation of $\mathfrak{g}$ in M (see [1]) given by $dS_X = \sum_n dS^n_X$, $X \in \mathfrak{g}$, where

$$dS^n_X = \frac{d}{dt} S^n_{\exp tX}|_{t=0}.$$

Put $M_n = \bigoplus_{i=1}^n M^i$ with $M^i = \hat{\bigotimes}_i M$ and denote by $\widetilde{M}$ the strict inductive limit $\widetilde{M} = \bigcup_{n\geq 1} M_n$. Then $\widetilde{M}$ is a Fréchet space.

The mapping $A \colon M \to N$ is supposed to be differentiable in an open subset U of M, containing the origin. We denote by $\widetilde{S}$ (resp. $\widetilde{dS}$) the linear representation associated with S (resp. dS) defined as follows (see [1]): $\widetilde{S}_g$, $g \in \boldsymbol{G}$ (resp. $\widetilde{dS}_X$, $X \in \mathfrak{g}$) is the isomorphism (resp. endomorphism) of $\widetilde{M}$ leaving invariant the subspaces M_n, $n \geq 1$, defined on M^n by

$$\widetilde{S}_g(x_1 \otimes \cdots \otimes x_n) = \sum_{p=1}^{n} \sum_{i_1+\cdots+i_p=n} S^{i_1}_g \otimes \cdots \otimes S^{i_p}_g(\sigma_n(x_1 \otimes \cdots \otimes x_n))$$

$$(\text{resp. } \widetilde{dS}_X(x_1 \otimes \cdots \otimes x_n) = \sum_{p=1}^{n} \sum_{q=0}^{p-1} (I_q \otimes dS^{n-p+1}_X \otimes I_{p-q-1} \circ \sigma_n(x_1 \otimes \cdots \otimes x_n))$$

with $x_1, \ldots, x_n \in M$.

The continuous linear representation $\widetilde{S}$ of $\boldsymbol{G}$ in $\widetilde{M}$ is C^∞ and we have $d\widetilde{S} = \widetilde{dS}$ (see [1, 3]).

Suppose we are given a complex, separately continuous, bilinear form $(x, y) \mapsto \langle x, y\rangle$ on $M \times N$. We associate to the mapping A the differential form ω_A on U defined by $\omega_A(x)(\varphi) = \langle \varphi, A(x)\rangle$, $x \in U$, $\varphi \in M$.

The mapping A is assumed to be covariant in the sense that it commutes with the action of $\boldsymbol{G}$, that is,

$$A \circ S_g = T_g \circ A, \quad g \in \boldsymbol{G}. \tag{1.1}$$

By differentiation we obtain

$$dA_x\, dS_X(x) = dT_X(A(x)), \quad X \in \mathfrak{g} \text{ and } x \in B(0,r) \cap U \tag{1.2}$$

(where dA_x denotes the differential of A at x).

Furthermore the differential form ω_A associated to A is supposed to be closed, i.e.,

$$\langle x_1, dA_x(x_2)\rangle = \langle x_2, dA_x(x_1)\rangle, \quad x_1, x_2 \in M \text{ and } x \in U. \tag{1.3}$$

It has been demonstrated in [8] that there exists an inner product $(\ |\ )$ on N and a linear mapping $\kappa : M \to N$ such that

$$\langle x, y\rangle = (\kappa(x) \mid y) \quad \text{for every } (x, y) \in M \times N.$$

When the mapping A "describes" a boundary value problem, κ can have the meaning of a trace operator (see [5, 7]).

Suppose that the linear representations S^1 and T^1 are contragredient, that is to say,

$$\langle S^1_g(x), T^1_g(y)\rangle = \langle x, y\rangle \quad \text{for every } (x, y) \in M \times N \text{ and } g \in \boldsymbol{G}. \tag{1.4}$$

Then the operator κ intertwines S^1 and T^1 (i.e. $\kappa \circ S^1_g = T^1_g \circ \kappa$ for every $g \in \boldsymbol{G}$) provided the inner product is T^1-invariant (i.e. $(T^1(y_1) \mid T^1(y_2)) = (y_1 \mid y_2)$ for every $y_1, y_2 \in N$).

PROPOSITION 1.1. *Under conditions* (1.1), (1.3) *and* (1.4), *one has the following transversality property:*

$$\forall X \in \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}], \forall x \in U \quad \langle dS^1_X(x), A(x)\rangle = 0.$$

PROOF. From (1.1), (1.4) it easily follows that

$$\langle x_1, dA_{x_2} dS^1_X(x_2)\rangle = -\langle dS^1_X(x_1), A(x_2)\rangle \quad \text{for all } x_1, x_2 \in M \text{ and } X \in \mathfrak{g}.$$

In particular, for $x_1 = dS^1_Y(x_2)$ with $Y \in \mathfrak{g}$, we have

$$\langle dS^1_Y(x_2), dA_{x_2} dS^1_X(x_2)\rangle = -\langle dS^1_X dS^1_Y(x_2), A(x_2)\rangle.$$

Then, in view of (1.3),

$$\langle dS^1_X dS^1_Y(x_2), A(x_2)\rangle = \langle dS^1_Y dS^1_X(x_2), A(x_2)\rangle,$$

and thus $\langle dS_{[X,Y]}(x_2), A(x_2)\rangle = 0$. □

In the case of affine representations (i.e. when $dS_X(x) = dS^1_X(x) + a_X$ with $a_X = \frac{d}{dt} S^0_{\exp tX}|_{t=0}$) the result of Proposition 1.1 has been stated in [5] assuming that the mapping A admits a potential with respect to κ and $(\ |\ )$ (see [8]).

To show that the transversality property expressed in Proposition 1.1 holds for nonlinear representations we need to develop further the duality.

2. Transversality criteria for analytic mappings

In this section, we suppose that A is an analytic mapping of the form $A = \sum_{n\geq 1} A^n$, with $A^n \in \mathcal{L}_n(M, N)$.

The duality on $M \times N$ is extended to $\widetilde{M} \times \widetilde{N}$ in the following way. For $x_1, \ldots, x_n \in M$ and $y_1, \ldots, y_n \in N$, define

$$\langle x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_p \rangle = \prod_{i=1}^{n} \langle x_i, y_i \rangle \quad \text{if } p = n,$$

and $\langle x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_p \rangle = 0$ if $n \neq p$. Let $(\varphi, \psi) \in \widetilde{M} \times \widetilde{N}$, $\varphi = \sum_{i=1}^{n} \varphi^i$, $\psi = \sum_{i=1}^{m} \psi^i$ with $(\varphi^i, \psi^i) \in M^i \times N^i$; we set $\langle \varphi, \psi \rangle = \sum_{i=1}^{n} \langle \varphi^i, \psi^i \rangle$ (if $n \leq m$).

The linear representation $\widetilde{T}$ associated to T is supposed to be contragredient to $\widetilde{S}$ (the linear representation associated to S); namely, $\langle \widetilde{S}_g(\varphi), \widetilde{T}_g(\psi) \rangle = \langle \varphi, \psi \rangle$ (with $\varphi \in \widetilde{M}$ and $\psi \in \widetilde{N}$) or, equivalently,

$$\langle \widetilde{S}_g(\varphi), \psi \rangle = \langle \varphi, \widetilde{T}_{g^{-1}}(\psi) \rangle. \tag{2.1}$$

By differentiation we obtain, for every $X \in \mathfrak{g}$,

$$\langle d\widetilde{S}_X(\varphi), \psi \rangle = -\langle \varphi, d\widetilde{T}_X(\psi) \rangle. \tag{2.2}$$

We consider the linear operator $\widetilde{A}$ in $\mathcal{L}(\widetilde{M}, \widetilde{N})$ associated to A such that $\widetilde{A}_{|M_n} \in \mathcal{L}(M_n, N_n)$, which is defined as follows: for $x_1, \ldots, x_n \in M$,

$$\widetilde{A}(x_1 \otimes \cdots \otimes x_n) = \sum_{p=1}^{n} \sum_{i_1+\cdots+i_p=n} A^{i_1} \otimes \cdots \otimes A^{i_p} \circ \sigma_n(x_1 \otimes \cdots \otimes x_n).$$

In the present case, condition (1.1) means that A intertwines the representations S and T, that is, $A \circ S_g = T_g \circ A$ as power series, for every $g \in \boldsymbol{G}$. Now, since $\widetilde{A \circ S_g} = \widetilde{A} \circ \widetilde{S}_g$, $\widetilde{A}$ is an intertwining operator for the representations $(\widetilde{S}, \widetilde{M})$ and $(\widetilde{T}, \widetilde{N})$,

$$\widetilde{A} \circ \widetilde{S}_g = \widetilde{T}_g \circ \widetilde{A}, \quad g \in \boldsymbol{G}. \tag{2.3}$$

By differentiation we obtain

$$\widetilde{A} \circ d\widetilde{S}_X = d\widetilde{T}_X \circ \widetilde{A} \quad \text{for every } X \in \mathfrak{g}. \tag{2.4}$$

Proposition 2.1. *Under conditions* (1.1), (2.1) *and* (1.3), *we have the following transversality properties:*

(i) $\langle d\widetilde{S}_X(\varphi), \widetilde{A}(\varphi) \rangle = 0$ *for all* $\varphi \in \widetilde{U}$ *and* $X \in \mathfrak{g}'$*;*
(ii) $\langle dS_X(x), A(x) \rangle = 0$ *for all* $x \in B(0, r) \cap U$ *(with* r *small enough) and* $X \in \mathfrak{g}'$.

Proof. The differential form ω_A being closed, A^1 is symmetric with respect to the duality $\langle\, ,\, \rangle$; developing the expression $\langle \sigma_n(x_1 \otimes \cdots \otimes x_n), \widetilde{A}\sigma_n(x_1' \otimes \cdots \otimes x_n') \rangle$ with $x_i, x_i' \in M$, $i = 1, \ldots, n$, we see that $\widetilde{A}$ is symmetric, that is,

$$\langle \varphi_1, \widetilde{A}(\varphi_2) \rangle = \langle \varphi_2, \widetilde{A}(\varphi_1) \rangle \quad \text{for every } \varphi_1, \varphi_2 \in \widetilde{M}. \tag{2.5}$$

Since $\widetilde{A}$ intertwines $d\widetilde{S}$ with $d\widetilde{T}$ (condition (2.4)), we have

$$\langle \varphi_1, \widetilde{A} \circ d\widetilde{S}_X(\varphi_2)\rangle = \langle \varphi_1, d\widetilde{T}_X \circ \widetilde{A}(\varphi_2)\rangle,$$

and by (2.2),

$$\langle \varphi_1, d\widetilde{T}_X \circ \widetilde{A}(\varphi_2)\rangle = -\langle d\widetilde{S}_X(\varphi_1), A(\varphi_2)\rangle.$$

So, taking $\varphi_1 = d\widetilde{S}_Y(\varphi_2)$ with $Y \in \mathfrak{g}$, we have

$$\langle d\widetilde{S}_Y(\varphi_2), \widetilde{A} \circ d\widetilde{S}_X(\varphi_2)\rangle = -\langle d\widetilde{S}_X d\widetilde{S}_Y(\varphi_2), \widetilde{A}(\varphi_2)\rangle.$$

But $\widetilde{A}$ being symmetric we can write

$$\langle d\widetilde{S}_Y(\varphi_2), \widetilde{A} \circ d\widetilde{S}_X(\varphi_2)\rangle = \langle d\widetilde{S}_X(\varphi_2), \widetilde{A} \circ d\widetilde{S}_Y(\varphi_2)\rangle.$$

Thus $\langle d\widetilde{S}_X d\widetilde{S}_Y(\varphi_2), \widetilde{A}(\varphi_2)\rangle = \langle d\widetilde{S}_Y d\widetilde{S}_X(\varphi_2), \widetilde{A}(\varphi_2)\rangle$; hence

$$\langle [d\widetilde{S}_X, d\widetilde{S}_Y](\varphi_2), \widetilde{A}(\varphi_2)\rangle = 0,$$

and so we get (i).

The statement (ii) will be proved by induction. By (i) we have in particular $\langle dS^1_X(x), A^1(x)\rangle = 0$ for $x \in U$ and $X \in \mathfrak{g}'$; supposing that $\langle dS^p_X(x), A^k(x)\rangle = 0$ for $p \leq n-1$ and $k \leq m-1$, we now show that $\langle dS^n_X(x), A^m(x)\rangle = 0$. Indeed,

$$dS^n_X(x) = d\widetilde{S}_X(\otimes^n x) - \sum_{p=2}^{n}\sum_{q=0}^{p-1} I_q \otimes dS_X^{n-p+1} \otimes I_{p-q-1}(\otimes^n x)$$

and

$$A^m(x) = \widetilde{A}(\otimes^m x) - \sum_{k=2}^{m}\sum_{i_1+\cdots+i_k=m} A^{i_1}(x) \otimes \cdots \otimes A^{i_k}(x) \quad \text{for } x \in B(0,r) \cap U.$$

Hence

$$\begin{aligned}
&\langle dS^n(x), A^m(x)\rangle \\
&\quad = \langle d\widetilde{S}_X(\otimes^n x), \widetilde{A}(\otimes^m x)\rangle - \sum_{k=2}^{m}\sum_{i_1+\cdots+i_k=m} \langle dS^n_X(\otimes^n x), A^{i_1}(x) \otimes \cdots \otimes A^{i_k}(x)\rangle \\
&\qquad - \sum_{p=2}^{n}\sum_{q=0}^{p-1} \langle \otimes^q x \otimes dS_X^{n-p+1}(x) \otimes^{p-q-1} x, A^m(x)\rangle \\
&\qquad - \sum_{p=2}^{n}\sum_{q=0}^{p-1}\sum_{k=2}^{m}\sum_{i_1+\cdots+i_k=m} \langle \otimes^q x \otimes dS_X^{n-p+1}(x) \otimes^{p-q-1} x, A^{i_1}(x) \otimes \cdots \otimes A^{i_k}(x)\rangle.
\end{aligned}$$

The first term vanishes by (i), as does the second and third, and the last one is a sum of terms of the form $\langle \otimes^{i_1} x, A^{i_1}(x)\rangle \cdots \langle dS_X^{n-p+1}(x), A^{i_j}(x)\rangle \cdots$, which vanish by the induction hypothesis. Now, $\langle\ ,\ \rangle$ being separately continuous, we get

$$\langle dS_X(x), A(x)\rangle = \sum_{n,m} \langle dS^n_X(x), A^m(x)\rangle = 0. \qquad \square$$

3. TRANSVERSALITY CRITERIA FOR C^∞ MAPPINGS

Note that if T^1 is irreducible, the commutation relation (1.2) gives that $dT_X A(0) = 0$ for every $X \in \mathfrak{g}$; in particular $A(0)$ belongs to $\bigcap_{X\in\mathfrak{g}} \operatorname{Ker} dT^1_X$, which is an invariant subspace of N; so $A(0) = 0$. Note that if $\boldsymbol{G}$ is nilpotent and T^1 unitary then $T \sim T^1$ (see [1]) and (1.1) reduces to the following condition of covariance for A considered in [5]:

$$A \circ S_g = T^1_g \circ A, \quad g \in \boldsymbol{G}.$$

In the following we suppose that $A(0) = 0$.

PROPOSITION 3.1. *Suppose that A is C^∞. Then under conditions* (1.1), (2.1) *and* (1.3), *for any point $x \in B(0,r) \cap U$ (with r small enough) $A(x)$ belongs to the polar of the orbit of x under $\mathfrak{g}'$, that is,*

$$\langle dS_X(x), A(x)\rangle = 0 \quad \textit{for every } X \in \mathfrak{g}'.$$

PROOF. Put $A^n = (1/n!)A^{(n)}(0)$, where $A^{(n)}(0)$ is the n^{th} differential of A at 0, and consider the operator $\widetilde{A}$ in $L_g(\widetilde{M}, \widetilde{N})$, the space of linear mappings F from $\widetilde{M}$ to $\widetilde{N}$ such that $F_{|M_n} \in \mathcal{L}(M_n, N_n)$, which is defined as follows: for $x_1, \dots, x_n \in B(0,r) \cap U$,

$$\widetilde{A}(x_1 \otimes \cdots \otimes x_n) = \sum_{p=1}^{n} \sum_{i_1+\cdots+i_p=n} A^{i_1} \otimes \cdots \otimes A^{i_p} \circ \sigma_n(x_1 \otimes \cdots \otimes x_n).$$

To carry over the development of Section 2 to the present case, we must prove that $\widetilde{A}$ intertwines $d\widetilde{S}$ with $d\widetilde{T}$.

Consider the formal series $\widehat{A} = \sum_{n\geq 1} A^n$ and let us show that $\widehat{A}$ is a formal intertwining operator for S and T. Let $g \in \boldsymbol{G}$ and take $x \in U$ with $\|x\|$ small enough for the segment $[0, S_g(x)]$ to be contained in U. Taylor's formulas for the mappings $A \circ S_g$ and $T_g \circ A$ yield for the n^{th} order term

$$\sum_{p=1}^{n} A^p\Big(\sum_{i_1+\cdots+i_p=n} S_g^{i_1}(x) \otimes \cdots \otimes S_g^{i_p}(x)\Big) = \sum_{k=1}^{n} T_g^k\Big(\sum_{j_1+\cdots+j_k=n} A^{j_1}(x) \otimes \cdots \otimes A^{j_k}(x)\Big).$$

Therefore we have formally

$$\widehat{A} \circ S_g(x) = T_g \circ \widehat{A}(x).$$

Now, $\widetilde{\widehat{A}} = \widetilde{A}$; thus

$$\widetilde{\widehat{A} \circ S_g} = \widetilde{A} \circ \widetilde{S}_g = \widetilde{T}_g \circ \widetilde{A}.$$

Hence, according to the proof of Proposition 2.1, we have $\langle dS^p_X(x), A^n(x)\rangle = 0$ for all $X \in \mathfrak{g}'$ and all p, n. Since $\langle\ ,\ \rangle$ is separately continuous and dS^p_X is continuous if $p \geq 2$, writing again Taylor's formula for A up to order n, we obtain

$$|\langle dS^p_X(x), A(x)\rangle| = o(\|x\|^{2n+p}) \quad \text{for } p \geq 2 \text{ and every } n.$$

It follows that $\langle dS^p_X(x), A(x)\rangle = 0$ if $p \geq 2$, and according to Proposition 1.1 we have $\langle dS^1_X(x), A(x)\rangle = 0$; therefore $\langle dS_X(x), A(x)\rangle = 0$. $\square$

ACKNOWLEDGMENTS. This work has been supported by a research contract within the Università di Padova.

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Received 27 April 2006,
and in revised form 5 September 2006.

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