

Coupling of Two Partial Differential Equations and its Application, II — the Case of Briot-Bouquet Type PDEs —

By

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Abstract

Let $F(t, x, u, v)$ be a holomorphic function in a neighborhood of the origin of \mathbb{C}^4 satisfying $F(0, x, 0, 0) \equiv 0$ and $(\partial F/\partial v)(0, x, 0, 0) \equiv 0$; then the equation (A) $t\partial u/\partial t = F(t, x, u, \partial u/\partial x)$ is called a Briot-Bouquet type partial differential equation, and the function $\lambda(x) = (\partial F/\partial u)(0, x, 0, 0)$ is called the characteristic exponent. This paper considers a reduction of this equation (A) to a simple form (B) $t\partial w/\partial t = \lambda(x)w$ under the assumption $\lambda(0) \notin (-\infty, 0] \cup \{1, 2, \dots\}$. The reduction is done by considering the coupling of two equations (A) and (B), and by solving their coupling equations. The result is applied to the problem of finding all the singular solutions of (A).

§1. Introduction

Let (t, x) be the variables in $\mathbb{C}_t \times \mathbb{C}_x$, and let $F(t, x, u, v)$ be a holomorphic function defined in a polydisk Δ centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_u \times \mathbb{C}_v$. In the previous paper [3], we have established the equivalence of the following two partial differential equations

$$\frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right) \quad \text{and} \quad \frac{\partial w}{\partial t} = 0$$

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by considering the coupling of these two equations and by solving their coupling equations.

In this paper, we will apply similar arguments to the following nonlinear singular partial differential equation

$$(1.1) \quad t \frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right)$$

under the assumptions

- A₁) $F(t, x, u, v)$ is holomorphic in Δ ,
- A₂) $F(0, x, 0, 0) = 0$ in $\Delta_0 = \Delta \cap \{t = 0, u = 0, v = 0\}$, and
- A₃) $\frac{\partial F}{\partial v}(0, x, 0, 0) = 0$ in Δ_0 .

In the book of Gérard-Tahara [1], the equation (1.1) is called a *Briot-Bouquet type* partial differential equation with respect to t if it satisfies the conditions A₁), A₂) and A₃); the function

$$(1.2) \quad \lambda(x) = \frac{\partial F}{\partial u}(0, x, 0, 0)$$

is called *the characteristic exponent* (or *the characteristic exponent function*) of (1.1). About the structure of holomorphic and singular solutions of (1.1) in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$, one can refer to Gérard-Tahara [1], [2] and Yamazawa [4]. Among them, the following theorem is the most fundamental result:

Theorem 1.1 ([2]). *If $\lambda(0) \notin \{1, 2, \dots\}$ holds, the equation (1.1) has a unique holomorphic solution $u_0(t, x)$ in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$ satisfying $u_0(0, x) \equiv 0$ near $x = 0$.*

In this paper, from the standpoint of the transformation theory of partial differential equations we will consider the following problem:

Problem 1.2. Find a normal form of the equation (1.1) by considering the coupling of two partial differential equations.

As is seen in the case of Briot-Bouquet's ordinary differential equations (in chapter 4 of [1]), it will be reasonable to treat the equation

$$(1.3) \quad t \frac{\partial w}{\partial t} = \lambda(x)w$$

as a candidate of the normal form of (1.1) in a generic case. In order to justify this assertion, it is enough to discuss their coupling equations

$$\begin{aligned}
 (\Phi) \quad & t \frac{\partial \phi}{\partial t} + \sum_{m \geq 0} D^m [F](t, x, u_0, \dots, u_{m+1}) \frac{\partial \phi}{\partial u_m} = \lambda(x)\phi, \quad \text{and} \\
 (\Psi) \quad & t \frac{\partial \psi}{\partial t} + \sum_{m \geq 0} \left(\sum_{0 \leq i \leq m} \lambda_{m,i}(x) w_i \right) \frac{\partial \psi}{\partial w_m} = F(t, x, \psi, D[\psi]),
 \end{aligned}$$

where D is the vector field with infinitely many variables (x, u_0, u_1, \dots) (resp. (x, w_0, w_1, \dots)):

$$D = \frac{\partial}{\partial x} + \sum_{i \geq 0} u_{i+1} \frac{\partial}{\partial u_i} \quad \left(\text{resp. } D = \frac{\partial}{\partial x} + \sum_{i \geq 0} w_{i+1} \frac{\partial}{\partial w_i} \right),$$

and

$$\lambda_{m,i}(x) = \frac{m!}{i!(m-i)!} \left(\frac{\partial}{\partial x} \right)^{m-i} \lambda(x), \quad 0 \leq i \leq m.$$

In the equation (Φ) (resp. (Ψ)), $(t, x, u_0, u_1, \dots) \in \mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_u^\infty$ (resp. $(t, x, w_0, w_1, \dots) \in \mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_w^\infty$) are infinitely many complex variables, and $\phi = \phi(t, x, u_0, u_1, \dots)$ (resp. $\psi = \psi(t, x, w_0, w_1, \dots)$) is the unknown function. Roughly speaking, the role of these equations is explained as

Proposition 1.3. *Suppose the conditions $A_1), A_2), A_3)$ and that*

$$\begin{aligned}
 (1.4) \quad & |i + \lambda(x)(j-1)| \geq \sigma(i+j) \quad \text{on } D_R = \{x \in \mathbb{C}; |x| \leq R\} \\
 & \text{for any } (i, j) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0), (0, 1)\}
 \end{aligned}$$

for some $\sigma > 0$ and $R > 0$. Then, we have the following results.

(1) *The equation (Φ) (resp. (Ψ)) has a holomorphic solution $\phi = \phi(t, x, u_0, u_1, \dots)$ (resp. $\psi = \psi(t, x, w_0, w_1, \dots)$) in a suitable domain in $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_u^\infty$ (resp. $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_w^\infty$).*

(2) *If $u(t, x)$ (resp. $w(t, x)$) is a solution of (1.1) (resp. (1.3)), then the function $w(t, x) = \phi(t, x, u, \partial u / \partial x, \dots)$ (resp. $u(t, x) = \psi(t, x, w, \partial w / \partial x, \dots)$) is a solution of (1.3) (resp. (1.1)).*

(3) *The two transformations $u(t, x) \mapsto w(t, x) = \phi(t, x, u, \partial u / \partial x, \dots)$ and $w(t, x) \mapsto u(t, x) = \psi(t, x, w, \partial w / \partial x, \dots)$ are invertible, and one is the inverse of the other.*

Hence, we have the following main theorem of this paper.

Theorem 1.4 (Main theorem). *Suppose the conditions $A_1), A_2), A_3)$ and (1.4) (for some $\sigma > 0$ and $R > 0$). Then the following two equations are*

equivalent:

$$(1.5) \quad t \frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right) \quad \text{and} \quad t \frac{\partial w}{\partial t} = \lambda(x)w.$$

The meaning of the equivalence of two equations will be defined later (see Definition 2.7). We have used the notation: $\mathbb{N} = \{0, 1, 2, \dots\}$.

The paper is organized as follows. In the next section 2, we will present some basic results in the formal theory of the coupling of two partial differential equations (developed in [3]). In section 3, we will prove Theorem 1.4 by the following steps: in subsections 3.1 and 3.2 we will find formal solutions of the coupling equations (Φ) and (Ψ) : this shows the formal equivalence of the two partial differential equations in (1.5), in subsections 3.3 and 3.4 we will prove the convergence of the formal solutions, and in subsection 3.5 we will establish the equivalence of the two equations in (1.5) in a concrete sense. In the last section 4, by using our main theorem (Theorem 1.4) we will determine all the singular solutions of (1.1) in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$.

§2. Basics on the Coupling Equations

In this section, we will survey basic results in the formal theory of the coupling of two singular partial differential equations

$$(A) \quad t \frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right)$$

(where $(t, x) \in \mathbb{C}_t \times \mathbb{C}_x$ are variables and $u = u(t, x)$ is the unknown function) and

$$(B) \quad t \frac{\partial w}{\partial t} = G\left(t, x, w, \frac{\partial w}{\partial x}\right)$$

(where $(t, x) \in \mathbb{C}_t \times \mathbb{C}_x$ are variables and $w = w(t, x)$ is the unknown function). For simplicity we suppose that $F(t, x, u_0, u_1)$ (resp. $G(t, x, w_0, w_1)$) is a holomorphic function defined in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_{u_0} \times \mathbb{C}_{u_1}$ (resp. $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_{w_0} \times \mathbb{C}_{w_1}$).

For a function $\phi = \phi(t, x, u_0, u_1, \dots)$ with infinitely many variables (t, x, u_0, u_1, \dots) we define $D[\phi](t, x, u_0, u_1, \dots)$ by

$$D[\phi] = \frac{\partial \phi}{\partial x}(t, x, u_0, u_1, \dots) + \sum_{i \geq 0} u_{i+1} \frac{\partial \phi}{\partial u_i}(t, x, u_0, u_1, \dots).$$

For $m \geq 2$ we define $D^m[\phi]$ as follows: $D^2[\phi] = D[D[\phi]]$, $D^3[\phi] = D[D^2[\phi]]$, and so on. If ϕ is a function with $(p+3)$ -variables (t, x, u_0, \dots, u_p) then $D^m[\phi]$ is a function with $(p+m+3)$ -variables $(t, x, u_0, \dots, u_{p+m})$. It is clear that

$$D^m[\phi]\left(t, x, u, \frac{\partial u}{\partial x}, \dots\right) = \left(\frac{\partial}{\partial x}\right)^m \left[\phi\left(t, x, u, \frac{\partial u}{\partial x}, \dots\right)\right]$$

holds for any $m \in \mathbb{N}$ and any function $u(t, x)$.

Of course, if $\psi = \psi(t, x, w_0, w_1, \dots)$ is a function with variables (t, x, w_0, w_1, \dots) the notation $D[\psi](t, x, w_0, w_1, \dots)$ is read as

$$D[\psi] = \frac{\partial \psi}{\partial x}(t, x, w_0, w_1, \dots) + \sum_{i \geq 0} w_{i+1} \frac{\partial \psi}{\partial w_i}(t, x, w_0, w_1, \dots).$$

We can regard D as a vector field with infinitely many variables (x, u_0, u_1, \dots) (resp. (x, w_0, w_1, \dots)):

$$D = \frac{\partial}{\partial x} + \sum_{i \geq 0} u_{i+1} \frac{\partial}{\partial u_i} \quad \left(\text{resp. } D = \frac{\partial}{\partial x} + \sum_{i \geq 0} w_{i+1} \frac{\partial}{\partial w_i}\right).$$

Usually, this operator D is called *the totally derivative operator*.

Definition 2.1. The coupling of two partial differential equations (A) and (B) means that we consider the following partial differential equation with infinitely many variables (t, x, u_0, u_1, \dots)

$$(\Phi) \quad t \frac{\partial \phi}{\partial t} + \sum_{m \geq 0} D^m[F](t, x, u_0, \dots, u_{m+1}) \frac{\partial \phi}{\partial u_m} = G(t, x, \phi, D[\phi])$$

(where $\phi = \phi(t, x, u_0, u_1, \dots)$ is the unknown function), or the following partial differential equation with infinitely many variables (t, x, w_0, w_1, \dots)

$$(\Psi) \quad t \frac{\partial \psi}{\partial t} + \sum_{m \geq 0} D^m[G](t, x, w_0, \dots, w_{m+1}) \frac{\partial \psi}{\partial w_m} = F(t, x, \psi, D[\psi])$$

(where $\psi = \psi(t, x, w_0, w_1, \dots)$ is the unknown function).

§2.1. The formal meaning of the coupling equation

First, let us explain the meaning of the coupling equations (Φ) and (Ψ) in the formal sense. Here, “*in the formal sense*” means that the result is true if the formal calculation makes sense. For simplicity we write

$$\phi(t, x, u, \partial u / \partial x, \dots) = \phi\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots, \frac{\partial^n u}{\partial x^n}, \dots\right).$$

The convenience of considering the coupling equation lies in the following proposition.

Proposition 2.2. (1) *If $\phi(t, x, u_0, u_1, \dots)$ is a solution of (Φ) and if $u(t, x)$ is a solution of (A) , then the function $w(t, x) = \phi(t, x, u, \partial u/\partial x, \dots)$ is a solution of (B) .*

(2) *If $\psi(t, x, w_0, w_1, \dots)$ is a solution of (Ψ) and if $w(t, x)$ is a solution of (B) , then the function $u(t, x) = \psi(t, x, w, \partial w/\partial x, \dots)$ is a solution of (A) .*

Proof. This result corresponds to [Proposition 3.1.1 in [3]], the proof of which works also in this case; we have only to replace $\partial/\partial t$ by $t\partial/\partial t$. □

In order to state the relation between (Φ) and (Ψ) , we need the following notion of *the reversibility of $\phi(t, x, u_0, u_1, \dots)$* .

Definition 2.3. Let $\phi(t, x, u_0, u_1, \dots)$ be a function of (t, x, u_0, u_1, \dots) . We say that the relation $w = \phi(t, x, u, \partial u/\partial x, \dots)$ is reversible with respect to u and w if there is a function $\psi(t, x, w_0, w_1, \dots)$ of (t, x, w_0, w_1, \dots) such that the relation

$$(2.1) \quad \begin{cases} w_0 = \phi(t, x, u_0, u_1, u_2, \dots), \\ w_1 = D[\phi](t, x, u_0, u_1, u_2, \dots), \\ w_2 = D^2[\phi](t, x, u_0, u_1, u_2, \dots), \\ \dots\dots \\ \dots\dots \end{cases}$$

is equivalent to

$$(2.2) \quad \begin{cases} u_0 = \psi(t, x, w_0, w_1, w_2, \dots), \\ u_1 = D[\psi](t, x, w_0, w_1, w_2, \dots), \\ u_2 = D^2[\psi](t, x, w_0, w_1, w_2, \dots), \\ \dots\dots \\ \dots\dots \end{cases}$$

In this case the function $\psi(t, x, w_0, w_1, \dots)$ is called the reverse function of $\phi(t, x, u_0, u_1, \dots)$.

By the definition, we see that the reverse function $\psi(t, x, w_0, w_1, \dots)$ of $\phi(t, x, u_0, u_1, \dots)$ is unique, if it exists.

The following proposition gives the relation between two coupling equations (Φ) and (Ψ) : we can say that the equation (Ψ) is the reverse of (Φ) .

Proposition 2.4. *If $\phi(t, x, u_0, u_1, \dots)$ is a solution of (Φ) and if the relation $w = \phi(t, x, u, \partial u/\partial x, \dots)$ is reversible with respect to u and w , then the reverse function $\psi(t, x, w_0, w_1, \dots)$ is a solution of (Ψ) .*

Proof. This result corresponds to [Proposition 3.1.4 in [3]], the proof of which works also in this case; we have only to replace $\partial/\partial t$ by $t\partial/\partial t$. \square

Recall that the following lemma played an important role in the proof of Proposition 2.4:

Lemma 2.5 ((2) of Lemma 3.1.5 in [3]). *Let $\phi(t, x, u_0, u_1, \dots)$ be a function with variables (t, x, u_0, u_1, \dots) and let $K(t, x, w_0, w_1, \dots)$ be a function with variables (t, x, w_0, w_1, \dots) . Then, for any $m = 0, 1, 2, \dots$ we have*

$$D^m[K(t, x, \phi, D[\phi], \dots)] = (D^m[K])(t, x, \phi, D[\phi], \dots)$$

as a function with respect to the variables (t, x, u_0, u_1, \dots) .

§2.2. Equivalence of (A) and (B)

Let \mathcal{F} and \mathcal{G} be function-spaces in which we can consider the following two partial differential equations:

$$[A] \quad t \frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right) \quad \text{in } \mathcal{F},$$

$$[B] \quad t \frac{\partial w}{\partial t} = G\left(t, x, w, \frac{\partial w}{\partial x}\right) \quad \text{in } \mathcal{G}.$$

Set

$$\mathcal{S}_A = \text{the set of all solutions of [A] in } \mathcal{F},$$

$$\mathcal{S}_B = \text{the set of all solutions of [B] in } \mathcal{G}.$$

Then, if the coupling equation (Φ) has a solution $\phi(t, x, u_0, u_1, \dots)$ and if $\phi(t, x, u, \partial u/\partial x, \dots) \in \mathcal{G}$ is well-defined for any $u \in \mathcal{S}_A$, we can define the mapping

$$(2.3) \quad \Phi : \mathcal{S}_A \ni u(t, x) \mapsto w(t, x) = \phi(t, x, u, \partial u/\partial x, \dots) \in \mathcal{S}_B.$$

If the relation $w = \phi(t, x, u, \partial u/\partial x, \dots)$ is reversible with respect to u and w , and if the reverse function $\psi(t, x, w_0, w_1, \dots)$ satisfies $\psi(t, x, w, \partial w/\partial x, \dots) \in \mathcal{F}$ for any $w \in \mathcal{S}_B$, we can also define the mapping

$$(2.4) \quad \Psi : \mathcal{S}_B \ni w(t, x) \mapsto u(t, x) = \psi(t, x, w, \partial w/\partial x, \dots) \in \mathcal{S}_A.$$

Set $w(t, x) = \phi(t, x, u, \partial u/\partial x, \dots)$; then by the definition of D we have

$$(2.5) \quad \begin{cases} w = \phi(t, x, u, \partial u/\partial x, \dots), \\ \partial w/\partial x = D[\phi](t, x, u, \partial u/\partial x, \dots), \\ \partial^2 w/\partial x^2 = D^2[\phi](t, x, u, \partial u/\partial x, \dots), \\ \dots\dots\dots \\ \dots\dots\dots \end{cases}$$

Similarly, if we set $u(t, x) = \psi(t, x, w, \partial w/\partial x, \dots)$ we have

$$(2.6) \quad \begin{cases} u = \phi(t, x, w, \partial w/\partial x, \dots), \\ \partial u/\partial x = D[\phi](t, x, w, \partial w/\partial x, \dots), \\ \partial^2 u/\partial x^2 = D^2[\phi](t, x, w, \partial w/\partial x, \dots), \\ \dots\dots\dots \\ \dots\dots\dots \end{cases}$$

Since (2.1) is equivalent to (2.2) we have the equivalence between (2.5) and (2.6); therefore we have $\Psi \circ \Phi = \text{identity}$ in \mathcal{S}_A and $\Phi \circ \Psi = \text{identity}$ in \mathcal{S}_B . Thus, we obtain

Theorem 2.6. *Suppose that the coupling equation (Φ) has a solution $\phi(t, x, u_0, u_1 \dots)$ and that the relation $w = \phi(t, x, u, \partial u/\partial x, \dots)$ is reversible with respect to u and w . If both mappings (2.3) and (2.4) are well-defined, we can conclude that the both mappings are bijective and one is the inverse of the other.*

By this theorem, we may say:

Definition 2.7. (1) If the coupling equation (Φ) (resp. (Ψ)) has a solution $\phi(t, x, u_0, u_1 \dots)$ (resp. $\psi(t, x, w_0, w_1 \dots)$) and if the relation $w = \phi(t, x, u, \partial u/\partial x, \dots)$ (resp. $u = \psi(t, x, w, \partial w/\partial x, \dots)$) is reversible with respect to u and w (or w and u), then we say that the two equations (A) and (B) are formally equivalent.

(2) In addition, if both mappings (2.3) and (2.4) are well-defined, we say that the two equations [A] and [B] are equivalent.

§2.3. A sufficient condition for the reversibility

As is seen above, the condition of the reversibility of $\phi(t, x, u_0, u_1, \dots)$ is very important. In [Proposition 3.3.1 in [3]] we gave one sufficient condition for the relation $w = \phi(t, x, u, \partial u/\partial x, \dots)$ to be reversible with respect to u and w . In this section, we will give another sufficient condition.

Let us introduce the notations: $D_R = \{x \in \mathbb{C}; |x| \leq R\}$, \mathcal{O}_R denotes the ring of holomorphic functions in a neighborhood of D_R , and $\mathcal{H}_{k,R}[t, u_0, \dots, u_p]$ denotes the set of all homogeneous polynomials of degree k in (t, u_0, \dots, u_p) with coefficients in \mathcal{O}_R .

Proposition 2.8. *Let $\phi(t, x, u_0, u_1, \dots)$ be of the form*

$$(2.7) \quad \phi = \sum_{k \geq 1} \phi_k(t, x, u_0, \dots, u_{k-1}) \in \sum_{k \geq 1} \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}].$$

If $(\partial\phi_1/\partial u_0) \neq 0$ on D_R , the relation $w = \phi(t, x, u, \partial u/\partial x, \dots)$ is reversible with respect to u and w , and the reverse function $\psi(t, x, w_0, w_1, \dots)$ is also of the form

$$(2.8) \quad \psi = \sum_{k \geq 1} \psi_k(t, x, w_0, \dots, w_{k-1}) \in \sum_{k \geq 1} \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$$

with $(\partial\psi_1/\partial w_0) \neq 0$ on D_R .

To prove this, let us consider the following equation with respect to the unknown function $\psi = \psi(t, x, w_0, w_1, \dots)$:

$$(2.9) \quad w_0 = \phi(t, x, \psi, D[\psi], D^2[\psi], \dots) \quad \text{in} \quad \sum_{k \geq 1} \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}].$$

We have

Lemma 2.9. *Let $\phi(t, x, u_0, u_1, \dots)$ be of the form (2.7), and suppose that $(\partial\phi_1/\partial u_0) \neq 0$ holds on D_R . Then the equation (2.9) has a unique solution $\psi(t, x, w_0, w_1, \dots)$ of the form (2.8) with $(\partial\psi_1/\partial w_0) \neq 0$ on D_R , and it satisfies also*

$$(2.10) \quad u_0 = \psi(t, x, \phi, D[\phi], D^2[\phi], \dots) \quad \text{in} \quad \sum_{k \geq 1} \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}].$$

Proof. For simplicity we set

$$\mathcal{M}^p = \sum_{k \geq p} \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}], \quad p = 1, 2, \dots$$

Set $\phi_1 = \alpha(x)t + \beta(x)u_0 \in \mathcal{H}_{1,R}[t, u_0]$; by the assumption we have $\beta(x) \neq 0$ on D_R . Then our equation (2.9) is written as

$$(2.11) \quad w_0 = \alpha(x)t + \beta(x)\psi + \sum_{k \geq 2} \phi_k(t, x, \psi, \dots, D^{k-1}[\psi]) \quad \text{in} \quad \mathcal{M}^1.$$

Let

$$\psi = \sum_{k \geq 1} \psi_k(t, x, w_0, \dots, w_{k-1}) \in \mathcal{M}^1$$

be the unknown function. By substituting this into the equation (2.11) and by considering the equation modulo \mathcal{M}^{p+1} we have

$$(2.12)_p \quad w_0 \equiv \alpha(x)t + \beta(x)(\psi_1 + \dots + \psi_p) + \sum_{k=2}^p \phi_k \left(t, x, \sum_{j=1}^{p+1-k} \psi_j, \dots, \sum_{j=1}^{p+1-k} D^{k-1}[\psi_j] \right) \pmod{\mathcal{M}^{p+1}}.$$

In the case $p = 1$, (2.12)₁ gives the equality

$$(2.13) \quad w_0 = \alpha(x)t + \beta(x)\psi_1$$

and so we have

$$\psi_1 = -(\alpha(x)/\beta(x))t + (1/\beta(x))w_0 \in \mathcal{H}_{1,R}[t, w_0].$$

Thus we have seen that $\psi_1 \in \mathcal{H}_{1,R}[t, w_0]$ is determined uniquely, and it satisfies $(\partial\psi_1/\partial w_0) \neq 0$ on D_R . If we take $p = 2$, we have also the equality

$$w_0 = \alpha(x)t + \beta(x)(\psi_1 + \psi_2) + \phi_2(t, x, \psi_1, D[\psi_1])$$

where $\psi_1 = -(\alpha(x)/\beta(x))t + (1/\beta(x))w_0$ and so $D[\psi_1] = -(\alpha(x)/\beta(x))'t + (1/\beta(x))'w_0 + (1/\beta(x))w_1$. Since (2.13) is true, we have

$$\psi_2 = -(1/\beta(x))\phi_2(t, x, \psi_1, D[\psi_1]) \in \mathcal{H}_{2,R}[t, w_0, w_1];$$

this proves that $\psi_2 \in \mathcal{H}_{2,R}[t, w_0, w_1]$ is also determined uniquely.

For a general $p \geq 3$, by (2.13) and (2.12)_p we have

$$(2.14)_p \quad \sum_{k=2}^p \psi_k(t, x, w_0, \dots, w_{k-1}) \\ \equiv -\frac{1}{\beta(x)} \sum_{k=2}^p \phi_k \left(t, x, \sum_{j=1}^{p+1-k} \psi_j, \dots, \sum_{j=1}^{p+1-k} D^{k-1}[\psi_j] \right) \pmod{\mathcal{M}^{p+1}}.$$

Since the right-hand side of (2.14)_p is determined only by $\psi_1, \dots, \psi_{p-1}$, the formula (2.14)_p guarantees that $\psi_p \in \mathcal{H}_{p,R}[t, w_0, \dots, w_{p-1}]$ is determined uniquely if $\psi_1, \dots, \psi_{p-1}$ are known.

Thus, we can conclude that all $\psi_p \in \mathcal{H}_{p,R}[t, w_0, \dots, w_{p-1}]$ ($p = 1, 2, \dots$) are determined uniquely by the formula (2.14)_p by induction on p . This proves the former half of Lemma 2.9.

Next let us show the latter half of this lemma: to do so it is sufficient to prove the following equality

$$(2.15)_p \quad u_0 \equiv \sum_{k=1}^p \psi_k \left(t, x, \sum_{h=1}^p \phi_h, \dots, \sum_{h=1}^p D^{k-1}[\phi_h] \right) \pmod{\mathcal{M}_u^{p+1}}$$

for all $p = 1, 2, \dots$, where \mathcal{M}_u^{p+1} is the same one as \mathcal{M}^{p+1} with (w_0, w_1, \dots) replaced by (u_0, u_1, \dots) . In the case $p = 1$ we have

$$\begin{aligned} \psi_1(t, x, \phi_1) &= -(\alpha(x)/\beta(x))t + (1/\beta(x))\phi_1 \\ &= -(\alpha(x)/\beta(x))t + (1/\beta(x))(\alpha(x)t + \beta(x)u_0) = u_0, \end{aligned}$$

which proves (2.15)₁.

Let $p \geq 2$ and suppose that (2.15) _{$p-1$} is already proved. Then, by Lemma 2.5 and the fact $D^j[u_0] = u_j$ we have

$$u_j \equiv \sum_{k=1}^{p-1} D^j[\psi_k] \left(t, x, \sum_{h=1}^{p-1} \phi_h, \dots, \sum_{h=1}^{p-1} D^{k-1+j}[\phi_h] \right) \pmod{\mathcal{M}_u^p}$$

for $j = 0, 1, 2, \dots$. Since this equality is considered modulo \mathcal{M}_u^p , this implies that

$$(2.16) \quad u_j \equiv \sum_{k=1}^{p-1} D^j[\psi_k] \left(t, x, \sum_{h=1}^p \phi_h, \dots, \sum_{h=1}^p D^{k-1+j}[\phi_h] \right) \pmod{\mathcal{M}_u^p}$$

holds for all $j = 0, 1, 2, \dots$. Since ψ_1, \dots, ψ_p are determined so that the relation (2.12) _{p} holds, we have

$$(2.17) \quad w_0 \equiv \alpha(x)t + \beta(x)(\psi_1 + \dots + \psi_p) + \sum_{k=2}^p \phi_k \left(t, x, \sum_{j=1}^{p-1} \psi_j, \dots, \sum_{j=1}^{p-1} D^{k-1}[\psi_j] \right) \pmod{\mathcal{M}^{p+1}}.$$

By substituting

$$w_i = \sum_{h=1}^p D^i[\phi_h](t, x, u_0, \dots, u_{h-1+i}), \quad i = 0, 1, 2, \dots$$

into the both sides of (2.17), we have the relation

$$(2.18) \quad \sum_{h=1}^p \phi_h(t, x, u_0, \dots, u_{h-1}) \equiv \alpha(x)t + \beta(x) \sum_{j=1}^p \psi_j \left(t, x, \sum_{h=1}^p \phi_h, \dots, \sum_{h=1}^p D^{j-1}[\phi_h] \right) + \sum_{k=2}^p \phi_k \left(t, x, \sum_{j=1}^{p-1} \psi_j \left(t, x, \sum_{h=1}^p \phi_h, \dots, \sum_{h=1}^p D^{j-1}[\phi_h] \right), \dots, \sum_{j=1}^{p-1} D^{k-1}[\psi_j] \left(t, x, \sum_{h=1}^p \phi_h, \dots, \sum_{h=1}^p D^{j-1+k-1}[\phi_h] \right) \right) \pmod{\mathcal{M}_u^{p+1}}$$

as functions with respect to the variables (t, x, u_0, u_1, \dots) . Therefore, by applying (2.16) to the right-hand side of (2.18) we obtain

$$(2.19) \quad \begin{aligned} & \sum_{h=1}^p \phi_h(t, x, u_0, \dots, u_{h-1}) \\ & \equiv \alpha(x)t + \beta(x) \sum_{j=1}^p \psi_j \left(t, x, \sum_{h=1}^p \phi_h, \dots, \sum_{h=1}^p D^{j-1}[\phi_h] \right) \\ & \quad + \sum_{k=2}^p \phi_k(t, x, u_0, \dots, u_{k-1}) \pmod{\mathcal{M}_u^{p+1}}, \end{aligned}$$

and so by (2.19) we have

$$\begin{aligned} & \phi_1(t, x, u_0) \\ & \equiv \alpha(x)t + \beta(x) \sum_{j=1}^p \psi_j \left(t, x, \sum_{h=1}^p \phi_h, \dots, \sum_{h=1}^p D^k[\phi_h] \right) \pmod{\mathcal{M}_u^{p+1}}. \end{aligned}$$

Thus, by applying $\phi_1 = \alpha(x)t + \beta(x)u_0$ and by using the condition $\beta(x) \neq 0$ on D_R we obtain

$$u_0 \equiv \sum_{j=1}^p \psi_j \left(t, x, \sum_{h=1}^p \phi_h, \dots, \sum_{h=1}^p D^k[\phi_h] \right) \pmod{\mathcal{M}_u^{p+1}}.$$

This proves $(2.15)_p$.

Thus, by induction on p we can prove $(2.15)_p$ for all $p = 1, 2, \dots$. This proves the latter half of Lemma 2.9. \square

Proof of Proposition 2.8. Let $\psi(t, x, w_0, w_1, \dots)$ be the solution of (2.9) in Lemma 2.9. Let us show the equivalence between the relations

$$(2.20) \quad u_j = D^j[\psi](t, x, w_0, w_1, \dots), \quad j = 0, 1, 2, \dots$$

and

$$(2.21) \quad w_j = D^j[\phi](t, x, u_0, u_1, \dots), \quad j = 0, 1, 2, \dots$$

Let us apply D^j to the both sides of the equality (2.9); by Lemma 2.5 we have

$$w_j = D^j[\phi](t, x, \psi, D[\psi], D^2[\psi], \dots), \quad j = 0, 1, 2, \dots$$

and therefore under the relation (2.20) we can get the relation (2.21). Similarly, by applying D^j to the both sides of the equality (2.10) we have

$$u_j = D^j[\psi](t, x, \phi, D[\phi], D^2[\phi], \dots), \quad j = 0, 1, 2, \dots$$

and therefore under the relation (2.21) we can get the relation (2.20). This proves Proposition 2.8. \square

§3. Proof of the Main Theorem

In this section, we will give a proof of Theorem 1.4 (Main theorem) in the introduction. Let us recall our situation again.

Let $(t, x) \in \mathbb{C}_t \times \mathbb{C}_x$ be the variables, and let $F(t, x, u, v)$ be a function defined in a polydisk Δ centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_u \times \mathbb{C}_v$ satisfying

- A₁) $F(t, x, u, v)$ is holomorphic in Δ ,
- A₂) $F(0, x, 0, 0) = 0$ in $\Delta_0 = \Delta \cap \{t = 0, u = 0, v = 0\}$, and
- A₃) $\frac{\partial F}{\partial v}(0, x, 0, 0) = 0$ in Δ_0 .

Under these condition, let us consider the following partial differential equation

$$(3.0.1) \quad t \frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right) :$$

this equation is called a Briot-Bouquet type partial differential equation with respect to t (in [1]), and the function

$$\lambda(x) = \frac{\partial F}{\partial u}(0, x, 0, 0)$$

is called the characteristic exponent of (3.0.1).

Let us seek for a reduction of the equation (3.0.1) to a simple form. As is seen in the case of Briot-Bouquet's ordinary differential equation (in chapter 4 of [1]), it will be reasonable to treat the equation

$$(3.0.2) \quad t \frac{\partial w}{\partial t} = \lambda(x)w$$

as a candidate of the reduced form of (3.0.1). In order to justify this assertion, it is enough to discuss the following coupling equation

$$(\Phi) \quad t \frac{\partial \phi}{\partial t} + \sum_{m \geq 0} D^m [F](t, x, u_0, \dots, u_{m+1}) \frac{\partial \phi}{\partial u_m} = \lambda(x)\phi, \quad \text{and}$$

$$(\Psi) \quad t \frac{\partial \psi}{\partial t} + \sum_{m \geq 0} \left(\sum_{0 \leq i \leq m} \lambda_{m,i}(x) w_i \right) \frac{\partial \psi}{\partial w_m} = F(t, x, \psi, D[\psi])$$

where

$$\lambda_{m,i}(x) = \frac{m!}{i!(m-i)!} \left(\frac{\partial}{\partial x} \right)^{m-i} \lambda(x), \quad 0 \leq i \leq m.$$

§3.1. Formal solutions of (Φ)

First, let us look for a formal solution of the coupling equation (Φ) of the form

$$\phi = \sum_{k \geq 1} \phi_k(t, x, u_0, \dots, u_{k-1}) \in \sum_{k \geq 1} \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}].$$

We have

Proposition 3.1.1. *Let $R > 0$ be sufficiently small. Suppose the conditions $A_1), A_2), A_3)$ and that*

$$(3.1.1) \quad |i + \lambda(x)(j - 1)| \neq 0 \quad \text{on } D_R$$

for any $(i, j) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0), (0, 1)\}$.

Then, the coupling equation (Φ) has a family of formal solutions of the form

$$(3.1.2) \quad \phi = \frac{-a(x)\beta(x)}{1 - \lambda(x)} t + \beta(x)u_0 + \sum_{k \geq 2} \phi_k(t, x, u_0, \dots, u_{k-1})$$

with $\phi_k(t, x, u_0, \dots, u_{k-1}) \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$ ($k = 2, 3, \dots$),

where $a(x) = (\partial F / \partial t)(0, x, 0, 0)$ and $\beta(x) \in \mathcal{O}_R$. Here, $\beta(x)$ can be chosen arbitrarily, and the other $\phi_k(t, x, u_0, \dots, u_{k-1})$ ($k = 2, 3, \dots$) are uniquely determined by $\beta(x)$.

In particular, if we take a function $\beta(x) \in \mathcal{O}_R$ so that $\beta(x) \neq 0$ holds on D_R , by Proposition 2.8 we see that the relation $w = \phi(t, x, u, \partial u / \partial x, \dots)$ is reversible with respect to u and w . Hence we obtain

Theorem 3.1.2. *Suppose the conditions $A_1), A_2), A_3)$ and (3.1.1). Then, the equation (3.0.1) is formally equivalent to (3.0.2).*

Proof of Proposition 3.1.1. By the conditions $A_1), A_2)$ and $A_3)$ we have the expression

$$(3.1.3) \quad F(t, x, u_0, u_1) = a(x)t + \lambda(x)u_0 + \sum_{i+j+\alpha \geq 2} c_{i,j,\alpha}(x)t^i u_0^j u_1^\alpha$$

where $a(x)$, $\lambda(x)$ and $c_{i,j,\alpha}(x)$ ($i + j + \alpha \geq 2$) are all holomorphic functions in a neighborhood of D_R . We set

$$R_p(t, x, u_0, u_1) = \sum_{i+j+\alpha=p} c_{i,j,\alpha}(x)t^i u_0^j u_1^\alpha \in \mathcal{H}_{p,R}[t, u_0, u_1], \quad p \geq 2.$$

Then, we have $F(t, x, u_0, u_1) = a(x)t + \lambda(x)u_0 + \sum_{p \geq 2} R_p(t, x, u_0, u_1)$ and so

$$(3.1.4) \quad \begin{aligned} D^m[F](t, x, u_0, \dots, u_{m+1}) \\ = a^{(m)}(x)t + \sum_{0 \leq i \leq m} \lambda_{m,i}(x)u_i + \sum_{p \geq 2} D^m[R_p](t, x, u_0, \dots, u_{m+1}) \end{aligned}$$

for any $m \in \mathbb{N}$, where $a^{(m)}(x) = (\partial/\partial x)^m a(x)$.

Thus, by substituting (3.1.4) into the coupling equation (Φ) we see that our coupling equation (Φ) is written in the form

$$(3.1.5) \quad (\tau - \lambda(x))\phi = - \sum_{m \geq 0} \sum_{p \geq 2} D^m[R_p](t, x, u_0, \dots, u_{m+1}) \frac{\partial \phi}{\partial u_m}$$

where

$$\tau = t \frac{\partial}{\partial t} + \sum_{m \geq 0} \left(a^{(m)}(x)t + \sum_{0 \leq i \leq m} \lambda_{m,i}(x)u_i \right) \frac{\partial}{\partial u_m}$$

a vector field with infinitely many variables (t, u_0, u_1, \dots) .

Now, let us solve the equation (3.1.5). Let

$$(3.1.6) \quad \phi = \sum_{k \geq 1} \phi_k(t, x, u_0, \dots, u_{k-1}) \in \sum_{k \geq 1} \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$$

be the unknown function. Since $D^m[R_p](t, x, u_0, \dots, u_{m+1})$ belongs in the class $\mathcal{H}_{p,R}[t, u_0, \dots, u_{m+1}]$, by substituting (3.1.6) into (3.1.5) and by comparing the homogeneous part of degree k with respect to (t, u_0, \dots, u_{k-1}) we see that (3.1.5) is decomposed into the following recurrent formulas:

$$(3.1.7) \quad (\tau_1 - \lambda(x))\phi_1 = 0 \quad \text{in } \mathcal{H}_{1,R}[t, u_0]$$

and for $k \geq 2$

$$(3.1.8) \quad \begin{aligned} (\tau_k - \lambda(x))\phi_k \\ = - \sum_{1 \leq q \leq k-1} \sum_{0 \leq m \leq q-1} D^m[R_{k-q+1}](t, x, u_0, \dots, u_{m+1}) \frac{\partial \phi_q}{\partial u_m} \\ \text{in } \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}], \end{aligned}$$

where

$$\tau_k = t \frac{\partial}{\partial t} + \sum_{0 \leq m \leq k-1} \left(a^{(m)}(x)t + \sum_{0 \leq i \leq m} \lambda_{m,i}(x)u_i \right) \frac{\partial}{\partial u_m}, \quad k = 1, 2, \dots$$

Thus, to complete the proof of Proposition 3.1.1 it is enough to show the following lemma:

Lemma 3.1.3. (1) If $\lambda(x) \neq 1$ on D_R , the equation $(\tau_1 - \lambda(x))\phi_1 = 0$ has a solution $\phi_1 \in \mathcal{H}_{1,R}[t, u_0]$ of the form

$$\phi_1 = \frac{-a(x)\beta(x)}{1 - \lambda(x)}t + \beta(x)u_0$$

and $\beta(x) \in \mathcal{O}_R$ can be chosen arbitrarily.

(2) Let $k \geq 2$. If $|i + \lambda(x)(j - 1)| \neq 0$ on D_R for any $(i, j) \in \mathbb{N} \times \mathbb{N}$ with $i + j = k$, then for any $f_k \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$ the equation $(\tau_k - \lambda(x))\phi_k = f_k$ has a unique solution $\phi_k \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$.

Proof. The condition $\phi_1 \in \mathcal{H}_{1,R}[t, u_0]$ means that ϕ_1 is expressed in the form $\phi_1 = \alpha(x)t + \beta(x)u_0$; therefore by substituting this into $(\tau_1 - \lambda(x))\phi_1 = 0$ we can easily verify the result (1).

Let us show (2). Set

$$\begin{aligned} f_k &= \sum_{i+|j|=k} f_{i,j}(x)t^i u_0^{j_0} \cdots u_{k-1}^{j_{k-1}}, \\ \phi_k &= \sum_{i+|j|=k} \phi_{i,j}(x)t^i u_0^{j_0} \cdots u_{k-1}^{j_{k-1}}, \end{aligned}$$

where $j = (j_0, \dots, j_{k-1})$ and $|j| = j_0 + \cdots + j_{k-1}$. Then, by using the condition $\lambda_{m,m}(x) = \lambda(x)$ (for all $m = 0, 1, 2, \dots$) we see that the equation $(\tau_k - \lambda(x))\phi_k = f_k$ is equivalent to

(3.1.9)

$$\begin{aligned} & \sum_{i+|j|=k} (i + \lambda(x)(|j| - 1))\phi_{i,j}(x)t^i u_0^{j_0} \cdots u_{k-1}^{j_{k-1}} \\ &= \sum_{i+|j|=k} f_{i,j}(x)t^i u_0^{j_0} \cdots u_{k-1}^{j_{k-1}} \\ & \quad - \sum_{i+|j|=k} \phi_{i,j}(x) \left[\sum_{0 \leq m \leq k-1} a^{(m)}(x)j_m t^{i+1} u_0^{j_0} \cdots u_m^{j_m-1} \cdots u_{k-1}^{j_{k-1}} \right. \\ & \quad \left. + \sum_{1 \leq m \leq k-1} \sum_{0 \leq i \leq m-1} \lambda_{m,i}(x)j_m t^i u_0^{j_0} \cdots u_i^{j_i+1} \cdots u_m^{j_m-1} \cdots u_{k-1}^{j_{k-1}} \right]. \end{aligned}$$

In order to handle this equation, the following total order relation \prec in $A_k = \{(i, j_0, \dots, j_{k-1}) \in \mathbb{N} \times \mathbb{N}^k; i + |j| = k\}$ will be very convenient: for $(i, j) = (i, j_0, \dots, j_{k-1}) \in A_k$ and $(p, q) = (p, q_0, \dots, q_{k-1}) \in A_k$ we write $(i, j) \prec (p, q)$ if one of the following conditions is satisfied;

- 1) $j_{k-1} < q_{k-1}$,
- 2) $j_{k-1} = q_{k-1}$ and $j_{k-2} < q_{k-2}$,
- 3) $j_{k-1} = q_{k-1}$, $j_{k-2} = q_{k-2}$ and $j_{k-3} < q_{k-3}$,
- 4) and so on.

Then, by comparing the coefficients of $t^i u_0^{j_0} \dots u_{k-1}^{j_{k-1}}$ in the both sides of (3.1.9) we have

$$(i + \lambda(x)(|j| - 1))\phi_{i,j}(x) = f_{i,j}(x) + G_{i,j}(\phi_{p,q}; (p, q) \succ (i, j))$$

where $G_{i,j}(\phi_{p,q}; (p, q) \succ (i, j))$ is the term determined by $\{\phi_{p,q}; (p, q) \succ (i, j)\}$. Moreover we see that $G_{0,(0,\dots,0,k)} \equiv 0$ holds.

Thus, by the condition $(i + \lambda(x)(|j| - 1)) \neq 0$ on D_R we can determine $\phi_{i,j}(x) \in \mathcal{O}_R$ inductively on $(i, j) \in A_k$ from above with respect to the order \prec in A_k . This proves the result (2). \square

By using Lemma 3.1.3 we can solve the equations (3.1.7) and (3.1.8). This completes the proof of Proposition 3.1.1. \square

Thus, we have proved Theorem 3.1.2 which asserts that the equation (3.0.1) is equivalent to (3.0.2); but it is only in a formal sense. To apply this reduction to the study of (3.0.1) in the holomorphic category, we need to prove the convergence of this formal reduction which will be done in subsections 3.3 and 3.4.

§3.2. Formal solutions of (Ψ)

Next, let us consider the second coupling equation (Ψ) . We have

Proposition 3.2.1. *Let $R > 0$ be sufficiently small. Suppose the conditions $A_1), A_2), A_3)$ and that*

$$(3.2.1) \quad |i + \lambda(x)(j - 1)| \neq 0 \quad \text{on } D_R$$

$$\text{for any } (i, j) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0), (0, 1)\}.$$

Then, the coupling equation (Ψ) has a family of formal solutions of the form

$$(3.2.2) \quad \psi = \frac{a(x)}{1 - \lambda(x)} t + \beta(x)w_0 + \sum_{k \geq 2} \psi_k(t, x, w_0, \dots, w_{k-1})$$

$$\text{with } \psi_k(t, x, w_0, \dots, w_{k-1}) \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}] \quad (k = 2, 3, \dots),$$

where $a(x) = (\partial F / \partial t)(0, x, 0, 0)$ and $\beta(x) \in \mathcal{O}_R$. Here, $\beta(x)$ can be chosen arbitrarily, and the other $\psi_k(t, x, w_0, \dots, w_{k-1})$ ($k = 2, 3, \dots$) are uniquely determined by $\beta(x)$.

Proof. As in (3.1.3), we have the expression

$$F(t, x, w_0, w_1) = a(x)t + \lambda(x)w_0 + \sum_{i+j+\alpha \geq 2} c_{i,j,\alpha}(x)t^i w_0^j w_1^\alpha$$

where $a(x)$, $\lambda(x)$ and $c_{i,j,\alpha}(x)$ ($i+j+\alpha \geq 2$) are all holomorphic functions in a neighborhood of D_R . Therefore, the coupling equation (Ψ) is expressed in the form

$$(3.2.3) \quad (\tau^* - \lambda(x))\psi = a(x)t + \sum_{i+j+\alpha \geq 2} c_{i,j,\alpha}(x)t^i \psi^j (D[\psi])^\alpha$$

where

$$\tau^* = t \frac{\partial}{\partial t} + \sum_{m \geq 0} \left(\sum_{0 \leq i \leq m} \lambda_{m,i}(x) w_i \right) \frac{\partial}{\partial w_m}$$

a vector field with infinitely many variables (t, w_0, w_1, \dots) , and $\lambda_{m,i}(x) = (m!/i!(m-i)!)(\partial/\partial x)^{m-i} \lambda(x)$ ($0 \leq i \leq m$). As in section 3.1 we note that $\lambda_{m,m}(x) = \lambda(x)$ holds for all $m = 0, 1, 2, \dots$

Let

$$(3.2.4) \quad \psi = \sum_{k \geq 1} \psi_k(t, x, w_0, \dots, w_{k-1}) \in \sum_{k \geq 1} \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$$

be the unknown function. Then, by substituting (3.2.4) into (3.2.3) and by comparing the homogeneous parts of degree k with respect to (t, w_0, \dots, w_{k-1}) we see that (3.2.3) is decomposed into the following recurrent formulas:

$$(3.2.5) \quad (\tau_1^* - \lambda(x))\psi_1 = a(x)t \quad \text{in } \mathcal{H}_{1,R}[t, w_0]$$

and for $k \geq 2$

$$(3.2.6) \quad (\tau_k^* - \lambda(x))\psi_k = \sum_{2 \leq i+j+\alpha \leq k} c_{i,j,\alpha}(x) t^i \left[\sum_{|p(j)|+|q(\alpha)|=k-i} \psi_{p_1} \times \dots \times \psi_{p_j} \times D[\psi_{q_1}] \times \dots \times D[\psi_{q_\alpha}] \right]$$

$$\text{in } \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}],$$

where

$$\tau_k^* = t \frac{\partial}{\partial t} + \sum_{0 \leq m \leq k-1} \left(\sum_{0 \leq i \leq m} \lambda_{m,i}(x) w_i \right) \frac{\partial}{\partial w_m}, \quad k = 1, 2, \dots,$$

$|p(j)| = p_1 + \dots + p_j$ and $|q(\alpha)| = q_1 + \dots + q_\alpha$. Thus, to complete the proof of Proposition 3.2.1 it is enough to show the following lemma:

Lemma 3.2.2. (1) *If $\lambda(x) \neq 1$ on D_R , the equation $(\tau_1^* - \lambda(x))\psi_1 = a(x)t$ has a solution $\psi_1 \in \mathcal{H}_{1,R}[t, w_0]$ of the form*

$$\psi_1 = \frac{a(x)}{1 - \lambda(x)}t + \beta(x)w_0$$

and $\beta(x) \in \mathcal{O}_R$ can be chosen arbitrarily.

(2) *Let $k \geq 2$. If $|i + \lambda(x)(j - 1)| \neq 0$ on D_R for any $(i, j) \in \mathbb{N} \times \mathbb{N}$ with $i + j = k$, then for any $f_k \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$ the equation $(\tau_k^* - \lambda(x))\psi_k = f_k$ has a unique solution $\psi_k \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$.*

The proof of this lemma is almost the same as that of Lemma 3.1.3 and so we may omit the details. By using this lemma we can solve the equations (3.2.5) and (3.2.6). This completes the proof of Proposition 3.2.1. \square

Now, let us give a more precise form of the formal solution of (Ψ) . For $j = (j_0, j_1, \dots, j_{k-1}) \in \mathbb{N}^k$ we write $\langle j \rangle = j_1 + 2j_2 + \dots + (k - 1)j_{k-1}$. For any $k \in \mathbb{N}^*(= \{1, 2, 3, \dots\})$ we set

$$S_k = \{(k, (0, \dots, 0)) \in \mathbb{N} \times \mathbb{N}^k\} \cup \{(i, j) \in \mathbb{N} \times \mathbb{N}^k; i + |j| = k, |j| \geq 1, i + 2(|j| - 1) \geq \langle j \rangle\}.$$

We have

Proposition 3.2.3. *The term $\psi_k(t, x, w_0, \dots, w_{k-1})$ (for $k \geq 2$) in (3.2.2) has the form*

$$(3.2.7) \quad \psi_k = \sum_{(i,j) \in S_k} \psi_{i,j}(x)t^i w_0^{j_0} w_1^{j_1} \dots w_{k-1}^{j_{k-1}}.$$

Proof. For $k \in \mathbb{N}^*(= \{1, 2, \dots\})$ we denote by \mathcal{G}_k the set of all the functions $g_k(t, x, w_0, \dots, w_{k-1}) \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$ of the form

$$g_k = \sum_{(i,j) \in S_k} g_{i,j}(x)t^i w_0^{j_0} w_1^{j_1} \dots w_{k-1}^{j_{k-1}}.$$

Then we have the following properties:

- 1) $\psi_1 = (a(x)/(1 - \lambda(x))t + \beta(x)w_0) \in \mathcal{G}_1$,
- 2) $t^i \in \mathcal{G}_i$ for any $i \in \mathbb{N}^*$,
- 3) $t(\partial/\partial t)\mathcal{G}_k \subset \mathcal{G}_k$ for any $k \in \mathbb{N}^*$,
- 4) $w_i(\partial/\partial w_m)\mathcal{G}_k \subset \mathcal{G}_k$ for any $k \in \mathbb{N}^*$ and $i, m \in \mathbb{N}$ with $0 \leq i \leq m$,
- 5) $(\partial/\partial x)\mathcal{G}_k \subset \mathcal{G}_k$ for any $k \in \mathbb{N}^*$,

- 6) $\mathcal{G}_k \times \mathcal{G}_l \subset \mathcal{G}_{k+l}$ for any $k, l \in \mathbb{N}^*$,
- 7) $w_{m+1}(\partial/\partial w_m)\mathcal{G}_k \times \mathcal{G}_l \subset \mathcal{G}_{k+l}$ for any $m \in \mathbb{N}$ and $k, l \in \mathbb{N}^*$,
- 8) $w_{m+1}(\partial/\partial w_m)\mathcal{G}_k \times w_{p+1}(\partial/\partial w_p)\mathcal{G}_l \subset \mathcal{G}_{k+l}$ for any $m, p \in \mathbb{N}$ and $k, l \in \mathbb{N}^*$.

The conditions 7) and 8) imply that $D[\mathcal{G}_k] \times \mathcal{G}_l \subset \mathcal{G}_{k+l}$ and $D[\mathcal{G}_k] \times D[\mathcal{G}_l] \subset \mathcal{G}_{k+l}$ hold for any $k, l \in \mathbb{N}^*$. By combining these conditions with the construction of the formal solution we can obtain the result in Proposition 3.2.3. \square

§3.3. Convergence of the formal solution of (Ψ)

For $s > 0$ and a function $\varphi(x) \in \mathcal{O}_R$ we define the norm $\|\varphi\|_s$ of $\varphi(x)$ by

$$\|\varphi\|_s = \max_{|x| \leq s} |\varphi(x)|.$$

For $c > 0, 0 < s \leq R$ and

$$f_k = \sum_{i+|j|=k} f_{i,j}(x) t^i w_0^{j_0} w_1^{j_1} \dots w_{k-1}^{j_{k-1}} \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$$

we define the norm $\|f_k\|_{c,s}$ of f_k by

$$\|f_k\|_{c,s} = \sum_{i+|j|=k} \|f_{i,j}\|_s \left(\frac{0!}{c^0}\right)^{j_0} \left(\frac{1!}{c^1}\right)^{j_1} \left(\frac{2!}{c^2}\right)^{j_2} \dots \left(\frac{(k-1)!}{c^{k-1}}\right)^{j_{k-1}}.$$

It is easy to see that $\|f_k g_l\|_{c,s} \leq \|f_k\|_{c,s} \|g_l\|_{c,s}$ holds for any $f_k \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$ and $g_l \in \mathcal{H}_{l,R}[t, u_0, \dots, u_{l-1}]$. We denote by $\mathbb{C}\{z\}$ the ring of convergent power series in z with complex coefficients.

Let

$$(3.3.1) \quad \psi = \sum_{k \geq 1} \psi_k(t, x, w_0, \dots, w_{k-1}) \in \sum_{k \geq 1} \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$$

be the formal solution constructed in Proposition 3.2.1. In this section we will show that this formal solution is convergent in the following sense:

Theorem 3.3.1. *Suppose the conditions $A_1), A_2), A_3)$ and that*

$$(3.3.2) \quad |i + \lambda(x)(j - 1)| \geq \sigma_0(i + j) \quad \text{on } D_R$$

for any $(i, j) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0), (0, 1)\}$

for some $\sigma_0 > 0$. Then we can find a constant $c_0 > 0$ such that the formal solution (3.3.1) satisfies

$$(3.3.3) \quad \sum_{k \geq 1} \|\psi_k\|_{c,s} z^k \in \mathbb{C}\{z\} \quad \text{for any } 0 < c \leq c_0 \text{ and } 0 < s < R.$$

Hence, we may say that *the formal solution* $\psi(t, x, w_0, w_1, \dots)$ *is convergent* if

$$\sum_{k \geq 1} \|\psi_k\|_{c,s} z^k \in \mathbb{C}\{z\}$$

holds for some $c > 0$ and $s > 0$.

The rest part of this section will be used to prove Theorem 3.3.1. First we note:

Lemma 3.3.2. *Suppose that the condition (3.3.2) holds for some $\sigma_0 > 0$. Then we can find a constant $c_0 > 0$ which satisfies the following: if $k \geq 2$, $0 < c \leq c_0$ and $0 < s \leq R$, the unique solution $\psi_k \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$ of the equation $(\tau_k^* - \lambda(x))\psi_k = f_k \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$ satisfies*

$$(3.3.4) \quad \sigma k \|\psi_k\|_{c,s} \leq \|f_k\|_{c,s}$$

where $\sigma = \sigma_0/2$.

Proof. We recall that $\lambda(x)$ is a holomorphic function in a neighborhood of D_R and so we have

$$(3.3.5) \quad \|\lambda_{m,p}\|_R = \frac{m!}{p!(m-p)!} \left\| \left(\frac{\partial}{\partial x} \right)^{m-p} \lambda \right\|_R \leq \frac{m!}{p!} \frac{\Lambda}{\eta^{m-p}}$$

for some $\Lambda > 0$ and $\eta > 0$ which are independent of m and p . We set $c_0 = \sigma_0\eta/(2\Lambda + \sigma_0)$; then we see that the condition $0 < c \leq c_0$ implies $0 < c < \eta$ and $\Lambda(c/\eta)/(1 - (c/\eta)) = \Lambda c/(\eta - c) \leq \sigma_0/2$.

Let $\psi_k \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$ and $f_k \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$ satisfy the equation $(\tau_k^* - \lambda(x))\psi_k = f_k$. Set

$$\begin{aligned} \psi_k &= \sum_{i+|j|=k} \psi_{i,j}(x) t^i w_0^{j_0} \dots w_{k-1}^{j_{k-1}}, \\ f_k &= \sum_{i+|j|=k} f_{i,j}(x) t^i w_0^{j_0} \dots w_{k-1}^{j_{k-1}}. \end{aligned}$$

Then the equation $(\tau_k^* - \lambda(x))\psi_k = f_k$ is equivalent to

$$\begin{aligned} &\sum_{i+|j|=k} (i + \lambda(x)(|j| - 1)) \psi_{i,j}(x) t^i w_0^{j_0} \dots w_{k-1}^{j_{k-1}} \\ &= \sum_{i+|j|=k} f_{i,j}(x) t^i w_0^{j_0} \dots w_{k-1}^{j_{k-1}} \\ &\quad - \sum_{i+|j|=k} \sum_{1 \leq m \leq k-1} \sum_{0 \leq p \leq m-1} \lambda_{m,p}(x) \psi_{i,j}(x) \times \\ &\quad \quad \times j_m t^i w_0^{j_0} \dots w_p^{j_p+1} \dots w_m^{j_m-1} \dots w_{k-1}^{j_{k-1}}. \end{aligned}$$

Therefore, by using the condition (3.3.2) and (3.3.5) we have

(3.3.6)

$$\begin{aligned}
& \sigma_0 k \|\psi_k\|_{c,s} \\
& \leq \|f_k\|_{c,s} + \sum_{i+|j|=k} \sum_{1 \leq m \leq k-1} \sum_{0 \leq p \leq m-1} \frac{m!}{p!} \frac{\Lambda}{\eta^{m-p}} \|\psi_{i,j}\|_s \times \\
& \quad \times j_m \left(\frac{0!}{c^0}\right)^{j_0} \cdots \left(\frac{p!}{c^p}\right)^{j_p+1} \cdots \left(\frac{m!}{c^m}\right)^{j_{m-1}} \cdots \left(\frac{(k-1)!}{c^{k-1}}\right)^{j_{k-1}} \\
& = \|f_k\|_{c,s} + \sum_{i+|j|=k} \sum_{1 \leq m \leq k-1} \sum_{0 \leq p \leq m-1} \frac{\Lambda}{\eta^{m-p}} c^{m-p} \|\psi_{i,j}\|_s \times \\
& \quad \times j_m \left(\frac{0!}{c^0}\right)^{j_0} \cdots \left(\frac{p!}{c^p}\right)^{j_p} \cdots \left(\frac{m!}{c^m}\right)^{j_m} \cdots \left(\frac{(k-1)!}{c^{k-1}}\right)^{j_{k-1}}.
\end{aligned}$$

Here, we note: if $0 < c \leq c_0$ we have

$$\sum_{0 \leq p \leq m-1} \frac{\Lambda}{\eta^{m-p}} c^{m-p} \leq \sum_{q \geq 1} \Lambda(c/\eta)^q = \Lambda(c/\eta)/(1 - (c/\eta)) \leq \sigma_0/2;$$

by applying this to (3.3.6) and by using $|j| \leq k$ we obtain

$$\begin{aligned}
& \sigma_0 k \|\psi_k\|_{c,s} \\
& \leq \|f_k\|_{c,s} + \frac{\sigma_0}{2} \sum_{i+|j|=k} \|\psi_{i,j}\|_s \left[\sum_{1 \leq m \leq k-1} j_m \right] \left(\frac{0!}{c^0}\right)^{j_0} \left(\frac{1!}{c^1}\right)^{j_1} \cdots \left(\frac{(k-1)!}{c^{k-1}}\right)^{j_{k-1}} \\
& \leq \|f_k\|_{c,s} + \frac{\sigma_0}{2} k \sum_{i+|j|=k} \|\psi_{i,j}\|_s \left(\frac{0!}{c^0}\right)^{j_0} \left(\frac{1!}{c^1}\right)^{j_1} \cdots \left(\frac{(k-1)!}{c^{k-1}}\right)^{j_{k-1}} \\
& = \|f_k\|_{c,s} + \frac{\sigma_0}{2} k \|\psi_k\|_{c,s}.
\end{aligned}$$

This leads us to the estimate (3.3.4). □

The following result is a modification of Nagumo's lemma.

Lemma 3.3.3. *Let $c > 0$ be fixed. If $\psi_k \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$ (with $k \geq 2$) has the form (3.2.7) and if*

$$(3.3.7) \quad \|\psi_k\|_{c,s} \leq \frac{C}{(R-s)^a} \quad \text{for any } 0 < s < R$$

for some $C > 0$ and $a \geq 0$, then we have

$$(3.3.8) \quad \|D[\psi_k]\|_{c,s} \leq \frac{(a+1)eC}{(R-s)^{a+1}} + \frac{3k}{c} \frac{C}{(R-s)^a} \quad \text{for any } 0 < s < R.$$

Proof. Set

$$\psi_k = \sum_{(i,j) \in S_k} \psi_{i,j}(x) t^i w_0^{j_0} w_1^{j_1} \cdots w_{k-1}^{j_{k-1}};$$

we have

$$\begin{aligned} D[\psi_k] &= \sum_{(i,j) \in S_k} \frac{\partial \psi_{i,j}}{\partial x}(x) t^i w_0^{j_0} w_1^{j_1} \cdots w_{k-1}^{j_{k-1}} \\ &+ \sum_{\substack{(i,j) \in S_k \\ |j| \geq 1}} \left[\sum_{0 \leq m \leq k-2} \psi_{i,j}(x) j_m t^i w_0^{j_0} \cdots w_m^{j_m-1} w_{m+1}^{j_{m+1}+1} \cdots w_{k-1}^{j_{k-1}} \right. \\ &\quad \left. + \psi_{i,j}(x) j_{k-1} t^i w_0^{j_0} w_1^{j_1} \cdots w_{k-2}^{j_{k-2}} w_{k-1}^{j_{k-1}-1} w_k \right]. \end{aligned}$$

Therefore, for any $0 < s < R$ and any $0 < h < R - s$ we have

$$\begin{aligned} (3.3.9) \quad & \|D[\psi_k]\|_{c,s} \\ & \leq \sum_{(i,j) \in S_k} \left\| \frac{\partial \psi_{i,j}}{\partial x} \right\|_s \left(\frac{0!}{c^0} \right)^{j_0} \left(\frac{1!}{c^1} \right)^{j_1} \cdots \left(\frac{(k-1)!}{c^{k-1}} \right)^{j_{k-1}} \\ & + \sum_{\substack{(i,j) \in S_k \\ |j| \geq 1}} \left[\sum_{0 \leq m \leq k-2} \|\psi_{i,j}\|_s j_m \left(\frac{0!}{c^0} \right)^{j_0} \cdots \left(\frac{m!}{c^m} \right)^{j_m-1} \times \right. \\ & \quad \left. \times \left(\frac{(m+1)!}{c^{m+1}} \right)^{j_{m+1}+1} \cdots \left(\frac{(k-1)!}{c^{k-1}} \right)^{j_{k-1}} \right. \\ & \quad \left. + \|\psi_{i,j}\|_s j_{k-1} \left(\frac{0!}{c^0} \right)^{j_0} \cdots \left(\frac{(k-2)!}{c^{k-2}} \right)^{j_{k-2}} \left(\frac{(k-1)!}{c^{k-1}} \right)^{j_{k-1}-1} \left(\frac{k!}{c^k} \right) \right] \\ & \leq \sum_{(i,j) \in S_k} \frac{1}{h} \|\psi_{i,j}\|_{s+h} \left(\frac{0!}{c^0} \right)^{j_0} \left(\frac{1!}{c^1} \right)^{j_1} \cdots \left(\frac{(k-1)!}{c^{k-1}} \right)^{j_{k-1}} \\ & \quad + \sum_{\substack{(i,j) \in S_k \\ |j| \geq 1}} \sum_{0 \leq m \leq k-1} \|\psi_{i,j}\|_s j_m \frac{(m+1)}{c} \left(\frac{0!}{c^0} \right)^{j_0} \cdots \left(\frac{m!}{c^m} \right)^{j_m} \times \\ & \quad \times \left(\frac{(m+1)!}{c^{m+1}} \right)^{j_{m+1}} \cdots \left(\frac{(k-1)!}{c^{k-1}} \right)^{j_{k-1}}. \end{aligned}$$

Since $(i, j) \in S_k$ and $|j| \geq 1$ imply $i + 2(|j| - 1) \geq \langle j \rangle$, we have

$$\sum_{0 \leq m \leq k-1} j_m(m+1) = \langle j \rangle + |j| \leq i + 2(|j| - 1) + |j| \leq 3(i + |j|) = 3k$$

and so by (3.3.7) and (3.3.9) we see

$$\begin{aligned} \|D[\psi_k]\|_{c,s} &\leq \sum_{(i,j) \in S_k} \frac{1}{h} \|\psi_{i,j}\|_{s+h} \left(\frac{0!}{c^0}\right)^{j_0} \left(\frac{1!}{c^1}\right)^{j_1} \cdots \left(\frac{(k-1)!}{c^{k-1}}\right)^{j_{k-1}} \\ &\quad + \frac{3k}{c} \sum_{\substack{(i,j) \in S_k \\ |j| \geq 1}} \|\psi_{i,j}\|_s \left(\frac{0!}{c^0}\right)^{j_0} \left(\frac{1!}{c^1}\right)^{j_1} \cdots \left(\frac{(k-1)!}{c^{k-1}}\right)^{j_{k-1}} \\ &\leq \frac{1}{h} \|\psi_k\|_{c,s+h} + \frac{3k}{c} \|\psi_k\|_{c,s} \\ &\leq \frac{1}{h} \frac{C}{(R-s-h)^a} + \frac{3k}{c} \frac{C}{(R-s)^a}. \end{aligned}$$

In the case $a = 0$, by letting $h \rightarrow R - s$ we have the result

$$\|D[\psi_k]\|_{c,s} \leq \frac{C}{(R-s)} + \frac{3k}{c} C,$$

which proves (3.3.8) with $a = 0$. In the case $a > 0$, by taking $h = (R-s)/(a+1)$ we obtain

$$\begin{aligned} \|D[\psi_k]\|_{c,s} &\leq \frac{(a+1)C}{(R-s)^{a+1}} \frac{1}{(1-1/(a+1))^a} + \frac{3k}{c} \frac{C}{(R-s)^a} \\ &= \frac{(a+1)C}{(R-s)^{a+1}} \left(1 + \frac{1}{a}\right)^a + \frac{3k}{c} \frac{C}{(R-s)^a} \\ &\leq \frac{(a+1)eC}{(R-s)^{a+1}} + \frac{3k}{c} \frac{C}{(R-s)^a}, \end{aligned}$$

which proves (3.3.8). □

Now, let us prove Theorem 3.3.1. Recall that our coupling equation (Ψ) is written in the form

$$(\tau^* - \lambda(x))\psi = a(x)t + \sum_{i+j+\alpha \geq 2} c_{i,j,\alpha}(x)t^i \psi^j (D[\psi])^\alpha.$$

We set $\|c_{i,j,\alpha}\|_R = A_{i,j,\alpha}$ (for $i + j + \alpha \geq 2$); then we see that the series

$$\sum_{i+j+\alpha \geq 2} A_{i,j,\alpha} t^i Y^{j+\alpha}$$

is convergent in a neighborhood of $(t, Y) = (0, 0) \in \mathbb{C} \times \mathbb{C}$. For simplicity we may suppose that $0 < R \leq 1$ holds.

Let

$$(3.3.10) \quad \psi = \sum_{k \geq 1} \psi_k(t, x, w_0, \dots, w_{k-1}) \in \sum_{k \geq 1} \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$$

be the formal solution constructed in Proposition 3.2.1. We know that ψ_k (for $k \geq 2$) are determined by the recurrent formulas:

$$(3.3.11) \quad (\tau_k^* - \lambda(x))\psi_k = \sum_{2 \leq i+j+\alpha \leq k} c_{i,j,\alpha}(x) t^i \left[\sum_{|p(j)|+|q(\alpha)|=k-i} \psi_{p_1} \times \dots \times \psi_{p_j} \times D[\psi_{q_1}] \times \dots \times D[\psi_{q_\alpha}] \right].$$

Let $c_0 > 0$ be the constant in Lemma 3.3.2, and let $0 < c \leq c_0$ be fixed. Take a positive constant $A > 0$ so that

$$(3.3.12) \quad \|\psi_1\|_{c,R} \leq A \quad \text{and} \quad \|D[\psi_1]\|_{c,R} \leq (e + 3/c)A$$

hold. As a majorant equation of the formal solution (3.3.10), we consider the following functional equation with respect to Y :

$$(3.3.13) \quad Y = Az + \frac{1}{\sigma} \frac{1}{(R-s)} \sum_{i+j+\alpha \geq 2} \frac{A_{i,j,\alpha}}{(R-s)^{i+j+\alpha-2}} z^i Y^j ((e + 3/c)Y)^\alpha$$

where $\sigma = \sigma_o/2$, and s is a parameter with $0 < s < R$. It is easy to see that (3.3.13) is an analytic functional equation with respect to (z, Y) . The implicit function theorem tells us that for any $0 < s < R$ this equation (3.3.13) has a unique holomorphic solution $Y = Y(z)$ of the form

$$Y = \sum_{k \geq 1} Y_k z^k.$$

Moreover, Y_k ($k = 1, 2, \dots$) are determined by the following recurrent formulas:

$$Y_1 = A$$

and for $k \geq 2$

$$(3.3.14) \quad Y_k = \frac{1}{\sigma} \frac{1}{(R-s)} \sum_{2 \leq i+j+\alpha \leq k} \frac{A_{i,j,\alpha}}{(R-s)^{i+j+\alpha-2}} \left[\sum_{|p(j)|+|q(\alpha)|=k-i} Y_{p_1} \times \dots \times Y_{p_j} \times ((e + 3/c)Y_{q_1}) \dots ((e + 3/c)Y_{q_\alpha}) \right].$$

By induction we can see also that Y_k has the form

$$(3.3.15) \quad Y_k = \frac{C_k}{(R-s)^{k-1}}, \quad k = 1, 2, \dots$$

for some constants $C_1 = A$ and $C_k \geq 0$ ($k \geq 2$) which are independent of the parameter s . We often write $Y_k = Y_k(s)$; this emphasizes that Y_k depends on the parameter s .

The following lemma guarantees that $Y(z)$ is a majorant series of the formal solution (3.3.10).

Lemma 3.3.4. *Let $\psi = \sum_{k \geq 1} \psi_k$ be the formal solution (3.3.10). Then, for any $k = 1, 2, \dots$ we have*

$$(3.3.16)_k \quad \|\psi_k\|_{c,s} \leq Y_k(s) \quad \text{for any } 0 < s < R,$$

$$(3.3.17)_k \quad \|D[\psi_k]\|_{c,s} \leq (e + 3/c)Y_k(s) \quad \text{for any } 0 < s < R.$$

Proof. Since $Y_1 = A$, the case $k = 1$ is clear from (3.3.12). The general case is proved by induction on k .

Let $k \geq 2$ and suppose that (3.3.16) $_l$ and (3.3.17) $_l$ are already proved for $l = 1, 2, \dots, k-1$. Then, by applying Lemma 3.3.2 to (3.3.11) and by using the induction hypothesis we have

$$\begin{aligned} \|\psi_k\|_{c,s} &\leq \frac{1}{\sigma k} \sum_{2 \leq i+j+\alpha \leq k} A_{i,j,\alpha} \left[\sum_{|p(j)|+|q(\alpha)|=k-i} \|\psi_{p_1}\|_{c,s} \times \right. \\ &\quad \left. \times \cdots \times \|\psi_{p_j}\|_{c,s} \times \|D[\psi_{q_1}]\|_{c,s} \cdots \|D[\psi_{q_\alpha}]\|_{c,s} \right] \\ &\leq \frac{1}{\sigma k} \sum_{2 \leq i+j+\alpha \leq k} A_{i,j,\alpha} \left[\sum_{|p(j)|+|q(\alpha)|=k-i} Y_{p_1}(s) \times \right. \\ &\quad \left. \times \cdots \times Y_{p_j}(s) \times (e + 3/c)Y_{q_1}(s) \cdots (e + 3/c)Y_{q_\alpha}(s) \right]. \end{aligned}$$

Hence, comparing this with (3.3.14) we obtain

$$(3.3.18) \quad \|\psi_k\|_{c,s} \leq \frac{(R-s)}{k} Y_k(s) = \frac{1}{k} \frac{C_k}{(R-s)^{k-2}}$$

for any $0 < s < R$. Since $1/k \leq 1$ and $1/(R-s) \geq 1$ hold, by (3.3.18) we have

$$\|\psi_k\|_{c,s} \leq \frac{C_k}{(R-s)^{k-1}} = Y_k(s) \quad \text{for any } 0 < s < R$$

which proves (3.3.16)_k. Moreover, by applying Lemma 3.3.3 to (3.3.18) we have

$$\begin{aligned} \|D[\psi_k]\|_{c,s} &\leq \frac{(k-1)e}{k} \frac{C_k}{(R-s)^{k-1}} + \frac{3k}{ck} \frac{C_k}{(R-s)^{k-2}} \\ &\leq e \frac{C_k}{(R-s)^{k-1}} + \frac{3}{c} \frac{C_k}{(R-s)^{k-1}} \\ &= (e + 3/c)Y_k(s) \quad \text{for any } 0 < s < R \end{aligned}$$

which proves (3.3.17)_k.

Thus, by induction on k we have proved Lemma 3.3.4. □

Let us return to the proof of Theorem 3.3.1. By Lemma 3.3.4 we have

$$\sum_{k \geq 1} \|\psi_k\|_{c,s} z^k \ll \sum_{k \geq 1} Y_k(s) z^k.$$

Since the right-hand side is convergent in a neighborhood of $z = 0$, this completes the proof of Theorem 3.3.1. □

Let us give a variant. Let

$$(3.3.19) \quad \tau^0 = t \frac{\partial}{\partial t} + \sum_{m \geq 0} w_m \frac{\partial}{\partial w_m}$$

and let $G(t, x, u_0, u_1)$ be a holomorphic function defined in a neighborhood of the origin $(t, x, u_0, u_1) = (0, 0, 0, 0) \in \mathbb{C}^4$ whose Taylor expansion in (t, u_0, u_1) has the form

$$(3.3.20) \quad G(t, x, u_0, u_1) = \sum_{i+j+\alpha \geq 2} \frac{A_{i,j,\alpha}}{(R-x)} t^i u_0^j u_1^\alpha$$

with $A_{i,j,\alpha} \geq 0$ ($i + j + \alpha \geq 2$) and $0 < R \leq 1$. Let us consider the following equation

$$(3.3.21) \quad (\tau^0 - 1)\psi = G(t, x, \psi, D[\psi]).$$

For $R > 0$ we write

$$\mathcal{H}_{k,(R)}[t, w_0, \dots, w_{k-1}] = \bigcap_{0 < \eta < R} \mathcal{H}_{k,\eta}[t, w_0, \dots, w_{k-1}].$$

We have

Theorem 3.3.5. *The equation (3.3.21) has a unique formal solution of the form $\psi = w_0 + \sum_{k \geq 2} \psi_k(t, x, w_0, \dots, w_{k-1}) \in \sum_{k \geq 1} \mathcal{H}_{k, (R)}[t, w_0, \dots, w_{k-1}]$. Moreover we have*

$$\sum_{k \geq 1} \|\psi_k\|_{c,s} z^k \in \mathbb{C}\{z\} \quad \text{for any } c > 0 \text{ and } 0 < s < R$$

where $\psi_1 = w_0$.

Proof. Set

$$(3.3.22) \quad \psi = \sum_{k \geq 1} \psi_k(t, x, w_0, \dots, w_{k-1}) \in \sum_{k \geq 1} \mathcal{H}_{k, (R)}[t, w_0, \dots, w_{k-1}].$$

It is easy to see that under the condition $\psi_1 = w_0$ the equation (3.3.21) has a unique formal solution of the form (3.3.22), and that $\psi_k(t, x, w_0, \dots, w_{k-1})$ (for $k \geq 2$) are determined by the following recurrent formulas:

$$\psi_k = \frac{1}{k-1} \sum_{2 \leq i+j+\alpha \leq k} \frac{A_{i,j,\alpha}}{(R-x)} t^i \sum_{|p(j)|+|q(\alpha)|=k-i} \psi_{p_1} \cdots \psi_{p_j} \times D[\psi_{q_1}] \cdots D[\psi_{q_\alpha}].$$

Therefore, for any $c > 0$ and any $0 < s < R$ we have

$$(3.3.23) \quad \|\psi_k\|_{c,s} \leq \frac{1}{k-1} \sum_{2 \leq i+j+\alpha \leq k} \frac{A_{i,j,\alpha}}{(R-s)} \left[\sum_{|p(j)|+|q(\alpha)|=k-i} \|\psi_{p_1}\|_{c,s} \times \right. \\ \left. \times \cdots \times \|\psi_{p_j}\|_{c,s} \times \|D[\psi_{q_1}]\|_{c,s} \cdots \|D[\psi_{q_\alpha}]\|_{c,s} \right].$$

Let us consider now the functional equation with respect to Y :

$$(3.3.24) \quad Y = z + \frac{1}{(R-s)} \sum_{i+j+\alpha \geq 2} \frac{A_{i,j,\alpha}}{(R-s)^{1+2(i+j+\alpha-2)}} z^i Y^j ((2e + 6/c)Y)^\alpha$$

where s is a parameter with $0 < s < R$. It is easy to see that (3.3.24) is an analytic functional equation with respect to (z, Y) . The implicit function theorem tells us that for any $0 < s < R$ this equation (3.3.24) has a unique holomorphic solution $Y = Y(z)$ of the form

$$Y = \sum_{k \geq 1} Y_k z^k.$$

Moreover, Y_k ($k = 1, 2, \dots$) are determined by the following recurrent formulas:

$$Y_1 = 1$$

and for $k \geq 2$

$$(3.3.25) \quad Y_k = \frac{1}{(R-s)} \sum_{2 \leq i+j+\alpha \leq k} \frac{A_{i,j,\alpha}}{(R-s)^{1+2(i+j+\alpha-2)}} \left[\sum_{|p(j)|+|q(\alpha)|=k-i} Y_{p_1} \times \dots \times Y_{p_j} \times ((2e+6/c)Y_{q_1}) \dots ((2e+6/c)Y_{q_\alpha}) \right].$$

By induction we can see also that Y_k has the form

$$Y_k = \frac{C_k}{(R-s)^{2(k-1)}}, \quad k = 1, 2, \dots$$

for some constants $C_1 = 1$ and $C_k \geq 0$ ($k \geq 2$) which are independent of the parameter s . We write $Y_k = Y_k(s)$ when we emphasize the fact that Y_k depends on the parameter s .

Since $\psi_1 = w_0$ we have $D[\psi_1] = w_1$; therefore $\|\psi_1\|_{c,s} = 1 = Y_1$ and $\|D[\psi_1]\|_{c,s} = 1/c \leq (2e+6/c) = (2e+6/c)Y_1$ hold for any $0 < s < R$. Moreover, for $k = 2, 3, \dots$ we can prove the estimates

$$\begin{aligned} \|\psi_k\|_{c,s} &\leq Y_k(s) \quad \text{for any } 0 < s < R, \\ \|D[\psi_k]\|_{c,s} &\leq (2e+6/c)Y_k(s) \quad \text{for any } 0 < s < R \end{aligned}$$

by induction on k in the same way as in the proof of Lemma 3.3.4. The factor $(2e+6/c)$ comes from the estimate $((2k-2)e+3k/c)/(k-1) = 2e + (k/(k-1))3/c \leq 2e+6/c$ for any $k \geq 2$.

Thus, we have

$$\sum_{k \geq 1} \|\psi_k\|_{c,s} z^k \ll \sum_{k \geq 1} Y_k(s) z^k.$$

Since the right-hand side is convergent in a neighborhood of $z = 0$, this completes the proof of Theorem 3.3.5. □

§3.4. Convergence of the formal solution of (Φ)

For $r > 0, c > 0, 0 < s \leq R$ and

$$f_k = \sum_{i+|j|=k} f_{i,j}(x) t^i u_0^{j_0} u_1^{j_1} \dots u_{k-1}^{j_{k-1}} \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$$

we define the norm $\|f_k\|_{r,c,s}$ of f_k by

$$\|f_k\|_{r,c,s} = \sum_{i+|j|=k} \|f_{i,j}\|_s r^i \left(\frac{0!}{c^0}\right)^{j_0} \left(\frac{1!}{c^1}\right)^{j_1} \left(\frac{2!}{c^2}\right)^{j_2} \dots \left(\frac{(k-1)!}{c^{k-1}}\right)^{j_{k-1}}.$$

In Proposition 3.1.1 we have shown that under the assumption (3.1.1) the coupling equation (Φ) has a family of formal solutions of the form

$$(3.4.1) \quad \phi = -\alpha(x)\beta(x)t + \beta(x)u_0 + \sum_{k \geq 2} \phi_k(t, x, u_0, \dots, u_{k-1})$$

with $\alpha(x) = a(x)/(1 - \lambda(x))$, $\beta(x) \in \mathcal{O}_R$ is arbitrary, and $\phi_k(t, x, u_0, \dots, u_{k-1}) \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$ ($k = 2, 3, \dots$): we know that the terms ϕ_k ($k \geq 2$) are determined by the recurrent formulas

$$(3.4.2) \quad \begin{aligned} & (\tau_k - \lambda(x))\phi_k \\ &= - \sum_{1 \leq q \leq k-1} \sum_{0 \leq m \leq q-1} D^m [R_{k-q+1}](t, x, u_0, \dots, u_{m+1}) \frac{\partial \phi_q}{\partial u_m} \end{aligned}$$

where

$$\tau_k = t \frac{\partial}{\partial t} + \sum_{0 \leq m \leq k-1} \left(a^{(m)}(x)t + \sum_{0 \leq i \leq m} \lambda_{m,i}(x)u_i \right) \frac{\partial}{\partial u_m}, \quad k \geq 2.$$

In this section we will prove the convergence of this formal solution (3.4.1) in the following sense:

Theorem 3.4.1. *Suppose the conditions $A_1), A_2), A_3)$ and that*

$$(3.4.3) \quad \begin{aligned} & |i + \lambda(x)(j-1)| \geq \sigma_0(i+j) \quad \text{on } D_R \\ & \text{for any } (i, j) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0), (0, 1)\} \end{aligned}$$

for some $\sigma_0 > 0$. Then we can find constants $r_0 > 0$ and $0 < c_0 < R$ such that the formal solution (3.4.1) satisfies

$$\sum_{k \geq 1} \|\phi_k\|_{r,c,s} z^k \in \mathbb{C}\{z\}$$

(with $\phi_1 = -\alpha(x)\beta(x)t + \beta(x)u_0$) for any $0 < r \leq r_0$, $0 < c \leq c_0$ and $0 < s < R$ satisfying $c + s = R$.

The rest part of this section is used to prove this result. In the proof, we may suppose:

$$\begin{aligned} \|a^{(m)}\|_R &\leq \frac{m!A}{\eta^m}, \quad m = 0, 1, 2, \dots, \\ \|\lambda_{m,p}\|_R &\leq \frac{m!}{p!} \frac{\Lambda}{\eta^{m-p}}, \quad m = 0, 1, 2, \dots \quad \text{and } 0 \leq p \leq m \end{aligned}$$

for some $A > 0$, $\eta > 0$ and $\Lambda > 0$ which are independent of m and p . As is proved later, it is sufficient to take $r_0 > 0$ and $0 < c_0 < R$ so that

$$(3.4.4) \quad 0 < c_0 < \eta \text{ and } r_0 A + (\Lambda c_0 / (\eta - c_0)) \leq \sigma_0 / 2$$

hold: we fix such $r_0 > 0$ and $0 < c_0 < R$ from now.

3.4.1. A reduction

First, let us recall that $R_p(t, x, u_0, u_1)$ in the right-hand side of (3.4.2) is expressed in the form

$$R_p(t, x, u_0, u_1) = \sum_{i+j+\alpha=p} c_{i,j,\alpha}(x) t^i u_0^j u_1^\alpha, \quad p \geq 2,$$

where $c_{i,j,\alpha}(x)$ ($i + j + \alpha \geq 2$) are holomorphic functions in a neighborhood of D_R . Without loss of generality we may suppose: $0 < R \leq 1$. We set $\|c_{i,j,\alpha}\|_R = A_{i,j,\alpha}$; then we have $c_{i,j,\alpha}(x) \ll A_{i,j,\alpha} / (1 - x/R) \ll A_{i,j,\alpha} / (R - x)$. We set

$$(3.4.5) \quad G(t, x, w_0, w_1) = \sum_{i+j+\alpha \geq 2} \frac{A_{i,j,\alpha}}{(R-x)} t^i w_0^j w_1^\alpha :$$

we have $R_p(t, x, w_0, w_1) \ll G(t, x, w_0, w_1)$ for all $p \geq 2$ as power series with respect to (t, x, w_0, w_1) . As in section 3.3 we use the notation

$$\mathcal{H}_{k,(R)}[t, w_0, \dots, w_{k-1}] = \bigcap_{0 < \eta < R} \mathcal{H}_{k,\eta}[t, w_0, \dots, w_{k-1}].$$

Under these notations, let us consider the following equation

$$(3.4.6) \quad (\tau^0 - 1)\psi^+ = G(t, x, \psi^+, D[\psi^+]),$$

where τ^0 is the vector field in (3.3.19), that is,

$$\tau^0 = t \frac{\partial}{\partial t} + \sum_{m \geq 0} w_m \frac{\partial}{\partial w_m}.$$

It is easy to see that this equation (3.4.6) has a family of formal solutions of the form

$$(3.4.7) \quad \psi^+ = \sum_{k \geq 1} \psi_k^+(t, x, w_0, \dots, w_{k-1}) \in \sum_{k \geq 1} \mathcal{H}_{k,(R)}[t, w_0, \dots, w_{k-1}] :$$

moreover we see that $\psi_1^+ \in \mathcal{H}_{1,(R)}[t, w_0]$ can be chosen arbitrarily and the other terms ψ_k^+ ($k \geq 2$) are determined by the recurrent formulas:

$$(3.4.8) \quad \psi_k^+ = \frac{1}{k-1} \sum_{2 \leq i+j+\alpha \leq k} \frac{A_{i,j,\alpha}}{(R-x)} t^i \left[\sum_{|p(j)|+|q(\alpha)|=k-i} \psi_{p_1}^+ \times \dots \times \psi_{p_j}^+ \times D[\psi_{q_1}^+] \times \dots \times D[\psi_{q_\alpha}^+] \right].$$

We take $\psi_1^+ = w_0$ from now. Then, on the convergence of this formal solution (3.4.7) (with $\psi_1^+ = w_0$), by Theorem 3.3.5 we have

Proposition 3.4.2. *For any $r > 0$, $c > 0$ and $0 < s < R$ we have*

$$\sum_{k \geq 1} \|\psi_k^+\|_{r,c,s} z^k \in \mathbb{C}\{z\}.$$

By combining this with the Stirling's formula we can see also

$$\sum_{k \geq 1} \frac{(k-1)^{k-1}}{(k-1)!} \|\psi_k^+\|_{r,c,s} z^k \in \mathbb{C}\{z\},$$

where in the case $k = 1$ the notation $(k-1)^{k-1}$ should be read as $0^0 = 1$. Thus, to prove Theorem 3.4.1 it is sufficient to show the following inequality (3.4.9) and (3.4.10):

Proposition 3.4.3. *Let $\phi = \sum_{k \geq 1} \phi_k$ be the formal solution (3.4.1) with $\phi_1 = -\alpha(x)\beta(x)t + \beta(x)u_0$, and let $\psi^+ = \sum_{k \geq 1} \psi_k^+$ be the formal solution of (3.4.6) with $\psi_1^+ = w_0$. Then, we have*

$$(3.4.9) \quad \|\phi_1\|_{r,c,s} \leq \|\beta\|_s (\|\alpha\|_s r + \|\psi_1^+\|_{r,c,s}),$$

$$(3.4.10) \quad \|\phi_k\|_{r,c,s} \leq \frac{(k-1)^{k-1}}{\sigma^{k-1}(k-1)!} \|\beta\|_s \|\psi_k^+\|_{r,c,s} \quad \text{for } k \geq 2$$

for any $0 < r \leq r_0$, $0 < c \leq c_0$, $0 < s < R$ satisfying $c + s = R$, and $0 < \sigma \leq \min\{1, \sigma_0/2\}$.

3.4.2. Basic properties of ψ^+

Let $\psi^+ = \sum_{p \geq 1} \psi_p^+(t, x, w_0, \dots, w_{p-1})$ be the formal solution of (3.4.6) with $\psi_1^+ = w_0$ and $\psi_p^+(t, x, w_0, \dots, w_{p-1}) \in \mathcal{H}_{p,(R)}[t, w_0, \dots, w_{p-1}]$ (for $p \geq 2$). We have

Lemma 3.4.4. (1) $\psi_k^+(t, x, w_0, \dots, w_{k-1})$ (for $k \geq 2$) has the form

$$\psi_k^+ = \sum_{i+|j|=k} \sum_{1 \leq l \leq 2k-3} \frac{C_{i,j,l}}{(R-x)^l} t^i w_0^{j_0} \dots w_{k-1}^{j_{k-1}} \gg 0$$

with $C_{i,j,l} \geq 0$ ($i + |j| = k$ and $1 \leq l \leq 2k - 3$).

(2) If $r > 0$, $c > 0$ and $s > 0$ hold, we have

$$(3.4.11) \quad \rho \ll_{\rho} \frac{c^m}{m!} D^m[\psi^+] \left(r\rho, s, \frac{0!}{c^0}\rho, \frac{1!}{c^1}\rho, \frac{2!}{c^2}\rho, \dots \right)$$

for any $m = 0, 1, 2, \dots$, where \ll_{ρ} denotes the majorant relation in $\mathbb{C}[[\rho]]$ the ring of formal power series in ρ .

(3) If $r > 0$, $0 < c < R$, $0 < s < R$ and $c + s = R$ hold, we have

$$(3.4.12) \quad D^i[\psi^+] \left(r\rho, s, \frac{0!}{c^0}\rho, \frac{1!}{c^1}\rho, \frac{2!}{c^2}\rho, \dots \right) \\ \ll_{\rho} \frac{i!}{m!} c^{m-i} D^m[\psi^+] \left(r\rho, s, \frac{0!}{c^0}\rho, \frac{1!}{c^1}\rho, \frac{2!}{c^2}\rho, \dots \right)$$

for any $0 \leq i \leq m$.

Proof. The part (1) is verified by induction on k ; in the argument we should use (3.4.8) and the following facts $\psi_1^+ = w_0$, $D[\psi_1^+] = w_1$, $2k - 3 \leq 2(k - 1)$, $(2k - 3) + 1 = 2(k - 1)$, and

$$1 + \sum_{1 \leq m \leq j} 2(p_m - 1) + \sum_{1 \leq n \leq \alpha} 2(q_n - 1) \\ = 1 + 2|p(j)| - 2j + 2|q(\alpha)| - 2\alpha \\ = 1 + 2(i + |p(j)| + |q(\alpha)|) - 2i - 2j - 2\alpha = 1 + 2k - 2i - 2j - 2\alpha \\ = (2k - 3) - 2(i + j + \alpha - 2) \leq 2k - 3.$$

Since $\psi_1^+ = w_1$, we have $D^m[\psi_1^+] = w_m$ and so

$$D^m[\psi^+] \left(r\rho, s, \frac{0!}{c^0}\rho, \frac{1!}{c^1}\rho, \frac{2!}{c^2}\rho, \dots \right) \\ \gg_{\rho} D^m[\psi_1^+] \left(r\rho, s, \frac{0!}{c^0}\rho, \frac{1!}{c^1}\rho, \frac{2!}{c^2}\rho, \dots \right) = \frac{m!}{c^m} \rho;$$

this proves the part (2).

Let us show (3). We note that to verify the condition (3.4.12) it is enough to discuss only the following two cases:

Case (i) $\psi^+ = w_0$, and

Case (ii) $q + |j| = k \geq 2$ and ψ^+ has the form

$$(3.4.13) \quad \psi^+ = \frac{C}{(R-x)^l} t^q w_0^{j_0} \cdots w_{k-1}^{j_{k-1}} \quad \text{with } C \geq 0 \text{ and } l \geq 1.$$

Take any $r > 0$, $0 < c < R$ and $0 < s < R$ with $c + s = R$; for simplicity we write

$$(*_\rho) = \left(r\rho, s, \frac{0!}{c^0}\rho, \frac{1!}{c^1}\rho, \frac{2!}{c^2}\rho, \dots \right).$$

In the case $\psi^+ = w_0$ we have $D^i[\psi^+] = w_i$ and $D^m[\psi^+] = w_m$; therefore if $0 \leq i \leq m$ we have (3.4.12) by

$$D^i[\psi^+](*_\rho) = \frac{i!}{c^i}\rho = \frac{i!}{m!}c^{m-i} \frac{m!}{c^m}\rho = \frac{i!}{m!}c^{m-i}D^m[\psi^+](*_\rho).$$

Let us show the case (ii). Suppose that $q + |j| = k \geq 2$ and that ψ^+ has the form (3.4.13). Set $u(x) = c/(R-x)$; then we have $u^{(n)}(x) = n!c/(R-x)^{n+1}$ for $n = 0, 1, 2, \dots$, and therefore

$$\begin{aligned} & \psi^+(t, x, u(x), u^{(1)}(x), \dots, u^{(k-1)}(x)) \\ &= \psi^+\left(t, x, \frac{0!c}{(R-x)}, \frac{1!c}{(R-x)^2}, \dots, \frac{(k-1)!c}{(R-x)^k}\right) \\ &= \frac{C}{(R-x)^l} t^q \left(\frac{0!c}{(R-x)}\right)^{j_0} \cdots \left(\frac{(k-1)!c}{(R-x)^k}\right)^{j_{k-1}} = \frac{M}{(R-x)^p} t^q \end{aligned}$$

where $M = C(0!c)^{j_0} \cdots ((k-1)!c)^{j_{k-1}}$ and $p = l + j_0 + 2j_1 + \cdots + kj_{k-1} \geq 1$. By the definition of D we have

$$\begin{aligned} & D^i[\psi^+]\left(t, x, \frac{0!c}{(R-x)}, \frac{1!c}{(R-x)^2}, \dots, \frac{(k-1+i)!c}{(R-x)^{k+i}}\right) \\ &= D^i[\psi^+](t, x, u(x), u^{(1)}(x), \dots, u^{(k-1+i)}(x)) \\ &= \left(\frac{\partial}{\partial x}\right)^i \left[\psi^+(t, x, u(x), u^{(1)}(x), \dots, u^{(k-1)}(x)) \right] \\ &= \left(\frac{\partial}{\partial x}\right)^i \frac{M}{(R-x)^p} t^q = \frac{p(p+1) \cdots (p+i-1)M}{(R-x)^{p+i}} t^q, \end{aligned}$$

by setting $t = r$, $x = s$ and by using $c = R - s$ we have

$$D^i[\psi^+]\left(r, s, \frac{0!}{c^0}, \frac{1!}{c^1}, \dots, \frac{(k+i-1)!}{c^{k+i-1}}\right) = \frac{p(p+1) \cdots (p+i-1)M}{c^{p+i}} r^q.$$

Similarly we have

$$D^m[\psi^+]\left(r, s, \frac{0!}{c^0}, \frac{1!}{c^1}, \dots, \frac{(k+m-1)!}{c^{k+m-1}}\right) = \frac{p(p+1) \cdots (p+m-1)M}{c^{p+m}} r^q.$$

Moreover, by the conditions $0 \leq i \leq m$ and $p \geq 1$ we have

$$\frac{m!}{i!} = (i + 1) \cdots m \leq (p + i) \cdots (p + m - 1) = \frac{p(p + 1) \cdots (p + m - 1)}{p(p + 1) \cdots (p + i - 1)}.$$

Thus, we obtain

$$\begin{aligned} D^i[\psi^+] \left(r, s, \frac{0!}{c^0}, \frac{1!}{c^1}, \dots, \frac{(k-1)!}{c^{k-1}} \right) \\ \leq \frac{i!}{m!} c^{m-i} D^m[\psi^+] \left(r, s, \frac{0!}{c^0}, \frac{1!}{c^1}, \dots, \frac{(k-1)!}{c^{k-1}} \right). \end{aligned}$$

Since in this case ψ^+ is homogeneous of degree k with respect to (t, w_0, \dots, w_{k-1}) , this proves the relation

$$D^i[\psi^+](*_\rho) \ll_\rho \frac{i!}{m!} c^{m-i} D^m[\psi^+](*_\rho).$$

Thus, (3.4.12) is also proved in the case (ii). This completes the proof of (3) of Lemma 3.4.4. \square

3.4.3. On the equation $(\tau_k - \lambda(x))\phi_k = f_k$

For $0 < s \leq R$ and

$$f = \sum_{i+|j|=k} f_{i,j}(x) t^i u_0^{j_0} u_1^{j_1} \cdots u_{k-1}^{j_{k-1}} \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$$

we write $|f|_s(t, u_0, \dots, u_{k-1})$ by

$$|f|_s(t, u_0, \dots, u_{k-1}) = \sum_{i+|j|=k} \|f_{i,j}\|_s t^i u_0^{j_0} u_1^{j_1} \cdots u_{k-1}^{j_{k-1}}.$$

Let $k \geq 2$, and let us consider the equation

$$(3.4.14) \quad (\tau_k - \lambda(x))\phi_k = f_k \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}],$$

where τ_k is the same as in (3.4.2). We have

Lemma 3.4.5. *Suppose that*

$$(3.4.15) \quad |i + \lambda(x)(j - 1)| \geq \sigma_0(i + j) \quad \text{on } D_R \quad \text{for any } i + j \geq 2$$

holds for some $\sigma_0 > 0$. Let $k \geq 2$, and let $r_0 > 0$ and $0 < c_0 < R$ be the ones in (3.4.4). Then we have the following results.

(1) The equation $(\tau_k - \lambda(x))\phi_k = f_k \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$ has a unique solution $\phi_k \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$.

(2) Moreover, if $0 < r \leq r_0$, $0 < c \leq c_0$, $0 < s < R$, $c + s = R$ and $0 < \sigma \leq \sigma_0/2$ hold, we have

$$(3.4.16) \quad \sigma k |\phi_k|_s(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\ \ll_\rho |f_k|_s(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)).$$

Proof. (1) is clear from (2) of Lemma 3.1.3. Let us show (2). Set

$$\phi_k = \sum_{i+|j|=k} \phi_{i,j}(x) t^i u_0^{j_0} \dots u_{k-1}^{j_{k-1}}, \\ f_k = \sum_{i+|j|=k} f_{i,j}(x) t^i u_0^{j_0} \dots u_{k-1}^{j_{k-1}}.$$

Since the relation $(\tau_k - \lambda(x))\phi_k = f_k$ is written in the form (3.1.9), by using (3.4.15) and estimates $\|a^{(m)}\|_R \leq m!A/\eta^m$ and $\|\lambda_{m,p}\|_R \leq (m!/p!)(\Lambda/\eta^{m-p})$ we have

$$\sigma_0 k |\phi_k|_s(t, u_0, \dots, u_{k-1}) \\ \ll \sum_{i+|j|=k} \|\phi_{i,j}\|_s \left[\sum_{0 \leq m \leq k-1} \frac{m!A}{\eta^m} j_m t^{i+1} u_0^{j_0} \dots u_m^{j_m-1} \dots u_{k-1}^{j_{k-1}} \right] \\ + \sum_{i+|j|=k} \|\phi_{i,j}\|_s \left[\sum_{1 \leq m \leq k-1} \sum_{0 \leq p \leq m-1} \frac{m!}{p!} \frac{\Lambda}{\eta^{m-p}} j_m \times \right. \\ \left. \times t^i u_0^{j_0} \dots u_p^{j_p+1} \dots u_m^{j_m-1} \dots u_{k-1}^{j_{k-1}} \right] \\ + |f_k|_s(t, u_0, \dots, u_{k-1})$$

(as formal power series in (t, u_0, \dots, u_{k-1})), and by setting $t = r\rho$ and $u_p = D^p[\psi^+](*_\rho)$ ($p = 0, 1, 2, \dots, k-1$) we obtain

(3.4.17)

$$\sigma_0 k |\phi_k|_s(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\ \ll_\rho \sum_{i+|j|=k} \|\phi_{i,j}\|_s \left[\sum_{0 \leq m \leq k-1} \frac{m!A}{\eta^m} j_m (r\rho)^{i+1} \psi^+(*_\rho)^{j_0} \dots D^m[\psi^+](*_\rho)^{j_m-1} \right. \\ \left. \times \dots \times D^{k-1}[\psi^+](*_\rho)^{j_{k-1}} \right]$$

$$\begin{aligned}
 & + \sum_{i+|j|=k} \|\phi_{i,j}\|_s \left[\sum_{1 \leq m \leq k-1} \sum_{0 \leq p \leq m-1} \frac{m!}{p!} \frac{\Lambda}{\eta^{m-p}} j_m (r\rho)^i \psi^+(*_\rho)^{j_0} \dots \right. \\
 & \quad \left. \times D^p[\psi^+](*_\rho)^{j_{p+1}} \dots D^m[\psi^+](*_\rho)^{j_{m-1}} \dots D^{k-1}[\psi^+](*_\rho)^{j_{k-1}} \right] \\
 & + |f_k|_s(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)).
 \end{aligned}$$

Here, we note that by (2) of Lemma 3.4.4 we have $(r\rho)^{i+1} \times D^m[\psi^+](*_\rho)^{j_{m-1}} \ll_\rho r(c^m/m!)(\rho r)^i \times D^m[\psi^+](*_\rho)^{j_m}$ and so

$$\begin{aligned}
 & \frac{m!A}{\eta^m} (r\rho)^{i+1} \psi^+(*_\rho)^{j_0} \dots D^m[\psi^+](*_\rho)^{j_{m-1}} \\
 & \ll_\rho \frac{m!A}{\eta^m} r \frac{c^m}{m!} \times \left[(r\rho)^i \psi^+(*_\rho)^{j_0} \dots D^m[\psi^+](*_\rho)^{j_m} \right] \\
 & \ll_\rho rA \times \left[(r\rho)^i \psi^+(*_\rho)^{j_0} \dots D^m[\psi^+](*_\rho)^{j_m} \right],
 \end{aligned}$$

and by (3) of Lemma 3.4.4 we have $D^p[\psi^+](*_\rho)^{j_{p+1}} \times D^m[\psi^+](*_\rho)^{j_{m-1}} \ll_\rho (p!/m!)c^{m-p}D^p[\psi^+](*_\rho)^{j_p} \times D^m[\psi^+](*_\rho)^{j_m}$ and so

$$\begin{aligned}
 & \sum_{0 \leq p \leq m-1} \frac{m!}{p!} \frac{\Lambda}{\eta^{m-p}} (r\rho)^i \psi^+(*_\rho)^{j_0} \dots D^p[\psi^+](*_\rho)^{j_{p+1}} \dots D^m[\psi^+](*_\rho)^{j_{m-1}} \\
 & \ll_\rho \sum_{0 \leq p \leq m-1} \frac{m!}{p!} \frac{\Lambda}{\eta^{m-p}} \frac{p!}{m!} c^{m-p} \times \left[(r\rho)^i \psi^+(*_\rho)^{j_0} \dots D^m[\psi^+](*_\rho)^{j_m} \right] \\
 & \ll_\rho (\Lambda c/(\eta - c)) \left[(r\rho)^i \psi^+(*_\rho)^{j_0} \dots D^m[\psi^+](*_\rho)^{j_m} \right].
 \end{aligned}$$

Hence, by applying these estimates to (3.4.17) and then by using the inequalities $rA + (\Lambda c/(\eta - c)) \leq \sigma_0/2$ (by (3.4.4)) and $j_0 + j_1 + j_2 + \dots + j_{k-1} = |j| \leq k$ we obtain

$$\begin{aligned}
 & \sigma_0 k |\phi_k|_s(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\
 & \ll_\rho \sum_{i+|j|=k} \|\phi_{i,j}\|_s \left[\sum_{0 \leq m \leq k-1} rA j_m \times (r\rho)^i \psi^+(*_\rho)^{j_0} \dots D^{k-1}[\psi^+](*_\rho)^{j_{k-1}} \right. \\
 & \quad \left. + \sum_{1 \leq m \leq k-1} (\Lambda c/(\eta - c)) j_m \times (r\rho)^i \psi^+(*_\rho)^{j_0} \dots D^{k-1}[\psi^+](*_\rho)^{j_{k-1}} \right] \\
 & + |f_k|_s(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho))
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i+|j|=k} \|\phi_{i,j}\|_s (rA + (\Lambda c/(\eta - c))) \left[\sum_{0 \leq m \leq k-1} j_m \right] \times \\
&\quad \times (r\rho)^i \psi^+(*_\rho)^{j_0} \cdots D^{k-1}[\psi^+](*_\rho)^{j_{k-1}} \\
&\quad + |f_k|_s (r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\
&\ll_\rho \sum_{i+|j|=k} \|\phi_{i,j}\|_s (\sigma_0/2) k \times (r\rho)^i \psi^+(*_\rho)^{j_0} \cdots D^{k-1}[\psi^+](*_\rho)^{j_{k-1}} \\
&\quad + |f_k|_s (r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\
&= \frac{\sigma_0}{2} k |\phi_k|_s (r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\
&\quad + |f_k|_s (r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)).
\end{aligned}$$

Thus we have proved the inequality

$$\begin{aligned}
&\frac{\sigma_0}{2} k |\phi_k|_s (r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\
&\quad \ll_\rho |f_k|_s (r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)).
\end{aligned}$$

This proves the part (2) Lemma 3.4.5. □

3.4.4. Proof of Proposition 3.4.3

We will give a proof of Proposition 3.4.3 from now. The essential part of the proof consists of two propositions (Proposition 3.4.6 and Proposition 3.4.8) given below.

Proposition 3.4.6. *Let $\psi^+ = \sum_{p \geq 1} \psi_p^+(t, x, w_0, \dots, w_{p-1})$ be the formal solution of (3.4.6) with $\psi_1^+ = w_0$.*

(1) *We have the following equalities:*

$$(3.4.18) \quad G(t, x, \psi^+, D[\psi^+]) = \psi_2^+ + 2\psi_3^+ + 3\psi_4^+ + \cdots = \sum_{p \geq 1} (p-1)\psi_p^+,$$

$$(3.4.19) \quad D^m[G](t, x, \psi^+, \dots, D^{m+1}[\psi^+]) = \sum_{p \geq 1} (p-1)D^m[\psi_p^+].$$

(2) *Let $\varphi(t, x, u_0, \dots, u_{k-1}) \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$. If*

$$\begin{aligned}
(3.4.20) \quad &\varphi(t, x, \psi^+, D[\psi^+], \dots, D^{k-1}[\psi^+]) \\
&= \sum_{p \geq k} h_p(t, x, w_0, \dots, w_{p-1}) \in \sum_{p \geq k} \mathcal{H}_{p,(R)}[t, w_0, \dots, w_{p-1}]
\end{aligned}$$

holds, we have

$$(3.4.21) \quad \sum_{0 \leq m \leq k-1} D^m [G](t, x, \psi^+, D[\psi^+], \dots, D^{m+1}[\psi^+]) \times \\ \times \frac{\partial \varphi}{\partial u_m}(t, x, \psi^+, D[\psi^+], \dots, D^{k-1}[\psi^+]) \\ = \sum_{p \geq k} (p-k) h_p(t, x, w_0, \dots, w_{p-1}) \in \sum_{p \geq k} \mathcal{H}_{p, (R)}[t, w_0, \dots, w_{p-1}].$$

(3) Let $\varphi(t, x, u_0, \dots, u_{k-1}) \in \mathcal{H}_{k, R}[t, u_0, \dots, u_{k-1}]$. If $c > 0$, $0 < s < R$, and

$$(3.4.22) \quad |\varphi|_s(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \ll_\rho \sum_{p \geq k} B_p \rho^p$$

holds for some $B_p \geq 0$ ($p \geq k$), we have

$$(3.4.23) \quad \sum_{0 \leq m \leq k-1} D^m [G](r\rho, s, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{m+1}[\psi^+](*_\rho)) \times \\ \times \frac{\partial |\varphi|_s}{\partial u_m}(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\ \ll_\rho \sum_{p \geq k} (p-k) B_p \rho^p.$$

Proof. For a function $K(t, x, u_0, \dots, u_{k-1})$ we write $K^* = K^*(t, x, w_0, w_1, \dots) = K(t, x, \psi^+, D[\psi^+], \dots, D^{k-1}[\psi^+])$.

Since τ^0 has the form $\tau^0 = t\partial/\partial t + \sum_{m \geq 0} w_m \partial/\partial w_m$, by (3.4.6) and the Euler's identity we have

$$G^* = (\tau^0 - 1)\psi^+ = \sum_{p \geq 1} (\tau^0 - 1)\psi_p^+ = \sum_{p \geq 1} (p-1)\psi_p^+ :$$

this proves (3.4.18). Moreover, by Lemma 2.5 and (3.4.18) we can verify the equality (3.4.19) in the following way:

$$(D^m [G])^* = D^m [G^*] = D^m \left[\sum_{p \geq 1} (p-1)\psi_p^+ \right] = \sum_{p \geq 1} (p-1) D^m [\psi_p^+].$$

Let us show (2). By applying τ^0 to the both sides of (3.4.20) we have

$$(3.4.24) \quad \sum_{p \geq k} p h_p(t, x, w_0, \dots, w_{p-1}) = \tau^0 \sum_{p \geq k} h_p(t, x, w_0, \dots, w_{p-1}) \\ = \tau^0 \varphi(t, x, \psi^+, D[\psi^+], \dots, D^{k-1}[\psi^+])$$

$$\begin{aligned}
&= \left(t \frac{\partial \varphi}{\partial t}\right)^* + \sum_{0 \leq m \leq k-1} \left(\frac{\partial \varphi}{\partial u_m}\right)^* \times t \frac{\partial}{\partial t} D^m [\psi^+] \\
&\quad + \sum_{i \geq 0} \sum_{0 \leq m \leq k-1} \left(\frac{\partial \varphi}{\partial u_m}\right)^* \times w_i \frac{\partial}{\partial w_i} D^m [\psi^+] \\
&= \left(t \frac{\partial \varphi}{\partial t}\right)^* + \sum_{0 \leq m \leq k-1} \left(\frac{\partial \varphi}{\partial u_m}\right)^* \times \tau^0 D^m [\psi^+].
\end{aligned}$$

On the other hand, since $\varphi(t, x, u_0, \dots, u_{k-1})$ is homogeneous of degree k with respect to (t, u_0, \dots, u_{k-1}) , by the *Euler's identity* we have

$$k \varphi(t, x, u_0, \dots, u_{k-1}) = \tau^0 \varphi(t, x, u_0, \dots, u_{k-1}) = t \frac{\partial \varphi}{\partial t} + \sum_{0 \leq m \leq k-1} u_m \frac{\partial \varphi}{\partial u_m}$$

and so by (3.4.20) we have

$$\begin{aligned}
(3.4.25) \quad \sum_{p \geq k} k h_p(t, x, w_0, \dots, w_{p-1}) &= k \varphi(t, x, \psi^+, D[\psi^+], \dots, D^{k-1}[\psi^+]) \\
&= \left(t \frac{\partial \varphi}{\partial t}\right)^* + \sum_{0 \leq m \leq k-1} D^m [\psi^+] \times \left(\frac{\partial \varphi}{\partial u_m}\right)^*.
\end{aligned}$$

Hence, by (3.4.24) and (3.4.25) we have

$$\begin{aligned}
(3.4.26) \quad \sum_{p \geq k} (p-k) h_p(t, x, w_0, \dots, w_{p-1}) \\
= \sum_{0 \leq m \leq k-1} (\tau^0 - 1) D^m [\psi^+] \times \left(\frac{\partial \varphi}{\partial u_m}\right)^*.
\end{aligned}$$

Here we note:

Lemma 3.4.7. *For any function $f(t, x, w_0, w_1, \dots)$ we have*

$$(3.4.27) \quad (\tau^0 \circ D) f(t, x, w_0, w_1, \dots) = (D \circ \tau^0) f(t, x, w_0, w_1, \dots).$$

Proof of Lemma 3.4.7. Since

$$\begin{aligned}
(\tau^0 \circ D) f - (D \circ \tau^0) f &= \left(t \frac{\partial}{\partial t} + \sum_{j \geq 0} w_j \frac{\partial}{\partial w_j}\right) \left(\frac{\partial f}{\partial x} + \sum_{i \geq 0} w_{i+1} \frac{\partial f}{\partial w_i}\right) \\
&\quad - \left(\frac{\partial}{\partial x} + \sum_{i \geq 0} w_{i+1} \frac{\partial}{\partial w_i}\right) \left(t \frac{\partial f}{\partial t} + \sum_{j \geq 0} w_j \frac{\partial f}{\partial w_j}\right),
\end{aligned}$$

we can verify (3.4.27) by a direct calculation. \square

Hence, by applying this lemma to (3.4.26) we have

$$\begin{aligned} & \sum_{p \geq k} (p - k) h_p(t, x, w_0, \dots, w_{p-1}) \\ &= \sum_{0 \leq m \leq k-1} D^m [(\tau^0 - 1)(\psi^+)] \times \left(\frac{\partial \varphi}{\partial u_m} \right)^* \\ &= \sum_{0 \leq m \leq k-1} D^m [G(t, x, \psi^+, D[\psi^+])] \times \left(\frac{\partial \varphi}{\partial u_m} \right)^* \\ &= \sum_{0 \leq m \leq k-1} D^m [G](t, x, \psi^+, \dots, D^{m+1}[\psi^+]) \times \left(\frac{\partial \varphi}{\partial u_m} \right)^*. \end{aligned}$$

This proves (3.4.21): the part (2) of Proposition 3.4.6 is proved.

Let us show (3). Let $\varphi(t, x, u_0, \dots, u_{k-1}) \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$. Since $|\varphi|_s(t, u_0, \dots, u_{k-1}) \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$ holds, we can set

$$\begin{aligned} (3.4.28) \quad & |\varphi|_s(t, \psi^+, D[\psi^+], \dots, D^{k-1}[\psi^+]) \\ &= \sum_{p \geq k} h_p(t, x, w_0, \dots, w_{p-1}) \in \sum_{p \geq k} \mathcal{H}_{p,(R)}[t, w_0, \dots, w_{p-1}], \end{aligned}$$

and by the part (2) of this proposition we have

$$\begin{aligned} (3.4.29) \quad & \sum_{0 \leq m \leq k-1} D^m [G](t, x, \psi^+, D[\psi^+], \dots, D^{m+1}[\psi^+]) \times \\ & \quad \times \frac{\partial |\varphi|_s}{\partial u_m}(t, \psi^+, D[\psi^+], \dots, D^{k-1}[\psi^+]) \\ &= \sum_{p \geq k} (p - k) h_p(t, x, w_0, \dots, w_{p-1}) \in \sum_{p \geq k} \mathcal{H}_{p,(R)}[t, w_0, \dots, w_{p-1}]. \end{aligned}$$

By setting $t = r\rho$, $x = s$ and $w_p = (p!/c^p)\rho$ ($p = 0, 1, 2, \dots$) in (3.4.28) and (3.4.29) we have

$$\begin{aligned} (3.4.30) \quad & |\varphi|_s(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\ &= \sum_{p \geq k} h_p(*)\rho^p \in \mathbb{C}[[\rho]], \end{aligned}$$

$$\begin{aligned} (3.4.31) \quad & \sum_{0 \leq m \leq k-1} D^m [G](r\rho, s, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{m+1}[\psi^+](*_\rho)) \times \\ & \quad \times \frac{\partial |\varphi|_s}{\partial u_m}(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\ &= \sum_{p \geq k} (p - k) h_p(*)\rho^p \in \mathbb{C}[[\rho]], \end{aligned}$$

where

$$(*) = \left(r, s, \frac{0!}{c^0}, \frac{1!}{c^1}, \frac{2!}{c^2}, \dots \right).$$

Hence, by comparing (3.4.22) and (3.4.30) we have $h_p(*) \leq B_p$ for $p \geq k$, and by applying this to the right-hand side of (3.4.31) we obtain the majorant relation (3.4.23). \square

Let $\phi = \sum_{k \geq 1} \phi_k(t, x, u_0, \dots, u_{k-1})$ be the formal solution (3.4.1) with $\phi_1 = -\alpha(x)\beta(x)t + \beta(x)u_0$ and $\phi_p(t, x, u_0, \dots, u_{k-1}) \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$ (for $k \geq 2$).

Proposition 3.4.8. *Suppose the conditions (3.4.3) and (3.4.4). Then, we have the following results:*

(1) *For any $r > 0$, $c > 0$ and $0 < s < R$, we have*

$$(3.4.32) \quad |\phi_1|_s(r\rho, \psi^+(*_\rho)) \\ \ll_\rho \|\beta\|_s (\|\alpha\|_s r\rho + \psi_1^+(*)\rho + \psi_2^+(*)\rho^2 + \psi_3^+(*)\rho^3 + \dots).$$

(2) *For any $k \geq 2$, $0 < r \leq r_0$, $0 < c \leq c_0$, $0 < s < R$ satisfying $c + s = R$ and $0 < \sigma \leq \min\{1, \sigma_0/2\}$ we have*

$$(3.4.33)_k \quad \sum_{1 \leq q \leq k-1} \sum_{0 \leq m \leq q-1} D^m [G](r\rho, s, \psi^+(*_\rho), \dots, D^{m+1}[\psi^+](*_\rho)) \times \\ \times \frac{\partial |\phi_q|_s}{\partial u_m}(r\rho, \psi^+(*_\rho), \dots, D^{q-1}[\psi^+](*_\rho)) \\ \ll_\rho \|\beta\|_s \sum_{2 \leq p \leq k-1} \frac{(p-1)^{p-1}}{\sigma^{p-2} (p-2)!} \psi_p^+(*)\rho^p \\ + \|\beta\|_s \sum_{p \geq k} \frac{(p-1)^{k-1}}{\sigma^{k-2} (k-2)!} \psi_p^+(*)\rho^p,$$

$$(3.4.34)_k \quad |\phi_k|_s(r\rho, \psi^+(*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\ \ll_\rho \|\beta\|_s \sum_{p \geq k} \frac{(p-1)^{k-1}}{\sigma^{k-1} (k-1)!} \psi_p^+(*)\rho^p.$$

Proof. Since $|\phi_1|_s(t, u_0) = \|\alpha\beta\|_s t + \|\beta\|_s u_0$, we have

$$|\phi_1|_s(r\rho, \psi^+(*_\rho)) = \|\alpha\beta\|_s r\rho + \|\beta\|_s \psi^+(*_\rho) \\ \ll_\rho \|\beta\|_s (\|\alpha\|_s r\rho + \psi^+(*_\rho)) \\ = \|\beta\|_s (\|\alpha\|_s r\rho + \psi_1^+(*)\rho + \psi_2^+(*)\rho^2 + \psi_3^+(*)\rho^3 + \dots);$$

this proves the part (1).

Let us show (2) by induction on k . The condition (3.4.33)₂ is verified by (3.4.18) and

$$\begin{aligned} G(r\rho, s, \psi^+(*_\rho), D[\psi^+](*_\rho)) \frac{\partial |\phi_1|_s}{\partial u_0} &= G(r\rho, s, \psi^+(*_\rho), D[\psi^+](*_\rho)) \|\beta\|_s \\ &= \|\beta\|_s (\psi_2^+(*)\rho^2 + 2\psi_3^+(*)\rho^3 + \dots) = \|\beta\|_s \sum_{p \geq 2} (p-1)\psi_p^+(*)\rho^p. \end{aligned}$$

By (3.4.2) we have

$$(\tau_2 - \lambda(x))\phi_2 = -R_2(t, x, u_0, u_1) \frac{\partial \phi_1}{\partial u_0} = -R_2(t, x, u_0, u_1)\beta(x);$$

by applying (2) of Lemma 3.4.5 to this equation and by using the majorant relation $R_2(t, x, u_0, u_1) \ll G(t, x, u_0, u_1)$ (by (3.4.5)) we have

$$\begin{aligned} 2\sigma |\phi_2|_s(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho)) &\ll_\rho \|\beta\|_s |R_2|_s(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho)) \\ &\ll_\rho \|\beta\|_s G(r\rho, s, \psi^+(*_\rho), D[\psi^+](*_\rho)). \end{aligned}$$

Here, we used the following fact: if $f(x) \ll g(x)$ holds we have $\|f\|_s \leq g(s)$, and so we have $|R_2|_s(t, u_0, u_1) \ll G(t, s, u_0, u_1)$ (as power series in (t, u_0, u_1)). Thus, the condition (3.4.34)₂ is verified by (3.4.18) and

$$\begin{aligned} |\phi_2|_s(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho)) &\ll_\rho \frac{1}{2\sigma} \|\beta\|_s G(r\rho, s, \psi^+(*_\rho), D[\psi^+](*_\rho)) \\ &\ll_\rho \frac{1}{\sigma} \|\beta\|_s (\psi_2^+(*)\rho^2 + 2\psi_3^+(*)\rho^3 + \dots) = \|\beta\|_s \sum_{p \geq 2} \frac{(p-1)}{\sigma} \psi_p^+(*)\rho^p. \end{aligned}$$

Now, let $k \geq 3$ and suppose that (3.4.33)_{k-1} and (3.4.34)_{k-1} are already proved. Note

$$\begin{aligned} &\sum_{1 \leq q \leq k-1} \sum_{0 \leq m \leq q-1} D^m [G] \frac{\partial |\phi_q|_s}{\partial u_m} \\ &= \sum_{1 \leq q \leq k-2} \sum_{0 \leq m \leq q-1} D^m [G] \frac{\partial |\phi_q|_s}{\partial u_m} + \sum_{0 \leq m \leq k-2} D^m [G] \frac{\partial |\phi_{k-1}|_s}{\partial u_m}. \end{aligned}$$

To the first sum of the right-hand side we can apply the induction hypothesis (3.4.33)_{k-1}, and to the second sum we can apply the induction hypothesis

(3.4.34)_{k-1} and (3) of Proposition 3.4.6 (with φ replaced by ϕ_{k-1}): thus we have

(3.4.35)

$$\begin{aligned}
& \sum_{1 \leq q \leq k-1} \sum_{0 \leq m \leq q-1} D^m[G](r\rho, s, \psi^+(*_\rho), \dots, D^{m+1}[\psi^+](*_\rho)) \times \\
& \quad \times \frac{\partial|\phi_q|_s}{\partial u_m}(r\rho, \psi^+(*_\rho), D[\psi^+](*_\rho), \dots, D^{q-1}[\psi^+](*_\rho)) \\
& \ll_{\rho} \|\beta\|_s \sum_{2 \leq p \leq k-2} \frac{(p-1)^{p-1}}{\sigma^{p-2}(p-2)!} \psi_p^+(*)\rho^p + \|\beta\|_s \sum_{p \geq k-1} \frac{(p-1)^{k-2}}{\sigma^{k-3}(k-3)!} \psi_p^+(*)\rho^p \\
& \quad + \|\beta\|_s \sum_{p \geq k-1} (p-k+1) \frac{(p-1)^{k-2}}{\sigma^{k-2}(k-2)!} \psi_p^+(*)\rho^p \\
& = \|\beta\|_s \sum_{2 \leq p \leq k-1} \frac{(p-1)^{p-1}}{\sigma^{p-2}(p-2)!} \psi_p^+(*)\rho^p \\
& \quad + \|\beta\|_s \sum_{p \geq k} \left(\frac{(p-1)^{k-2}}{\sigma^{k-3}(k-3)!} + \frac{(p-k+1)(p-1)^{k-2}}{\sigma^{k-2}(k-2)!} \right) \psi_p^+(*)\rho^p.
\end{aligned}$$

Since $0 < \sigma \leq 1$ is assumed and since

$$\begin{aligned}
& \frac{(p-1)^{k-2}}{\sigma^{k-3}(k-3)!} + \frac{(p-k+1)(p-1)^{k-2}}{\sigma^{k-2}(k-2)!} \\
& \leq \frac{(p-1)^{k-2}}{\sigma^{k-2}(k-2)!} ((k-2) + (p-k+1)) = \frac{(p-1)^{k-1}}{\sigma^{k-2}(k-2)!},
\end{aligned}$$

by (3.4.35) we obtain (3.4.33)_k.

Let us show (3.4.34)_k. Since $R_{k-q+1}(t, x, u_0, u_1) \ll G(t, x, u_0, u_1)$ (by (3.4.5)) we have $D^m[R_{k-q+1}](t, x, u_0, \dots, u_{m+1}) \ll D^m[G](t, x, u_0, \dots, u_{m+1})$ and so

$$|D^m[R_{k-q+1}]|_s(t, u_0, \dots, u_{m+1}) \ll D^m[G](t, s, u_0, \dots, u_{m+1})$$

as power series in (t, u_0, \dots, u_{m+1}) . By using this fact, (3.4.2), (2) of Lemma 3.4.5 and the relation

$$\left| \frac{\partial \phi_q}{\partial u_m} \right|_s(t, u_0, \dots, u_{q-1}) = \frac{\partial |\phi_q|_s}{\partial u_m}(t, u_0, \dots, u_{q-1})$$

we have

$$\begin{aligned} & |\phi_k|_s(r\rho, \psi^+(*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\ & \ll_\rho \frac{1}{\sigma k} \sum_{1 \leq q \leq k-1} \sum_{0 \leq m \leq q-1} |D^m[R_{k-q+1}]|_s(r\rho, \psi^+(*_\rho), \dots, D^{m+1}[\psi^+](*_\rho)) \\ & \quad \times \frac{\partial |\phi_q|_s}{\partial u_m}(r\rho, \psi^+(*_\rho), \dots, D^{q-1}[\psi^+](*_\rho)) \\ & \ll_\rho \frac{1}{\sigma(k-1)} \sum_{1 \leq q \leq k-1} \sum_{0 \leq m \leq q-1} D^m[G](r\rho, s, \psi^+(*_\rho), \dots, D^{m+1}[\psi^+](*_\rho)) \\ & \quad \times \frac{\partial |\phi_q|_s}{\partial u_m}(r\rho, \psi^+(*_\rho), \dots, D^{q-1}[\psi^+](*_\rho)). \end{aligned}$$

Therefore, by applying (3.4.33)_k to this inequality we have

$$\begin{aligned} & |\phi_k|_s(r\rho, \psi^+(*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \\ & \ll_\rho \|\beta\|_s \sum_{2 \leq p \leq k-1} \frac{(p-1)^{p-1}}{\sigma^{p-1}(k-1)(p-2)!} \psi_p^+(*_\rho) \rho^p \\ & \quad + \|\beta\|_s \sum_{p \geq k} \frac{(p-1)^{k-1}}{\sigma^{k-1}(k-1)!} \psi_p^+(*_\rho) \rho^p. \end{aligned}$$

Since the degree of the left-hand side with respect to ρ is greater than or equal to k we obtain

$$|\phi_k|_s(r\rho, \psi^+(*_\rho), \dots, D^{k-1}[\psi^+](*_\rho)) \ll_\rho \|\beta\|_s \sum_{p \geq k} \frac{(p-1)^{k-1}}{\sigma^{k-1}(k-1)!} \psi_p^+(*_\rho) \rho^p :$$

this proves (3.4.34)_k. □

Proof of Proposition 3.4.3. By comparing the coefficients of ρ in the both sides of (3.4.32) we have

$$|\phi_1|_s(r, \psi_1^+(*)) \leq \|\beta\|_s (\|\alpha\|_s r + \psi_1^+(*)) :$$

this implies the inequality (3.4.9). Note that $\phi_1 = -\alpha(x)\beta(x)t + \beta(x)u_0$ and $\psi_1^+ = w_0$ hold.

Similarly, by comparing the homogeneous part of degree k with respect to ρ in the both sides of (3.4.34)_k we have

$$|\phi_k|_s(r, \psi_1^+(*), \dots, D^{k-1}[\psi_1^+](*)) \leq \|\beta\|_s \frac{(k-1)^{k-1}}{\sigma^{k-1}(k-1)!} \psi_k^+(*)$$

for any $k \geq 2$. Since $D^j[\psi_1^+](*) = (D^j[w_0])(*) = (w_j)(*) = j!/c^j$ holds for all $j \in \mathbb{N}$, we obtain

$$\begin{aligned} |\phi_k|_s \left(r, \frac{0!}{c^0}, \frac{1!}{c^1}, \dots, \frac{(k-1)!}{c^{k-1}} \right) \\ \leq \|\beta\|_s \frac{(k-1)^{k-1}}{\sigma^{k-1} (k-1)!} \psi_k^+ \left(r, s, \frac{0!}{c^0}, \frac{1!}{c^1}, \dots, \frac{(k-1)!}{c^{k-1}} \right). \end{aligned}$$

Since $|\psi_k^+|_s(t, w_0, \dots, w_{k-1}) = \psi_k^+(t, s, w_0, \dots, w_{k-1})$ holds, we have the inequality (3.4.10). \square

§3.5. Equivalence of two PDEs

Let us find suitable function-spaces \mathcal{F} and \mathcal{G} so that the following two partial differential equations

$$(3.5.1) \quad t \frac{\partial u}{\partial t} = F \left(t, x, u, \frac{\partial u}{\partial x} \right) \quad \text{in } \mathcal{F},$$

$$(3.5.2) \quad t \frac{\partial w}{\partial t} = \lambda(x)w \quad \text{in } \mathcal{G}$$

are equivalent in the sense defined below (in Definition 3.5.1). Set

$\mathcal{S}_A =$ the set of all solutions of (3.5.1) in \mathcal{F} ,

$\mathcal{S}_B =$ the set of all solutions of (3.5.2) in \mathcal{G} .

Definition 3.5.1. Let $\phi(t, x, u_0, u_1, \dots)$ and $\psi(t, x, w_0, w_1, \dots)$ be solutions of (Φ) and (Ψ) , respectively. We say that the two equations (3.5.1) and (3.5.2) are equivalent, if the two mappings

$$\Phi : \mathcal{S}_A \ni u(t, x) \mapsto w(t, x) = \phi(t, x, u, \partial u / \partial x, \dots) \in \mathcal{S}_B,$$

$$\Psi : \mathcal{S}_B \ni w(t, x) \mapsto u(t, x) = \psi(t, x, w, \partial w / \partial x, \dots) \in \mathcal{S}_A$$

are well-defined, bijective and one is the inverse of the other.

For $r > 0$, $c > 0$, $s > 0$ and $\varepsilon > 0$ we write

$$\begin{aligned} U_k(r, c, s, \varepsilon) &= \left\{ (t, x, u_0, \dots, u_{k-1}) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^k; |t| \leq r\varepsilon, |x| \leq s, \right. \\ &\quad \left. |u_0| \leq 0!\varepsilon, |u_1| \leq 1!\varepsilon/c, \dots, |u_{k-1}| \leq (k-1)!\varepsilon/c^{k-1} \right\}, \\ W_k(c, s, \varepsilon) &= \left\{ (t, x, w_0, \dots, w_{k-1}) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^k; |t| \leq \varepsilon, |x| \leq s, \right. \\ &\quad \left. |w_0| \leq 0!\varepsilon, |w_1| \leq 1!\varepsilon/c, \dots, |w_{k-1}| \leq (k-1)!\varepsilon/c^{k-1} \right\} \end{aligned}$$

($k = 1, 2, \dots$). For a holomorphic function $f(t, x, u_0, \dots, u_{k-1})$ on $U_k = U_k(r, c, s, \varepsilon)$ we define the norm $\|f\|_{U_k}$ by

$$\|f\|_{U_k} = \max_{U_k} |f(t, x, u_0, \dots, u_{k-1})|.$$

The norm $\|g\|_{W_k}$ is defined in the same way. Then, by the results in subsections 3.1~3.4 we have:

Theorem 3.5.2. *Suppose the conditions $A_1), A_2), A_3)$ and that*

$$(3.5.3) \quad |i + \lambda(x)(j - 1)| \geq \sigma_0(i + j) \quad \text{on } D_R$$

for any $(i, j) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0), (0, 1)\}$

for some $\sigma_0 > 0$. Then, we can find constants $r_0 > 0$ and $0 < c_0 < R$ so that the following results hold.

(1) *The coupling equation (Φ) has a unique formal solution of the form*

$$(3.5.4) \quad \phi = \frac{-a(x)}{1 - \lambda(x)} t + u_0 + \sum_{k \geq 2} \phi_k(t, x, u_0, \dots, u_{k-1})$$

with $\phi_k(t, x, u_0, \dots, u_{k-1}) \in \mathcal{H}_{k,R}[t, u_0, \dots, u_{k-1}]$ ($k = 2, 3, \dots$);

moreover, for any $0 < r \leq r_0$, $0 < c \leq c_0$ and $s = R - c$ there is an $\varepsilon > 0$ such that

$$\sum_{k \geq 1} \|\phi_k\|_{U_k} \quad \text{with } U_k = U_k(r, c, s, \varepsilon)$$

is convergent, where $\phi_1 = -a(x)/(1 - \lambda(x))t + u_0$.

(2) *The coupling equation (Ψ) has a unique formal solution of the form*

$$(3.5.5) \quad \psi = \frac{a(x)}{1 - \lambda(x)} t + w_0 + \sum_{k \geq 2} \psi_k(t, x, w_0, \dots, w_{k-1})$$

with $\psi_k(t, x, w_0, \dots, w_{k-1}) \in \mathcal{H}_{k,R}[t, w_0, \dots, w_{k-1}]$ ($k = 2, 3, \dots$);

moreover, for any $r > 0$, $0 < c \leq c_0$ and $0 < s < R$ there is an $\varepsilon > 0$ such that

$$\sum_{k \geq 1} \|\psi_k\|_{W_k} \quad \text{with } W_k = W_k(c, s, \varepsilon)$$

is convergent, where $\psi_1 = a(x)/(1 - \lambda(x))t + w_0$.

(3) *In addition, the relation $w = \phi(t, x, u, \partial u / \partial x, \dots)$ is reversible with respect to u and w , and the function $\psi(t, x, w_0, w_1, \dots)$ is the reverse function of $\phi(t, x, u_0, u_1, \dots)$.*

Thus, if we can find function-spaces \mathcal{F} and \mathcal{G} so that the two mappings

$$\begin{aligned}\Phi : \mathcal{F} \ni u(t, x) &\longmapsto w(t, x) = \phi(t, x, u, \partial u / \partial x, \dots) \in \mathcal{G}, \\ \Psi : \mathcal{G} \ni w(t, x) &\longmapsto u(t, x) = \psi(t, x, w, \partial w / \partial x, \dots) \in \mathcal{F}\end{aligned}$$

are well-defined, we can conclude (by Theorem 2.6) that the two equations (3.5.1) and (3.5.2) are really equivalent. Let us define such function-spaces.

For simplicity, we denote by:

- $\mathcal{R}(\mathbb{C} \setminus \{0\})$ the universal covering space of $\mathbb{C} \setminus \{0\}$,
- $S_\theta = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |\arg t| < \theta\}$ a sector in $\mathcal{R}(\mathbb{C} \setminus \{0\})$,
- $S_\theta(r) = \{t \in S_\theta; 0 < |t| \leq r\}$,
- $S(\varepsilon(s)) = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}); 0 < |t| < \varepsilon(\arg t)\}$, where $\varepsilon(s)$ is a positive-valued continuous function on \mathbb{R}_s ,

Definition 3.5.3. (1) We denote by $\tilde{\mathcal{O}}_+$ the set of all $u(t, x)$ satisfying the following i) and ii): i) $u(t, x)$ is a holomorphic function on $S(\varepsilon(s)) \times D_R$ for some positive-valued continuous function $\varepsilon(s)$ on \mathbb{R}_s and $R > 0$; and ii) there is an $a > 0$ such that for any $\theta > 0$ we have

$$\max_{x \in D_R} |u(t, x)| = O(|t|^a) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta).$$

(2) We denote by \mathcal{S}_+ the set of all $u(t, x)$ satisfying the following i) and ii): i) $u(t, x)$ is a holomorphic function on $S_\theta(r) \times D_R$ for some $\theta > 0$, $r > 0$ and $R > 0$; and ii) there is an $a > 0$ such that

$$\max_{x \in D_R} |u(t, x)| = O(|t|^a) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta(r)).$$

Similarly we define

Definition 3.5.4. (1) We denote by $\tilde{\mathcal{O}}_{zero}$ the set of all $u(t, x)$ satisfying the following i) and ii): i) $u(t, x)$ is a holomorphic function on $S(\varepsilon(s)) \times D_R$ for some positive-valued continuous function $\varepsilon(s)$ on \mathbb{R}_s and $R > 0$; and ii) for any $\theta > 0$ we have

$$\max_{x \in D_R} |u(t, x)| = o(1) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta).$$

(2) We denote by \mathcal{S}_{zero} the set of all $u(t, x)$ satisfying the following i) and ii): i) $u(t, x)$ is a holomorphic function on $S_\theta(r) \times D_R$ for some $\theta > 0$, $r > 0$ and $R > 0$; and ii) we have

$$\max_{x \in D_R} |u(t, x)| = o(1) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta(r)).$$

Then, by Theorem 3.5.2 we can easily see:

Proposition 3.5.5. *Set $\mathcal{F} = \tilde{\mathcal{O}}_+, \mathcal{S}_+, \tilde{\mathcal{O}}_{zero}$ or \mathcal{S}_{zero} . Then, the following two mappings are well-defined:*

$$\begin{aligned} \Phi : \mathcal{F} \ni u(t, x) &\longmapsto w(t, x) = \phi(t, x, u, \partial u / \partial x, \dots) \in \mathcal{F}, \\ \Psi : \mathcal{F} \ni w(t, x) &\longmapsto u(t, x) = \psi(t, x, w, \partial w / \partial x, \dots) \in \mathcal{F}. \end{aligned}$$

Thus, we obtain

Theorem 3.5.6. *Suppose the conditions $A_1), A_2), A_3)$ and that*

$$(3.5.6) \quad \begin{aligned} |i + \lambda(x)(j - 1)| &\geq \sigma_0(i + j) \quad \text{on } D_R \\ &\text{for any } (i, j) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0), (0, 1)\} \end{aligned}$$

for some $\sigma_0 > 0$. Set $\mathcal{F} = \tilde{\mathcal{O}}_+, \mathcal{S}_+, \tilde{\mathcal{O}}_{zero}$ or \mathcal{S}_{zero} . Then, the following two equations are equivalent:

$$(3.5.7) \quad t \frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right) \quad \text{in } \mathcal{F},$$

$$(3.5.8) \quad t \frac{\partial w}{\partial t} = \lambda(x)w \quad \text{in } \mathcal{F}.$$

§4. Application

In this section, by using the equivalence theorem (Theorem 3.5.6) we will investigate the structure of solutions of Briot-Bouquet type partial differential equation

$$(4.1) \quad t \frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right)$$

under the same assumptions $A_1), A_2)$ and $A_3)$.

§4.1. Solutions in a sectorial domain

First, we note:

Lemma 4.1. *The following two conditions are equivalent:*

- (1) $\lambda(0) \notin (-\infty, 0] \cup \{1, 2, \dots\}$.
- (2) *There are $\sigma_0 > 0$ and $R > 0$ such that $|i + \lambda(x)(j - 1)| \geq \sigma_0(i + j)$ holds on D_R for any $(i, j) \in \mathbb{N}^2 \setminus \{(0, 0), (0, 1)\}$.*

Proof. Denote by $L(1, \lambda(x))$ the segment in the complex plane joining the two points 1 and $\lambda(x)$. For $(i, j) \in \mathbb{N}^2 \setminus \{(0, 0), (0, 1)\}$ we set

$$z_{i,j}(x) = \frac{i + \lambda(x)j}{i + j}$$

which is the point on $L(1, \lambda(x))$ such that $|1 - z_{i,j}(x)| : |z_{i,j}(x) - \lambda(x)| = j : i$.

Let us show that (1) implies (2). Suppose the condition (1); then we have $\lambda(x) \notin (-\infty, 0] \cup \{1, 2, \dots\}$ on D_R for some $R > 0$, and so $|i + \lambda(x)(j - 1)| \neq 0$ for any $(i, j) \in \mathbb{N}^2 \setminus \{(0, 0), (0, 1)\}$ and $x \in D_R$. Moreover, if we denote by $d(x)$ the distance from the origin to $L(1, \lambda(x))$, we have $d(x) \geq c$ on D_R for some $c > 0$; this implies $|z_{i,j}(x)| \geq c$ for any $x \in D_R$.

Choose a sufficiently large N so that $|\lambda(x)|/N \leq c/2$ on D_R . Then, if $(i, j) \in \mathbb{N}^2$ satisfies $i + j \geq N$ we see

$$\left| \frac{i + \lambda(x)(j - 1)}{i + j} \right| \geq |z_{i,j}(x)| - \frac{|\lambda(x)|}{i + j} \geq c - \frac{|\lambda(x)|}{N} \geq c - \frac{c}{2} \geq \frac{c}{2} \quad \text{on } D_R.$$

On the other hand, since the number of the elements of $I_N = \{(i, j) \in \mathbb{N}^2 \setminus \{(0, 0), (0, 1)\}; i + j < N\}$ is finite, and since $|i + \lambda(x)(j - 1)| \neq 0$ for any $(i, j) \in I_N$ and $x \in D_R$ we have

$$|i + \lambda(x)(j - 1)| \geq \sigma(i + j) \quad \text{on } D_R \text{ for any } (i, j) \in I_N$$

for some $\sigma > 0$. Thus, by setting $\sigma_0 = \min\{c/2, \sigma\}$ we have the condition (2).

Next let us show that (2) implies (1). Suppose the condition (2). Then, by considering the case $j = 0$ and $x = 0$ we have $|i - \lambda(0)| \geq \sigma_0 i$ for any $i \in \{1, 2, \dots\}$. This implies $\lambda(0) \notin \{1, 2, \dots\}$.

The condition $\lambda(0) \notin (-\infty, 0]$ is verified as follows. Choose a sufficiently large N so that $|\lambda(0)|/N \leq \sigma_0/2$. Then, if $(i, j) \in \mathbb{N}^2$ satisfies $i + j \geq N$ we have

$$|z_{i,j}(0)| \geq \left| \frac{i + \lambda(0)(j - 1)}{i + j} \right| - \frac{|\lambda(0)|}{i + j} \geq \sigma_0 - \frac{|\lambda(0)|}{N} \geq \sigma_0 - \frac{\sigma_0}{2} \geq \frac{\sigma_0}{2}.$$

This implies that the distance from the origin to $L(1, \lambda(0))$ is greater than or equal to $\sigma_0/2$, because the set $\{z_{i,j}(0); (i, j) \in \mathbb{N}^2 \text{ and } i + j \geq N\}$ is dense in $L(1, \lambda(0))$. Thus, we obtain the condition that $L(1, \lambda(0))$ does not contain the origin and therefore $\lambda(0) \notin (-\infty, 0]$. \square

Thus, by this lemma we see that all the results in section 3 are valid under the condition

$$(4.2) \quad \lambda(0) \notin (-\infty, 0] \cup \{1, 2, \dots\}.$$

Now, let us determine all the singular solutions of the equation (4.1). For a function-class \mathcal{F} we set:

$$\text{Sol}(\mathcal{F}) = \text{the set of all solutions of (4.1) in } \mathcal{F}.$$

Let $\psi(t, x, w_0, w_1, \dots)$ be the solution of (Ψ) in (2) of Theorem 3.5.2, and we write:

$$\Psi[w] = \psi(t, x, w, \partial w / \partial x, \dots).$$

We denote by $\mathbb{C}\{x\}$ the ring of convergent power series in x which is the set of all holomorphic functions in a neighborhood of $x = 0 \in \mathbb{C}$. We have

Theorem 4.2. *Suppose the conditions $A_1), A_2), A_3)$ and (4.2). Then, we have the following result.*

(1) *If $\text{Re}\lambda(0) > 0$, we have*

$$(4.3) \quad \begin{aligned} \text{Sol}(\tilde{\mathcal{O}}_+) &= \text{Sol}(\mathcal{S}_+) = \text{Sol}(\tilde{\mathcal{O}}_{zero}) = \text{Sol}(\mathcal{S}_{zero}) \\ &= \{ \Psi[h(x)t^{\lambda(x)}]; h(x) \in \mathbb{C}\{x\} \}. \end{aligned}$$

(2) *If $\text{Re}\lambda(0) \leq 0$, we have*

$$(4.4) \quad \text{Sol}(\tilde{\mathcal{O}}_+) = \text{Sol}(\mathcal{S}_+) = \text{Sol}(\tilde{\mathcal{O}}_{zero}) = \text{Sol}(\mathcal{S}_{zero}) = \{ \Psi[0] \}.$$

Note that $\Psi[0] = \psi(t, x, 0, 0, \dots)$ ($= u_0(t, x)$) is the unique holomorphic solution satisfying $u_0(0, x) = 0$ (obtained in Theorem 1.1).

Proof. By Theorem 3.5.6 it is sufficient to solve the equation

$$(4.5) \quad t \frac{\partial w}{\partial t} = \lambda(x)w \quad \text{in } \mathcal{F}$$

with $\mathcal{F} = \tilde{\mathcal{O}}_+, \mathcal{S}_+, \tilde{\mathcal{O}}_{zero}$ or \mathcal{S}_{zero} .

Let us denote by \mathcal{S}_* the set of all holomorphic functions $u(t, x)$ on $S_\theta(r) \times D_R$ for some $\theta > 0, r > 0$ and $R > 0$ (which may depend on $u(t, x)$). It is easy to see that the set $\text{Sol}((4.5), \mathcal{S}_*)$ of all the solutions of this equation (4.5) with $\mathcal{F} = \mathcal{S}_*$ is given by

$$\text{Sol}((4.5), \mathcal{S}_*) = \{ h(x)t^{\lambda(x)}; h(x) \in \mathbb{C}\{x\} \}.$$

Therefore, Theorem 4.2 follows from Lemma 4.3 given below and the following fact:

$$\tilde{\mathcal{O}}_+ \subset \mathcal{S}_+ \subset \mathcal{S}_{zero} \subset \mathcal{S}_*, \quad \text{and} \quad \tilde{\mathcal{O}}_+ \subset \tilde{\mathcal{O}}_{zero} \subset \mathcal{S}_{zero} \subset \mathcal{S}_*.$$

Lemma 4.3. (1) If $\operatorname{Re}\lambda(0) > 0$ and $h(x) \in \mathbb{C}\{x\}$, there are $a > 0$ and $R > 0$ such that

$$(4.6) \quad \max_{D_R} |h(x)t^{\lambda(x)}| = O(|t|^a) \quad (\text{as } t \rightarrow 0 \text{ in } S_\theta)$$

holds for any $\theta > 0$.

(2) If $\operatorname{Re}\lambda(0) \leq 0$ and

$$(4.7) \quad \max_{D_R} |h(x)t^{\lambda(x)}| = o(1) \quad (\text{as } t \rightarrow 0 \text{ in } S_\theta)$$

for some $R > 0$ and $\theta > 0$, we have $h(x) \equiv 0$.

Proof of Lemma 4.3. If $\operatorname{Re}\lambda(0) > 0$, we have $\operatorname{Re}\lambda(x) \geq a$ on D_R for some $a > 0$ and $R > 0$; then we can easily see the condition (4.6).

Let us show (2). If $\operatorname{Re}\lambda(0) < 0$, we may assume that $\operatorname{Re}\lambda(x) < -a$ on D_R for some $a > 0$, and then by (4.7) we have

$$\max_{D_R} |h(x)| \leq \max_{D_R} |h(x)t^{\lambda(x)}| \times \max_{D_R} |t^{-\lambda(x)}| = o(1) \times O(|t|^a)$$

(as $t \rightarrow 0$ in S_θ): this proves $h(x) \equiv 0$. If $\operatorname{Re}\lambda(0) = 0$ and $\operatorname{Re}\lambda(x) \neq 0$, we can find an $x_0 \in D_R$ such that $\operatorname{Re}\lambda(x_0) < 0$, and so by the same argument as above we have $h(x) \equiv 0$ in a neighborhood of $x = x_0$; this proves $h(x) \equiv 0$. If $\operatorname{Re}\lambda(x) \equiv 0$, we have $|t^{-\lambda(x)}| = O(1)$ (as $t \rightarrow 0$ in S_θ) and so by (4.7) we have

$$\max_{D_R} |h(x)| \leq \max_{D_R} |h(x)t^{\lambda(x)}| \times \max_{D_R} |t^{-\lambda(x)}| = o(1) \times O(1)$$

(as $t \rightarrow 0$ in S_θ): this proves $h(x) \equiv 0$. □

By this lemma we see: (1) if $\operatorname{Re}\lambda(0) > 0$ we have $h(x)t^{\lambda(x)} \in \tilde{\mathcal{O}}_+$ for any $h(x) \in \mathbb{C}\{x\}$, and (2) if $\operatorname{Re}\lambda(0) \leq 0$ and $h(x)t^{\lambda(x)} \in \mathcal{S}_{zero}$ hold we have $h(x) \equiv 0$. This completes the proof of Theorem 4.2. □

Since $\Psi[0] = \psi(t, x, 0, 0, \dots)$ is nothing but the unique holomorphic solution $u_0(t, x)$ obtained in Theorem 1.1, we have

Theorem 4.4 (Analytic continuation). *Suppose the conditions $A_1), A_2), A_3), \operatorname{Re}\lambda(0) \leq 0$ and $\lambda(0) \notin (-\infty, 0]$. If $u(t, x)$ is a holomorphic solution of (4.1) on $S_\theta(r) \times D_R$ satisfying $u(t, x) \rightarrow 0$ uniformly on D_R (as $t \rightarrow 0$ in $S_\theta(r)$), then $u(t, x)$ has an analytic continuation up to some neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$.*

Remark. The author believes that the analytic continuation in Theorem 4.4 is valid also in the case $\lambda(0) \in (-\infty, 0)$. But, to prove this we need a completely different proof, because the condition $\lambda(0) \notin (-\infty, 0]$ is the main part of our Poincaré condition (3.5.3) and it is essential to our transformation theory. In the case $\lambda(0) = 0$, we have the following example: the equation $t(\partial u/\partial t) = u(\partial u/\partial x)$ has a family of solutions $u = (x+c)/(-\log t)$ with $c \in \mathbb{C}$.

Now, let us see the concrete form of the solution $\Psi[h(x)t^{\lambda(x)}]$. By Proposition 3.2.3 we see that $\psi(t, x, w_0, w_1, \dots)$ has the form

$$\psi = \frac{a(x)}{1-\lambda(x)}t + w_0 + \sum_{k \geq 2} \left[\psi_{k,0}(x)t^k + \sum_{i+|j|=k, |j| \geq 1, i+2(|j|-1) \geq j} \psi_{i,j}(x)t^i w_0^{j_0} \dots w_{k-1}^{j_{k-1}} \right],$$

and so we have

$$(4.8) \quad \Psi[h(x)t^{\lambda(x)}] = \frac{a(x)}{1-\lambda(x)}t + h(x)t^{\lambda(x)} + \sum_{k \geq 2} \psi_{k,0}(x)t^k + \sum_{\substack{i+|j| \geq 2, |j| \geq 1 \\ i+2(|j|-1) \geq j}} \psi_{i,j}(x)t^i [h(x)t^{\lambda(x)}]^{j_0} \left[\left(\frac{\partial}{\partial x} \right) (h(x)t^{\lambda(x)}) \right]^{j_1} \times \dots \times \left[\left(\frac{\partial}{\partial x} \right)^{k-1} (h(x)t^{\lambda(x)}) \right]^{j_{k-1}}.$$

Since $(\partial/\partial x)^i(h(x)t^{\lambda(x)})$ is expressed in the form

$$\left(\frac{\partial}{\partial x} \right)^i (h(x)t^{\lambda(x)}) = \sum_{0 \leq p \leq i} h_p(x)t^{\lambda(x)}(\log t)^p, \quad 0 \leq i \leq k-1,$$

by substituting this into (4.8) we have the following result:

Proposition 4.5. *In the case $\text{Re}\lambda(0) > 0$, the $\tilde{\mathcal{O}}_+$ -solution $\Psi[h(x)t^{\lambda(x)}]$ (with $h(x) \in \mathbb{C}\{x\}$) has the expansion of the form*

$$(4.9) \quad \Psi[h(x)t^{\lambda(x)}] = \frac{a(x)}{1-\lambda(x)}t + \sum_{k \geq 2} \psi_{k,0}(x)t^k + h(x)t^{\lambda(x)} + \sum_{\substack{i+j \geq 2, j \geq 1 \\ i+2(j-1) \geq p}} h_{i,j,p}(x)t^{i+j\lambda(x)}(\log t)^p$$

for some holomorphic functions $\psi_{k,0}(x)$ ($k \geq 2$) and $h_{i,j,p}(x)$ ($i+j \geq 2, j \geq 1$ and $i+2(j-1) \geq p$) on D_R .

The case $\mathcal{F} = \tilde{\mathcal{O}}_+$ of Theorem 4.2 (and Proposition 4.5) is already written in the book of Gérard-Tahara [1] (chapter 5): the case $\mathcal{F} = \mathcal{S}_+$ is not written explicitly in [1], but one can easily see that the proof in [1] works also in the case $\mathcal{F} = \mathcal{S}_+$. Recall:

Theorem 4.6 (Theorem 5.2.3 in [1]). *Suppose the conditions $A_1), A_2), A_3)$ and $\lambda(0) \notin \{1, 2, \dots\}$. Then we have the following results.*

(1) *The equation (4.1) has a unique holomorphic solution $u_0(t, x)$ in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_x$ satisfying $u_0(0, x) \equiv 0$.*

(2) *We denote by $Sol(\tilde{\mathcal{O}}_+)$ (resp. $Sol(\mathcal{S}_+)$) the set of all the solutions of (4.1) belonging in the class $\tilde{\mathcal{O}}_+$ (resp. \mathcal{S}_+). We have:*

(2-1) *If $\operatorname{Re}\lambda(0) \leq 0$, we have*

$$(4.10) \quad Sol(\tilde{\mathcal{O}}_+) = Sol(\mathcal{S}_+) = \{u_0(t, x)\},$$

where $u_0(t, x)$ is the unique holomorphic solution obtained in (1).

(2-2) *If $\operatorname{Re}\lambda(0) > 0$, we have*

$$(4.11) \quad Sol(\tilde{\mathcal{O}}_+) = Sol(\mathcal{S}_+) = \{u_0(t, x)\} \cup \{U(\varphi); 0 \neq \varphi(x) \in \mathbb{C}\{x\}\},$$

where $U(\varphi)$ is an $\tilde{\mathcal{O}}_+$ -solution of (4.1) having the expansion of the following form:

$$U(\varphi) = \sum_{i \geq 1} u_i(x)t^i + \sum_{i+2j \geq p+2, j \geq 1} \varphi_{i,j,p}(x)t^{i+j\lambda(x)}(\log t)^p$$

with $\varphi_{0,1,0}(x) = \varphi(x)$.

§4.2. Solutions in a spiral domain

First, let us illustrate what we are thinking from now.

Example 4.7. Let us consider the equation

$$(4.12) \quad t \frac{du}{dt} = (-1 + i)u \quad (\text{where } i = \sqrt{-1}).$$

We have a general solution $u(t) = At^{-1+i}$ with an arbitrary constant $A \in \mathbb{C}$.

(1) As is seen in Lemma 4.3, if $At^{-1+i} = o(1)$ (as $t \rightarrow 0$ in S_θ) we have $A = 0$: therefore, if $u(t)$ is a solution of (4.12) belonging in the class \mathcal{S}_{zero} we have $u(t) \equiv 0$.

(2) But, if we consider the curve $C = \{t \in \mathcal{R}(\mathbb{C}_t \setminus \{0\}); 2 \log |t| + \arg t = 0\}$ we have $At^{-1+i} = o(1)$ (as $C \ni t \rightarrow 0$) for any $A \in \mathbb{C}$. This is verified as follows. Set $t = re^{i\phi}$; then we have

$$|t^{-1+i}| = |(re^{i\phi})^{-1+i}| = r^{-1}e^{-\phi} = e^{-\log r - \phi}$$

therefore $t^{-1+i} \rightarrow 0$ is equivalent to the condition $\log r + \phi \rightarrow \infty$. Since C is expressed as $C = \{t = re^{i\phi}; 2\log r + \phi = 0\}$, the condition $C \ni t \rightarrow 0$ implies that $\log r \rightarrow -\infty$ and so $\log r + \phi = -\log r \rightarrow \infty$ hold. This proves that $t^{-1+i} \rightarrow 0$ (as $C \ni t \rightarrow 0$).

(3) Thus, if we set

$$C = \{t \in \mathcal{R}(\mathbb{C}_t \setminus \{0\}); 2 \log |t| + \arg t = 0\},$$

$$\Omega = \{t \in \mathcal{R}(\mathbb{C}_t \setminus \{0\}); -1 < 2 \log |t| + \arg t < 1\}$$

we have the properties: i) C is a curve such that $t (\in C)$ can approach the origin along the curve C , ii) Ω is a neighborhood of C on which At^{-1+i} is a solution of (4.12), and iii) $At^{-1+i} \rightarrow 0$ (as $C \ni t \rightarrow 0$).

Motivated by this illustration, we will define:

Definition 4.8. We denote by \mathcal{C}_{zero} the set of all $u(t, x)$ such that there are a curve C in $\mathcal{R}(\mathbb{C} \setminus \{0\})$, a neighborhood Ω of C , and $R > 0$ which satisfy the following properties: i) $t (\in C)$ can approach the origin along the curve C , ii) $u(t, x)$ is a holomorphic function on $\Omega \times D_R$, and iii) we have

$$\max_{x \in D_R} |u(t, x)| = o(1) \quad (\text{as } C \ni t \rightarrow 0).$$

(Note that C , Ω and R may depend on $u(t, x)$.)

Then, we have

Theorem 4.9. Suppose the conditions $A_1), A_2), A_3)$ and (4.2). Then, the following two equations are equivalent:

$$(4.13) \quad t \frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right) \quad \text{in } \mathcal{C}_{zero},$$

$$(4.14) \quad t \frac{\partial w}{\partial t} = \lambda(x)w \quad \text{in } \mathcal{C}_{zero}.$$

As an application we have

Theorem 4.10. Suppose the conditions $A_1), A_2), A_3)$ and (4.2). Denote by $Sol(\mathcal{C}_{zero})$ the set of all the solutions of (4.1) belonging in the class \mathcal{C}_{zero} . Then we have

$$Sol(\mathcal{C}_{zero}) = \{\Phi[h(x)t^{\lambda(x)}]; h(x) \in \mathbb{C}\{x\}\}.$$

This follows from

Lemma 4.11. *If $\lambda(0) \notin (-\infty, 0]$, we can find a curve C in $\mathcal{R}(\mathbb{C} \setminus \{0\})$ such that $t \in C$ can approach the origin along the curve C and that*

$$(4.15) \quad \max_{x \in D_R} |t^{\lambda(x)}| = o(1) \quad (\text{as } C \ni t \rightarrow 0).$$

Proof. If $\operatorname{Re}\lambda(0) > 0$, by taking $C = \{t \in \mathbb{R}; 0 < t < \delta\}$ for a sufficiently small $\delta > 0$ we have (4.15). If $\operatorname{Re}\lambda(0) \leq 0$ and $\operatorname{Im}\lambda(0) > 0$, then we can find $a > 0, b > 0$ and $R > 0$ such that $|\operatorname{Re}\lambda(x)| \leq a$ and $\operatorname{Im}\lambda(x) > b$ on D_R hold; then for any $t = re^{i\phi}$ (with $\phi > 0$) we have

$$\max_{x \in D_R} |t^{\lambda(x)}| \leq r^{-a} e^{-b\phi} = e^{-a \log r - b\phi}$$

and so by setting $C = \{t = re^{i\phi}; 2a \log r + b\phi = 0, \phi > 0\}$ we have the condition (4.15). If $\operatorname{Re}\lambda(0) \leq 0$ and $\operatorname{Im}\lambda(0) < 0$, we can prove (4.15) in the same way. \square

Proof of Theorem 4.10. Since Theorem 4.9 is already known, we have only to notice the following facts: (i) in the case $|\operatorname{Re}\lambda(x)| \leq a$ and $\operatorname{Im}\lambda(x) > b$ on D_R (with $a > 0, b > 0$ and $R > 0$), the function $h(x)t^{\lambda(x)}$ is a solution of (4.14) in a spiral domain $\Omega = \{(t, x) \in \mathcal{R}(\mathbb{C}_t \setminus \{0\}) \times D_R; -1 < 2a \log |t| + b \arg t < 1, \log |t| < -A\}$ for some $A > 0$, and (ii) in the case $|\operatorname{Re}\lambda(x)| \leq a$ and $\operatorname{Im}\lambda(x) < -b$ on D_R (with $a > 0, b > 0$ and $R > 0$), the function $h(x)t^{\lambda(x)}$ is a solution of (4.14) in a spiral domain $\Omega = \{(t, x) \in \mathcal{R}(\mathbb{C}_t \setminus \{0\}) \times D_R; -1 < 2a \log |t| - b \arg t < 1, \log |t| < -A\}$ for some $A > 0$. \square

Similarly to Proposition 4.5 we have

Proposition 4.12. *In the case $\operatorname{Re}\lambda(0) \leq 0$ and $\lambda(0) \notin (-\infty, 0]$, the function $\Psi[h(x)t^{\lambda(x)}]$ (with $h(x) \in \mathbb{C}\{x\}$) defines a holomorphic solution of (4.1) in a spiral domain $\Omega_+ = \{(t, x) \in \mathcal{R}(\mathbb{C}_t \setminus \{0\}) \times D_R; -1 < 2a \log |t| + b \arg t < 1, \log |t| < -A\}$ or $\Omega_- = \{(t, x) \in \mathcal{R}(\mathbb{C}_t \setminus \{0\}) \times D_R; -1 < 2a \log |t| - b \arg t < 1, \log |t| < -A\}$ for some $R > 0, a > 0, b > 0$ and $A > 0$, and it has the expansion of the form*

$$(4.16) \quad \begin{aligned} \Psi[h(x)t^{\lambda(x)}] &= \frac{a(x)}{1 - \lambda(x)} t + \sum_{k \geq 2} \psi_{k,0}(x) t^k \\ &\quad + h(x)t^{\lambda(x)} + \sum_{\substack{i+j \geq 2, j \geq 1 \\ i+2(j-1) \geq p}} h_{i,j,p}(x) t^{i+j\lambda(x)} (\log t)^p \end{aligned}$$

for some holomorphic functions $\psi_{k,0}(x)$ ($k \geq 2$) and $h_{i,j,p}(x)$ ($i + j \geq 2, j \geq 1$ and $i + 2(j - 1) \geq p$) on D_R .

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