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Geometry. — *Configuration spaces of tori*, by YOEL FELER, communicated by F. Catanese.

ABSTRACT. — The n-point configuration spaces $\mathcal{E}^n(\mathbb{T}^2) = \{(q_1, \ldots, q_n) \in (\mathbb{T}^2)^n \mid q_i \neq q_j \ \forall\, i \neq j\}$ and $\mathcal{C}^n(\mathbb{T}^2) = \{Q \subset \mathbb{T}^2 \mid \#Q = n\}$ of a complex torus $\mathbb{T}^2$ are complex manifolds. We prove that for $n > 4$ any holomorphic self-map F of $\mathcal{C}^n(\mathbb{T}^2)$ either carries the whole of $\mathcal{C}^n(\mathbb{T}^2)$ into an orbit of the diagonal $(\operatorname{Aut}\mathbb{T}^2)$-action in $\mathcal{C}^n(\mathbb{T}^2)$ or is of the form $F(Q) = T(Q)Q$, where $T\colon \mathcal{C}^n(\mathbb{T}^2) \to \operatorname{Aut}\mathbb{T}^2$ is a holomorphic map. We also prove that for $n > 4$ any endomorphism of the torus braid group $B_n(\mathbb{T}^2) = \pi_1(\mathcal{C}^n(\mathbb{T}^2))$ with a non-abelian image preserves the pure torus braid group $P_n(\mathbb{T}^2) = \pi_1(\mathcal{E}^n(\mathbb{T}^2))$.

KEY WORDS: Configuration space; torus braid group; holomorphic endomorphism.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 32H02, 32H25, 32C18, 32M05, 14J50.

1. INTRODUCTION

The configuration space $\mathcal{C}^n(X)$ of a complex space X consists of all n-point subsets ("configurations") $Q = \{q_1, \ldots, q_n\} \subset X$, $\#Q = n$. The automorphism group $\operatorname{Aut} X$ acts in $\mathcal{C}^n(X)$ by $X \supset Q \mapsto AQ = \{Aq_1, \ldots, Aq_n\}$. If $\operatorname{Aut} X$ is a complex Lie group, any holomorphic map $T\colon \mathcal{C}^n(X) \to \operatorname{Aut} X$ produces the holomorphic self-map ("endomorphism") F_T of $\mathcal{C}^n(X)$, $F_T(Q) = T(Q)Q$; such a map F_T is called *tame*. Choosing a base point $Q^0 \in \mathcal{C}^n(X)$, define an endomorphism F_{T,Q^0} by $F_{T,Q^0}(Q) = T(Q)Q^0$; it *maps the whole configuration space into one orbit* $(\operatorname{Aut} X)Q^0$ *of the diagonal* $(\operatorname{Aut} X)$*-action in* $\mathcal{C}^n(X)$; maps that have the latter property are said to be *orbit-like*.

V. Lin [13, 15, 17] proved that when $n > 4$ and X is $\mathbb{C}$ or $\mathbb{CP}^1$, an endomorphism F of $\mathcal{C}^n(X)$ is either tame or orbit-like. The latter happens if and only if F is abelian, i.e. the image $F_*(\pi_1(\mathcal{C}^n(X)))$ under the induced endomorphism F_* of the fundamental group $\pi_1(\mathcal{C}^n(X))$ is abelian. (Recall that $\pi_1(\mathcal{C}^n(X))$ is the braid group $B_n(X)$ of X; it is non-abelian whenever $n \geq 3$.) Similar results were obtained by V. Zinde (see [22–26]) for $X = \mathbb{C}^*$.

Here we treat the endomorphisms of the configuration spaces of a torus $\mathbb{T}^2$, which completes the story for all non-hyperbolic Riemann surfaces.

Throughout the paper, $\operatorname{Aut}\mathbb{T}^2$ stands for the group of all biholomorphic ($\equiv$ biregular) self-mappings of $\mathbb{T}^2$.

DEFINITION 1.1. *A group homomorphism $\varphi\colon G \to H$ is called* abelian *if its image is abelian. A continuous map $F\colon X \to Y$ of path connected spaces is called* abelian *if the induced homomorphism $F_*\colon \pi_1(X) \to \pi_1(Y)$ is abelian.*

THEOREM 1.2. *For $n > 4$, each holomorphic map $F\colon \mathcal{C}^n(\mathbb{T}^2) \to \mathcal{C}^n(\mathbb{T}^2)$ is either tame or orbit-like; the latter happens exactly when F is abelian. Any automorphism of $\mathcal{C}^n(\mathbb{T}^2)$ is tame.*

COROLLARY 1.3. *For $n > 4$, the set $\mathcal{H}(\mathcal{C}^n(\mathbb{T}^2), \mathcal{C}^n(\mathbb{T}^2))$ of all holomorphic homotopy classes of non-abelian holomorphic endomorphisms of $\mathcal{C}^n(\mathbb{T}^2)$ is in natural one-to-one correspondence with the set $\mathcal{H}(\mathcal{C}^n(\mathbb{T}^2), \operatorname{Aut}\mathbb{T}^2)$ of all holomorphic homotopy classes of holomorphic maps $\mathcal{C}^n(\mathbb{T}^2) \to \operatorname{Aut}\mathbb{T}^2$.*

COROLLARY 1.4. *For $n > 4$, the orbits of the natural $(\operatorname{Aut}\mathcal{C}^n(\mathbb{T}^2))$-action in $\mathcal{C}^n(\mathbb{T}^2)$ coincide with the orbits of the diagonal $(\operatorname{Aut}\mathbb{T}^2)$-action in $\mathcal{C}^n(\mathbb{T}^2)$.*

Artin [1] proved that automorphisms of the braid group $B_n = B_n(\mathbb{C})$ preserve the pure braid group P_n. V. Lin [14–18] generalized this to non-abelian endomorphisms of $B_n(\mathbb{C})$ and $B_n(\mathbb{CP}^1)$; the case of $B_n(\mathbb{C}^*)$ was handled by V. Zinde [25, 26]. N. Ivanov [10] proved an analogue of Artin's theorem for automorphisms of braid groups of all Riemann surfaces of finite type but $\mathbb{CP}^1$. Our next theorem states an analogue of Lin's theorem for the torus braid group $B_n(\mathbb{T}^2) = \pi_1(\mathcal{C}^n(\mathbb{T}^2))$ and the pure torus braid group $P_n(\mathbb{T}^2)$, which is the fundamental group of the *ordered* configuration space $\mathcal{E}^n(\mathbb{T}^2) = \{(q_1, \dots, q_n) \in (\mathbb{T}^2)^n \mid q_i \neq q_j \ \forall i \neq j\}$. Part (b) of the next theorem is similar to results obtained in [14–26] for the braid groups of $\mathbb{C}$, $\mathbb{CP}^1$ and $\mathbb{C}^*$.

THEOREM 1.5. (a) *Let $n > 4$ and φ be a non-abelian endomorphism of $B_n(\mathbb{T}^2)$. Then $\varphi(P_n(\mathbb{T}^2)) \subseteq P_n(\mathbb{T}^2)$.*
(b) *For $n > \max\{m, 4\}$, any homomorphism $\varphi\colon B_n(\mathbb{T}^2) \to B_m(\mathbb{T}^2)$ is abelian.*

Let us outline the plan of the proof of Theorem 1.2. By Theorem 1.5(a), a non-abelian holomorphic self-map F of $\mathcal{C}^n(\mathbb{T}^2)$ fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{E}^n(\mathbb{T}^2) & \xrightarrow{f} & \mathcal{E}^n(\mathbb{T}^2) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{C}^n(\mathbb{T}^2) & \xrightarrow{F} & \mathcal{C}^n(\mathbb{T}^2) \end{array} \tag{1.1}$$

where $p\colon \mathcal{E}^n(\mathbb{T}^2) \ni q = (q_1, \dots, q_n) \mapsto \{q_1, \dots, q_n\} = Q \in \mathcal{C}^n(\mathbb{T}^2)$ is a Galois covering with Galois group $\mathbf{S}(n)$. The map f is non-constant, holomorphic and *strictly equivariant* with respect to the standard action of the symmetric group $\mathbf{S}(n)$ in $\mathcal{E}^n(\mathbb{T}^2)$, meaning that there is an automorphism α of $\mathbf{S}(n)$ such that $f(\sigma q) = \alpha(\sigma) f(q)$ for all $q \in \mathcal{E}^n(\mathbb{T}^2)$ and $\sigma \in \mathbf{S}(n)$. To study such maps f, we start with an explicit description of all non-constant holomorphic maps $\lambda\colon \mathcal{E}^n(\mathbb{T}^2) \to \mathbb{T}^2 \setminus \{0\}$. The set L of all such maps is finite and separates points of a certain submanifold $M \subset \mathcal{E}^n(\mathbb{T}^2)$ with codim $M = 1$; we endow L with a special simplicial structure. The action of $\mathbf{S}(n)$ in $\mathcal{E}^n(\mathbb{T}^2)$ induces a simplicial $\mathbf{S}(n)$-action in the complex L; the orbits of this action may be exhibited explicitly. A map f as above induces a simplicial self-map f^* of L, defined by $f^*\lambda = \lambda \circ f$ for $\lambda \in L$, which carries important information about f. Since f is strictly equivariant, f^* is nicely related to the $\mathbf{S}(n)$-action on L. Studying all these things together, we come to the desired conclusion.

2. SOME ALGEBRAIC PROPERTIES OF TORUS BRAID GROUPS

The main goal of this section is to prove Theorem 1.5.

By O. Zariski [21] (cf. J. Birman [2]), the torus braid group $B_n(\mathbb{T}^2)$ admits a presentation with $n+1$ generators $\sigma_1, \ldots, \sigma_{n-1}, a_1, a_2$ and the defining system of relations

(2.1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2,\ i, j = 1, \ldots, n-3$;

(2.2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, \ldots, n-2$;

(2.3) $\sigma_i a_k = a_k\sigma_i$ for $k = 1, 2$ and $i = 2, \ldots, n-1$;

(2.4) $(\sigma_1^{-1}a_k)^2 = (a_k\sigma_1^{-1})^2$ for $k = 1, 2$;

(2.5) $\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 = a_1a_2^{-1}a_1^{-1}a_2$;

(2.6) $a_2\sigma_1^{-1}a_1^{-1}\sigma_1a_2^{-1}\sigma_1^{-1}a_1\sigma_1 = \sigma_1^2$.

For $(a_1, \ldots, a_n) \in \mathcal{E}^n(\mathbb{T}^2)$ and $m = 1, \ldots, n-1$, set $\mathcal{E}^{n-m}(\mathbb{T}^2 \setminus \{a_1, \ldots, a_m\}) = \{(q_{m+1}, \ldots, q_n) \in (\mathbb{T}^2 \setminus \{a_1, \ldots, a_m\})^{n-m} \mid q_i \neq q_j\}$. For $m \leq n-2$, the maps $t_{m+1}\colon \mathcal{E}^{n-m}(\mathbb{T}^2 \setminus \{a_1, \ldots, a_m\}) \ni (q_{m+1}, \ldots, q_n) \mapsto q_{m+1} \in \mathbb{T}^2 \setminus \{a_1, \ldots, a_m\}$ and $t_1\colon \mathcal{E}^n(\mathbb{T}^2) \ni (q_1, \ldots, q_n) \mapsto q_1 \in \mathbb{T}^2$ define smooth locally trivial fibrings (see [6]) with fibres isomorphic respectively to $\mathcal{E}^{n-1}(\mathbb{T}^2 \setminus \{a_1\})$ and to $\mathcal{E}^{n-m-1}(\mathbb{T}^2 \setminus \{a_1, \ldots, a_{m+1}\})$. These spaces are aspherical and the final segments of the exact homotopy sequences of the above fibrings look as $1 \to P_{n-1;1} \to P_n(\mathbb{T}^2) \to \mathbb{Z}^2 \to 1$ and $1 \to P_{n-m-1;m+1} \to P_{n-m;m} \to \mathbb{F}_m \to 1$, where $P_{n-m;m} = \pi_1(\mathcal{E}^{n-m}(\mathbb{T}^2 \setminus \{a_1, \ldots, a_m\}))$ and $\mathbb{F}_m$ is a free group of rank m. This leads to the following well-known statement.

PROPOSITION 2.1. *The subgroups $P_{n-s;s}$ fit into the normal series* $\{1\} \subset P_{1;n-1} \subset \cdots \subset P_{n-m-1;m+1} \subset P_{n-m;m} \subset \cdots \subset P_{n-1;1} \subset P_{n;0} = P_n(\mathbb{T}^2)$ *with* $P_{1;n-1} \cong \mathbb{F}_{n-1}, \ldots, P_{n-m;m}/P_{n-m-1;m+1} \cong \mathbb{F}_m, \ldots, P_{n-1;1}/P_{n-2;2} \cong \mathbb{F}_2$, $P_n(\mathbb{T}^2)/P_{n-1;1} \cong \mathbb{Z}^2$.

COROLLARY 2.2. *Any non-trivial subgroup $H \subseteq P_n(\mathbb{T}^2)$ admits non-trivial homomorphisms to $\mathbb{Z}$. In particular, a group G with the finite abelianization $G/G' = G/[G,G]$ cannot have non-trivial homomorphisms to $P_n(\mathbb{T}^2)$.*

The exact homotopy sequence of the covering $p\colon \mathcal{E}^n(\mathbb{T}^2) \to \mathcal{C}^n(\mathbb{T}^2)$ looks as $1 \to P_n(\mathbb{T}^2) \xrightarrow{p_*} B_n(\mathbb{T}^2) \xrightarrow{\delta} \mathbf{S}(n) \to 1$, where $\delta(\sigma_i) = (i, i+1)$ for $i = 1, \ldots, n-1$ and $\delta(a_1) = \delta(a_2) = 1$. Let i be the homomorphism of the Artin braid group $B_n = \pi_1(\mathcal{C}^n(\mathbb{C}))$ to the torus braid group $B_n(\mathbb{T}^2)$ sending the standard generators $\sigma_1, \ldots, \sigma_{n-1}$ to the eponymous generators of $B_n(\mathbb{T}^2)$.

LEMMA 2.3. *Let $n > 4$ and let $\varphi\colon B_n(\mathbb{T}^2) \to B_m(\mathbb{T}^2)$ be a homomorphism such that the composition $\Phi = \delta \circ \varphi \circ i\colon B_n \xrightarrow{i} B_n(\mathbb{T}^2) \xrightarrow{\varphi} B_m(\mathbb{T}^2) \xrightarrow{\delta} \mathbf{S}(m)$ is abelian. Then φ is abelian. In particular, φ is abelian whenever $\delta \circ \varphi$ is.*

PROOF. Let $\Phi'\colon B_n' \to \mathbf{S}(m)$ be the restriction of Φ to the commutator subgroup $B_n' = [B_n, B_n]$. Since Φ is abelian, Φ' is trivial and hence $\varphi(i(B_n')) \subseteq \operatorname{Ker}\delta = P_m(\mathbb{T}^2)$. By the Gorin–Lin theorem [8], $B_n' = [B_n', B_n']$ for $n > 4$, and Corollary 2.2 shows that $\varphi(i(B_n')) = 1$. Hence $\varphi \circ i$ is abelian and (2.2) implies that

$\varphi(i(\sigma_1)) = \cdots = \varphi(i(\sigma_{n-1}))$; thus, (a) $\varphi(\sigma_1) = \cdots = \varphi(\sigma_{n-1})$. By (2.6), (a) and (2.3), we obtain (b) $\varphi(\sigma_2)^2 = \varphi(a_2)^{-1}\varphi(a_1)\varphi(a_2)\varphi(a_1)^{-1}$. By (a) and (2.5), we get (c) $(\varphi(\sigma_2))^{2(n-1)} = \varphi(a_1)\varphi(a_2)^{-1}\varphi(a_1)^{-1}\varphi(a_2)$. Multiplying the relations (b) and (c) we see that $(\varphi(\sigma_2))^{2n} = 1$. Since $B_m(\mathbb{T}^2)$ is torsion free (see [6, Theorem 8]), $\varphi(\sigma_2) = 1$ and, by (a) and (b), φ is abelian. □

PROOF OF THEOREM 1.5. Let $n > 4$ and let φ be a non-abelian endomorphism of $B_n(\mathbb{T}^2)$. By Lemma 2.3, the homomorphism $\Phi = \delta \circ \varphi \circ i\colon B_n \to \mathbf{S}(n)$ is non-abelian. By V. Lin's theorem (see [17, Sec. 4] or [15, 16, 18]), Φ coincides with the standard epimorphism $B_n \to \mathbf{S}(n)$ up to an automorphism of $\mathbf{S}(n)$; thus, the homomorphism $\delta \circ \varphi$ is surjective. N. Ivanov (see [10, Theorem 1]) proved that for $n > 4$ any non-abelian homomorphism $B_n(\mathbb{T}^2) \to \mathbf{S}(n)$ whose image is a primitive permutation group on n letters coincides with the standard epimorphism δ up to an automorphism of $\mathbf{S}(n)$. Therefore, $\mathrm{Ker}(\delta \circ \varphi) = P_n(\mathbb{T}^2) = \mathrm{Ker}\,\delta$, $\mathrm{Ker}(\delta \circ \varphi) = \mathrm{Ker}\,\delta = P_n(\mathbb{T}^2)$, which implies that $\varphi^{-1}(P_n(\mathbb{T}^2)) = P_n(\mathbb{T}^2)$ and a fortiori $\varphi(P_n(\mathbb{T}^2)) \subseteq P_n(\mathbb{T}^2)$.

To prove (b), we use another theorem of Lin ([17, Theorem 4.4]), which says that for $n > \max(m, 4)$ any homomorphism $B_n \to \mathbf{S}(m)$ is abelian; thus $\Phi = \delta\circ\varphi\circ i\colon B_n \to \mathbf{S}(m)$ is abelian. By Lemma 2.3, φ is abelian. □

3. ORDERED CONFIGURATION SPACES

3.1. *Holomorphic mappings $\mathcal{E}^n(\mathbb{T}^2) \to \mathbb{T}^2 \setminus \{0\}$*

DEFINITION 3.1. *We denote by $\mathfrak{M}$ the finite cyclic subgroup of* $\mathrm{Aut}\,\mathbb{T}^2$ *consisting of* $\pm\,\mathrm{id}$ *and all automorphisms of $\mathbb{T}^2$ induced by multiplication on the complex line by non-integral complex numbers. $\mathfrak{M}$ is isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$. Let $\mathfrak{M}_+$ consist of all $\mathfrak{m} \in \mathfrak{M}$ with* $0 \le \mathrm{Arg}\,\mathfrak{m} < \pi$, *i.e. $\mathfrak{M}_+$ consists of* 1, 2 *or* 3 *elements* (see [7, Chap. V, Sec. V.4.7]).

Let $\mathfrak{N}$ be a minimal generating set of the $\mathbb{Z}$-module of group endomorphisms of $\mathbb{T}^2$ (any endomorphism of the group $\mathbb{T}^2$ is induced by multiplication on the complex line by a complex number); $\#\,\mathfrak{N} = 2$ (see [20, Chap. VI, Sec. 5] and [12, Chap. 10]). *Moreover, either $\#\,\mathfrak{M}_+ = 1$ or we may assume that $\mathfrak{N} \subseteq \mathfrak{M}_+$.*

THEOREM 3.2. *Any non-constant holomorphic map $f : \mathcal{E}^n(\mathbb{T}^2) \to \mathbb{T}^2 \setminus \{0\}$ is of the form $f(q_1, \dots, q_n) = \mathfrak{m}(q_i - q_j)$ with certain $\mathfrak{m} \in \mathfrak{M}_+$ and $i \neq j$.*

To prove the theorem we need some preparation.

DEFINITION 3.3. *A configuration $(a_1, \dots, a_m) \in \mathcal{E}^m(\mathbb{T}^2)$ is called* exceptional *if there exist $i \neq j$ and an endomorphism λ of $\mathbb{T}^2$ such that $\lambda(a_i) = \lambda(a_j)$ and $\lambda^{-1}(\lambda(a_i)) \subseteq \{a_1, \dots, a_m\}$.*

LEMMA 3.4. (a) *The set A of all exceptional configurations $a \in \mathcal{E}^m(\mathbb{T}^2)$ is contained in a subvariety $M \subset \mathcal{E}^m(\mathbb{T}^2)$ of codimension* 1.

(b) *For any non-exceptional configuration $(a_1, \dots, a_m) \in \mathcal{E}^m(\mathbb{T}^2)$, every non-constant holomorphic map $\lambda\colon \mathbb{T}^2 \setminus \{a_1, \dots, a_m\} \to \mathbb{T}^2 \setminus \{0\}$ extends to a biregular automorphism of $\mathbb{T}^2$ sending a certain a_i to* 0.

PROOF. (a) Let N denote the union of all finite subgroups of order $\leq m$ in $\mathbb{T}^2$; this set is finite. Set $M = \{(a_1, \ldots, a_m) \in \mathcal{E}^m(\mathbb{T}^2) \mid a_j - a_i \in N$ for some $i \neq j\}$; then M is a subvariety in $\mathcal{E}^m(\mathbb{T}^2)$ of codimension 1. We show that $A \subseteq M$.

Let $a = (a_1, \ldots, a_m) \in A$ and let i, j, and λ be as in Definition 3.3. Set $\mu(t) = \lambda(t + a_i) - \lambda(a_i)$, $t \in \mathbb{T}^2$. Then $\mu(0) = 0$ and μ is a group homomorphism with finite kernel $\operatorname{Ker}\mu$ (see [3, Chap. 3, Sec. 3.1]). If $t \in \operatorname{Ker}\mu$, then $\lambda(t + a_i) = \lambda(a_i)$, $t + a_i \in \lambda^{-1}(\lambda(a_i)) \subseteq \{a_1, \ldots, a_m\}$ and $t \in \{a_1 - a_i, \ldots, 0, \ldots, a_m - a_i\}$, that is, $\operatorname{Ker}\mu \subseteq \{a_1 - a_i, \ldots, 0, \ldots, a_m - a_i\}$. In particular, $\#\operatorname{Ker}\mu \leq m$ and hence $\operatorname{Ker}\mu \subseteq N$. Since $\mu(a_j - a_i) = 0$, we have $a_j - a_i \in N$ and $a \in M$.

(b) Let $a = (a_1, \ldots, a_m) \notin A$. The map λ extends to a holomorphic self-map $\tilde{\lambda}$ of $\mathbb{T}^2$ (see H. Huber [9, §6, Satz 2]; also [11, Chap. VI, Sec. 2, remarks after Cor. 2.6]). By the Riemann–Hurwitz relation (see [7, Chap. I, Sec. I.2.7]), $\tilde{\lambda}$ is an unbranched regular covering map of degree $k < \infty$. Clearly $\tilde{\lambda}^{-1}(0) \subseteq \{a_1, \ldots, a_m\}$ and $\tilde{\lambda}(a_i) = 0$ for a certain i. Since $a \notin A$, for all $j \neq i$ we have $\tilde{\lambda}(a_j) \neq 0$, i.e. $\tilde{\lambda}^{-1}(0) = \{a_i\}$ and $\deg \tilde{\lambda} = 1$; thus, $\tilde{\lambda}$ is biregular. □

PROOF OF THEOREM 3.2. The proof is by induction on n. Since any holomorphic map $\mathcal{E}^1(\mathbb{T}^2) \cong \mathbb{T}^2 \to \mathbb{T}^2 \setminus \{0\}$ is constant, the base of induction is proved.

Assume that the assertion is already proved for some $n = m - 1 \geq 1$. For $a = (a_2, \ldots, a_m) \in \mathcal{E}^{m-1}(\mathbb{T}^2)$, denote by $\lambda_a = \lambda(\cdot, a_2, \ldots, a_m)$ the restriction of λ to the fibre $p^{-1}(a) = \mathbb{T}^2 \setminus \{a_2, \ldots, a_m\}$ of the map $p\colon \mathcal{E}^m(\mathbb{T}^2) \ni (q_1, q_2, \ldots, q_m) \mapsto (q_2, \ldots, q_m) \in \mathcal{E}^{m-1}(\mathbb{T}^2)$. It is clear that $S := \{a \in \mathcal{E}^{m-1}(\mathbb{T}^2) \mid \lambda_a = \text{const}\}$ is an analytic subset of $\mathcal{E}^{m-1}(\mathbb{T}^2)$, and either (i) $S = \mathcal{E}^{m-1}(\mathbb{T}^2)$ or (ii) $\dim_{\mathbb{C}} S \leq m - 2$. In case (i), $\lambda = \lambda(q_1, \ldots, q_m)$ does not depend on q_1 and may be considered as a holomorphic map $\mathcal{E}^{m-1}(\mathbb{T}^2) \to \mathbb{T}^2 \setminus \{0\}$; by the induction hypothesis, λ is of the desired form. Let us consider case (ii). By Lemma 3.4(a), the set A of all exceptional configurations is contained in a subvariety $M \subset \mathcal{E}^{m-1}(\mathbb{T}^2)$ of dimension $m - 2$. Let $a \in \mathcal{E}^{m-1}(\mathbb{T}^2) \setminus (S \cup M)$. Then $\lambda_a\colon \mathbb{T}^2 \setminus \{a_2, \ldots, a_m\}$ is a non-constant map. By Lemma 3.4(b), λ_a extends to an automorphism $\tilde{\lambda}_a$ of $\mathbb{T}^2$. Clearly, $\tilde{\lambda}_a(t) = \mathfrak{m}(t - a_i)$ with some $\mathfrak{m} = \mathfrak{m}_a \in \mathfrak{M}$ and $i = i_a$ (see [7, Chap. V, Sec. V.4.7]). Thus, for all $q = (q_1, \ldots, q_m)$ in the connected, everywhere dense set $\mathcal{E}^m(\mathbb{T}^2) \setminus p^{-1}(S \cup M)$ we have $(*)$ $\lambda(q) = \mathfrak{m}(q_1 - q_i)$ with certain $\mathfrak{m} = \mathfrak{m}_q \in \mathfrak{M}$ and $i = i_q$. Since $\mathfrak{M}$ is finite, $\mathfrak{m}$ and i do not depend on q, and $(*)$ holds true for all $q \in \mathcal{E}^m(\mathbb{T}^2)$, which completes the induction step, thus proving the theorem. □

DEFINITION 3.5. *For any* $\mathfrak{m} \in \mathfrak{M}_+$ *and* $i \neq j \in \{1, \ldots, n\}$, *the map* $e_{\mathfrak{m};i,j}\colon \mathcal{E}^n(\mathbb{T}^2) \ni (q_1, \ldots, q_n) \mapsto \mathfrak{m}(q_i - q_j) \in \mathbb{T}^2 \setminus \{0\}$ *is called a* difference. *For* $\mu = e_{\mathfrak{m};i,j}$, *the pair* $\{q_i, q_j\}$ *is called the* support *of* μ *and the automorphism* $\mathfrak{m} \in \mathfrak{M}_+$ *is called the* marker *of* μ. *We denote them by* $\operatorname{supp}\mu$ *and* $\mathfrak{m}_\mu$ *respectively.*

By Theorem 3.2, any non-constant holomorphic map $\mu : \mathcal{E}^n(\mathbb{T}^2) \to \mathbb{T}^2 \setminus \{0\}$ admits a unique representation in the form of a difference, i.e. $\mu = e_{\mathfrak{m};i,j}$ for some uniquely defined $\mathfrak{m} \in \mathfrak{M}_+$ and $i, j \in \{1, \ldots, n\}$.

3.2. *A simplicial structure on the set of differences*

For any connected complex space Y, V. Lin [17] introduced a natural simplicial structure on the set of all non-constant holomorphic functions $Y \to \mathbb{C} \setminus \{0, 1\}$. He used this structure in order to study $\mathbf{S}(n)$-equivariant endomorphisms of the ordered n-point configuration spaces of $\mathbb{C}$ and $\mathbb{CP}^1$. We modify this idea and define a similar simplicial structure on the set of all non-constant holomorphic maps $Y \to \mathbb{T}^2 \setminus \{0\}$. (V. Lin pointed out that the same construction applies to the set of all non-constant holomorphic maps $Y \to G \setminus \{\mathbf{e}\}$, where $\mathbf{e}$ is the unity element of a complex Lie group G.)

DEFINITION 3.6. *For a connected complex space Y, let $L(Y)$ denote the set of all non-constant holomorphic maps $\mu\colon Y \to \mathbb{T}^2 \setminus \{0\}$. For $\mu, \nu \in L(Y)$, we say that ν is a* proper reminder *of μ and write $\nu \mid \mu$ if $\mu - \nu \in L(Y)$. This relation is symmetric, i.e. $\nu \mid \mu$ is equivalent to $\mu \mid \nu$.*

*A subset $\Delta^m = \{\mu_0, \ldots, \mu_m\} \subseteq L(Y)$ is said to be an m-*simplex *if $\mu_i \mid \mu_j$ for all $i \neq j$. Since a subset of a simplex is also a simplex, we obtain a well-defined simplicial complex $L_\triangle(Y)$ with the set of vertices $L(Y)$.*

LEMMA 3.7. *Let $f\colon Z \to Y$ be a holomorphic map of connected complex spaces. Suppose that for each $\lambda \in L(Y)$ the map $f^*(\lambda) := \lambda \circ f\colon Z \xrightarrow{f} Y \xrightarrow{\lambda} \mathbb{T}^2 \setminus \{0\}$ is non-constant. Then $f^*\colon L(Y) \ni \lambda \mapsto \lambda \circ f \in L(Z)$ is a simplicial map and the restriction of f^* to $\Delta \in L_\triangle(Y)$ is injective. In particular,* $\dim f^*(\Delta) = \dim \Delta$.

PROOF. For any $\lambda \in L(Y)$, the map $f^*(\lambda)\colon Z \to \mathbb{T}^2 \setminus \{0\}$ is holomorphic and non-constant; hence $f^*(\lambda) \in L(Z)$. If $\mu, \nu \in L(Y)$ and $\mu \mid \nu$, then $\lambda = \mu - \nu \in L(Y)$ and $f^*(\mu) - f^*(\nu) = f^*(\mu - \nu) = f^*(\lambda) \in L(Z)$; consequently, $f^*(\mu) \mid f^*(\nu)$. This implies that f^* is simplicial and injective on any simplex. $\square$

REMARK 3.8. Clearly, for any regular dominant map $f\colon Y \to Z$ of non-singular irreducible algebraic varieties, we have $f^*(\lambda) \neq \text{const}$ for all $\lambda \in L(Y)$.

Notice that by Theorem 3.2, $L(\mathcal{E}^n(\mathbb{T}^2))$ is the set of all differences on $\mathcal{E}^n(\mathbb{T}^2)$.

LEMMA 3.9. *Suppose that either $\#\mathfrak{M}_+ < 3$ or $s > 1$. Let $\{\mu_0, \ldots, \mu_s\} \in L_\triangle(\mathcal{E}^n(\mathbb{T}^2))$ be an s-simplex. Then $\mathfrak{m}_{\mu_i} = \mathfrak{m}_{\mu_j}$, $\#(\operatorname{supp}\mu_i \cap \operatorname{supp}\mu_j) = 1$ for all $i \neq j$, and $\#(\operatorname{supp}\mu_0 \cap \cdots \cap \operatorname{supp}\mu_s) = 1$.*

PROOF. Let $\#\mathfrak{M}_+ < 3$, $i \neq j$ and let $\mu_i = \mathfrak{m}_i(q_{i'} - q_{i''})$ and $\mu_j = \mathfrak{m}_j(q_{j'} - q_{j''})$. Since $\mu_i \mid \mu_j$, we must have $\mu_i - \mu_j = \mathfrak{m}(q_{k'} - q_{k''})$ for some $\mathfrak{m} \in \mathfrak{M}_+$ and $k' \neq k''$. Thus, $\mathfrak{m}_i(q_{i'} - q_{i''}) - \mathfrak{m}_j(q_{j'} - q_{j''}) = \mathfrak{m}(q_{k'} - q_{k''})$. Since $\#\mathfrak{M}_+ < 3$, the latter relation can be fulfilled only if either $\mathfrak{m}_i q_{i'} - \mathfrak{m}_j q_{j'} = 0$ or $\mathfrak{m}_i q_{i''} - \mathfrak{m}_j q_{j''} = 0$. This implies $\mathfrak{m}_i = \mathfrak{m}_j$ and we have $(*)$ either $i' = j'$ or $i'' = j''$. If $s = 1$ we have finished the proof. If $s > 2$, then the property $\#(\operatorname{supp}\mu_i \cap \operatorname{supp}\mu_j) = 1$ implies immediately that $\#(\operatorname{supp}\mu_0 \cap \cdots \cap \operatorname{supp}\mu_s) = 1$. For $s = 2$ we have $\mu_0 = \mathfrak{m}(q_{i'} - q_{i''})$, $\mu_1 = \mathfrak{m}(q_{j'} - q_{j''})$ and $\mu_2 = \mathfrak{m}(q_{k'} - q_{k''})$. Since $\mu_0 \mid \mu_1$, $\mu_1 \mid \mu_2$ and $\mu_2 \mid \mu_0$, we obtain $\#(\operatorname{supp}\mu_0 \cap \operatorname{supp}\mu_1) = \#(\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2) = \#(\operatorname{supp}\mu_2 \cap \operatorname{supp}\mu_0) = 1$. Let $N = \#(\operatorname{supp}\mu_0 \cap \operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2)$. Clearly $N \leq 1$; let us show that $N \neq 0$. Suppose to the contrary that $N = 0$. Relations $(*)$ apply to μ_0 and μ_1, and without loss of generality we can assume that $i' = j'$. For μ_1 and μ_2 the same relations tell us that either $j' = k'$ or $j'' = k''$; since $N = 0$, the first case is impossible and we are left with $j'' = k''$. Finally, we apply $(*)$ to μ_0 and μ_2 and see that either $i' = k'$ or $i'' = k''$, which leads to a contradiction and completes the proof in the case $\#\mathfrak{M}_+ < 3$. By similar straightforward combinatorial computations, one can prove the lemma in the case $\#\mathfrak{M}_+ = 3$. □

The $\mathbf{S}(n)$-action in $\mathcal{E}^n(\mathbb{T}^2)$ induces an $\mathbf{S}(n)$-action in $L(\mathcal{E}^n(\mathbb{T}^2))$, defined by $(\sigma\lambda)(q) = \lambda(\sigma^{-1}q)$, which, in turn, induces a simplicial $\mathbf{S}(n)$-action in $L_\triangle(\mathcal{E}^n(\mathbb{T}^2))$ which preserves dimension of simplices; let us describe the orbits of this action.

DEFINITION 3.10. *We define the following* normal forms *of simplices of dimension* $s > 1$*:* $\Delta^s_{\mathfrak{m}} = \{e_{\mathfrak{m};1,2}, \dots, e_{\mathfrak{m};1,s+2}\}$, $\nabla^s_{\mathfrak{m}} = \{e_{\mathfrak{m};2,1}, \dots, e_{\mathfrak{m};s+2,1}\}$, *where* $\mathfrak{m} \in \mathfrak{M}_+$*; these simplices are called* normal.

LEMMA 3.11. *For* $s > 1$*, there are exactly* $\#\mathfrak{M}$ *orbits of the* $\mathbf{S}(n)$*-action on the set of all* s*-simplices. Every orbit contains exactly one normal simplex.*

PROOF. Since $e_{\mathfrak{m};a,b} \nmid e_{\mathfrak{m};b,c}$, Lemma 3.9 shows that for any s-simplex $\Delta \in L_\triangle(\mathcal{E}^n(\mathbb{T}^2))$ there exist $\mathfrak{m} \in \mathfrak{M}_+$ and distinct indices $a, b_0, \dots, b_s$ such that Δ equals either $\{e_{\mathfrak{m};a,b_0}, \dots, e_{\mathfrak{m};a,b_s}\}$ or $\{e_{\mathfrak{m};b_0,a}, \dots, e_{\mathfrak{m};b_s,a}\}$. An appropriate permutation $\sigma \in \mathbf{S}(n)$ carries Δ to a normal form. □

3.3. *Regular mappings* $\mathcal{E}^n(\mathbb{T}^2) \to \mathbb{T}^2$

LEMMA 3.12. *Any rational map* $\lambda\colon (\mathbb{T}^2)^n \to \mathbb{T}^2$ *is of the form*

$$\lambda(q_1, \dots, q_n) = \sum_{i=1}^{n} \sum_{\mathfrak{m}\in\mathfrak{N}} k_{i,\mathfrak{m}} \mathfrak{m} q_i + c,$$

where $k_{i,\mathfrak{m}} \in \mathbb{Z}$ *and* $c \in \mathbb{T}^2$*. In particular, it is regular.*

PROOF. The proof is by induction on n. Let $n = 1$. Since $\lambda\colon \mathbb{T}^2 \to \mathbb{T}^2$ is rational, it extends to a regular map (see [19, Chap. II, Sec. 3.1], Cor. 1). Any regular self-map of $\mathbb{T}^2$ is of the desired form (see Definition 3.1).

Assume that the theorem has already been proved for some $n = m - 1 \geq 1$. There is a subset $\Sigma \subset (\mathbb{T}^2)^m$ of codimension 1 such that λ is regular on $(\mathbb{T}^2)^m \setminus \Sigma$. Let $(t_0, z_0) \in (\mathbb{T}^2 \times (\mathbb{T}^2)^{m-1}) \setminus \Sigma$ and D be a small neighbourhood of z_0 in $(\mathbb{T}^2)^{m-1}$. Without loss of generality, we may assume that $t_0 = 0$ and $(0, z) \notin \Sigma$ for all $z \in D$. For $(t, z) \in (\mathbb{T}^2 \times D) \setminus \Sigma$, set $\mu(t, z) = \lambda(t, z) - \lambda(0, z)$ and $\nu(t, z) = \mu(t, z) - \mu(t, z_0)$. For any $z \in D$, we have $\nu(0, z) = 0$ and the map $t \mapsto \nu(t, z)$ extends to a holomorphic endomorphism ν_z of $\mathbb{T}^2$; moreover, $\nu_{z_0}(\mathbb{T}^2) = 0$. One can find a neighbourhood $D' \Subset D$ of z_0 and a compact subset $K \subset \mathbb{T}^2 \times D$ such that for all $z \in D'$ the set $K \cap (\mathbb{T}^2 \times \{z\})$ is a union of two loops that do not meet Σ and generate $\pi_1(\mathbb{T}^2 \times \{z\})$. Moreover, since $\nu(\mathbb{T}^2 \times \{z_0\}) = 0$, we may assume that $\nu(K)$ is contained in a small contractible neighbourhood of $0 \in \mathbb{T}^2$. Therefore for any $z \in D'$ the map ν_z is contractible and trivial. Thus, $\mu(t, z) - \mu(t, z_0) \equiv 0$ and $\lambda(t, z) \equiv \lambda(0, z) + \lambda(t, z_0) - \lambda(0, z_0)$ for all $z \in D'$. By the uniqueness theorem, the latter identity holds true for all $(t, z) \in (\mathbb{T}^2 \times (\mathbb{T}^2)^{m-1}) \setminus \Sigma$; the inductive hypothesis applies to $\lambda(0, z)$ and $\lambda(t, z_0)$, and completes the proof. □

4. HOLOMORPHIC MAPPINGS OF CONFIGURATION SPACES

The main goal of this section is to prove Theorem 1.2.

THEOREM 4.1. (a) *For $n > 4$ any non-abelian continuous map $F: \mathcal{C}^n(\mathbb{T}^2) \to \mathcal{C}^n(\mathbb{T}^2)$ admits a continuous lifting $f: \mathcal{E}^n(\mathbb{T}^2) \to \mathcal{E}^n(\mathbb{T}^2)$ (see diagram* (1.1)).
(b) *For $n > 4$ any continuous lifting $f: \mathcal{E}^n(\mathbb{T}^2) \to \mathcal{E}^n(\mathbb{T}^2)$ of a non-abelian continuous map $F: \mathcal{C}^n(\mathbb{T}^2) \to \mathcal{C}^n(\mathbb{T}^2)$ is strictly equivariant.*

PROOF. By the covering mapping theorem, (a) follows from Theorem 1.5. Let us prove (b). The diagram (1.1) for f and F implies that there is an epimorphism α of $\mathbf{S}(n)$ such that $\delta \circ F_* = \alpha \circ \delta$. Clearly, $f(\sigma q) = \alpha(\sigma) f(q)$ for all $q \in \mathcal{E}^n(\mathbb{T}^2)$ and $\sigma \in \mathbf{S}(n)$; moreover, α is an automorphism, otherwise its image is a non-trivial quotient of $\mathbf{S}(n)$, which must be abelian since $n > 4$. Then the homomorphism $\delta \circ F_* = \alpha \circ \delta$ is also abelian and, by Lemma 2.3, F_* is abelian, a contradiction. □

Let us show that every strictly equivariant map induces a simplicial map.

LEMMA 4.2. *Let $n > 2$ and $f = (f_1, \ldots, f_n): \mathcal{E}^n(\mathbb{T}^2) \to \mathcal{E}^n(\mathbb{T}^2)$ be a strictly equivariant holomorphic map. Then $f^*: L(\mathcal{E}^n(\mathbb{T}^2)) \ni \lambda \mapsto \lambda \circ f \in L(\mathcal{E}^n(\mathbb{T}^2))$ is a well-defined simplicial map; moreover, it preserves dimension of simplices.*

PROOF. By Lemma 3.7, we must only prove that $\mu \circ f \neq \text{const}$ for any $\mu \in L(\mathcal{E}^n(\mathbb{T}^2))$. Suppose to the contrary that $\mu \circ f = c \in \mathbb{T}^2$. Then $(\mu \circ f)(\sigma q) \equiv c$ for all $\sigma \in \mathbf{S}(n)$. Since f is strictly equivariant, there is $\alpha \in \operatorname{Aut} \mathbf{S}(n)$ such that $f(\sigma q) = \alpha(\sigma) f(q)$ for all $\sigma \in \mathbf{S}(n)$ and $q \in \mathcal{E}^n(X)$; thus $c \equiv \mu(f(\sigma q)) = \mu(\alpha(\sigma) f(q))$. By Theorem 3.2, $\mu = \mathfrak{m}(q_i - q_j)$ for some distinct i, j and $\mathfrak{m} \in \mathfrak{M}$; hence $c \equiv (\mu \circ f)(q) = \mathfrak{m}(f_i(q) - f_j(q))$. Since α is an automorphism and $n > 2$, there is $\sigma \in \mathbf{S}(n)$ such that $\alpha(\sigma^{-1})(i) = i$ and $\alpha(\sigma^{-1})(j) = k \neq j$; thus, $c \equiv \mu(\alpha(\sigma) f(q)) = \mathfrak{m}(f_{\alpha(\sigma^{-1})(i)}(q) - f_{\alpha(\sigma^{-1})(j)}(q)) = \mathfrak{m}(f_i(q) - f_k(q))$. Therefore, $\mathfrak{m}(f_i(q) - f_j(q)) = c = \mathfrak{m}(f_i(q) - f_k(q))$ and $f_j(q) = f_k(q)$, a contradiction. □

4.1. *Proof of Theorem 1.2*

We shall prove two theorems, which together yield Theorem 1.2.

THEOREM 4.3. *For $n > 4$, any non-abelian endomorphism F of $\mathcal{C}^n(\mathbb{T}^2)$ is tame.*

PROOF. By Theorems 1.5 and 4.1, the map F lifts to a strictly equivariant holomorphic map f that fits into the commutative diagram (1.1). Let α be the automorphism of $\mathbf{S}(n)$ corresponding to a strictly equivariant map f.

By Lemma 4.2, f^* is a dimension preserving simplicial self-map of $L_\Delta(\mathcal{E}^n(\mathbb{T}^2))$. Let $\Delta_1 = \{q_1 - q_2, \ldots, q_1 - q_n\}$ and $\Delta = f^*(\Delta_1)$. By Lemma 3.11, there is $\sigma \in \mathbf{S}(n)$ that brings Δ to its normal form; without loss of generality, we may assume that this normal form is $\nabla_{\mathfrak{m}} = \{\mathfrak{m}(q_2 - q_1), \ldots, \mathfrak{m}(q_n - q_1)\}$, where $\mathfrak{m} \in \mathfrak{M}_+$. Set $\tilde{f} = f \circ \sigma$; then $(*)$ $\tilde{f}_j = \tilde{f}_1 + \mathfrak{m}(q_1 - q_j)$ for any $j = 1, \ldots, n$. Define the holomorphic map $\tau: \mathcal{E}^n(\mathbb{T}^2) \to \operatorname{Aut}(\mathbb{T}^2)$ by the condition $\tau(q)(z) = \tau(q_1, \ldots, q_n)(z) = -\mathfrak{m}z + (\tilde{f}_1(q) + \mathfrak{m}q_1)$, where

$q = (q_1, \ldots, q_n) \in \mathcal{E}^n(\mathbb{T}^2)$ and $z \in \mathbb{T}^2$. Equations $(*)$ imply that $\tau(q)q_j = f_j(\sigma q)$ for all $j = 1, \ldots, n$ and $q = (q_1, \ldots, q_n) \in \mathcal{E}^n(\mathbb{T}^2)$; therefore $\tau(q)q = f(\sigma q) = \alpha(\sigma)f(q)$, or, what is the same, $f(q) = \alpha(\sigma^{-1})\tau(q)q$ for all $q \in \mathcal{E}^n(\mathbb{T}^2)$. To complete the proof, we must check that τ is $\mathbf{S}(n)$-invariant; that is, we must prove that $\tau(sq) = \tau(q)$ for all $q \in \mathcal{E}^n(\mathbb{T}^2)$ and all $s \in \mathbf{S}(n)$. For every $s \in \mathbf{S}(n)$ and all $q \in \mathcal{E}^n(\mathbb{T}^2)$ we have $\tau(sq)sq = f(\sigma sq) = f(\sigma s\sigma^{-1}\sigma q) = \alpha(\sigma s\sigma^{-1})f(\sigma q) = \alpha(\sigma s\sigma^{-1})\tau(q)q$. Thus, $(**)$ $[(\tau(sq))^{-1} \cdot \tau(q)]q = \alpha(\sigma s^{-1}\sigma^{-1})sq$, where $((\tau(sq))^{-1} \cdot \tau(q)) \in \operatorname{Aut}\mathbb{T}^2$ is the product in the group $\operatorname{Aut}\mathbb{T}^2$. Let us notice that for $n > 1$ there is a non-empty Zariski open subset $A \subset \mathcal{E}^n(\mathbb{T}^2)$ such that if $\theta q = \rho q$ for some $q \in A$, $\theta \in \operatorname{Aut}\mathbb{T}^2$ and $\rho \in \mathbf{S}(n)$, then $\theta = \mathrm{id}$ and $\rho = 1$. Therefore, equation $(**)$ implies $\tau(sq) = \tau(q)$ and $\alpha(\sigma s^{-1}\sigma^{-1})s = 1$ for all $q \in A$ and all $s \in \mathbf{S}(n)$. Clearly, $\alpha(s) = \sigma^{-1}s\sigma$ and the continuity of τ implies that $\tau(sq) = \tau(q)$ holds true for all $q \in \mathcal{E}^n(\mathbb{T}^2)$ and all $s \in \mathbf{S}(n)$. □

REMARK 4.4. Let $n = 3$ or 4. The statement of Theorem 4.3 still holds true if F is an automorphism. The only changes we need to make in the proof are as follows: instead of our Theorem 4.1, we use Theorem 2 from [10], which states that $P_n(\mathbb{T}^2)$ is a characteristic subgroup of $B_n(\mathbb{T}^2)$; moreover, instead of Lemma 4.2, we use Remark 3.8. The rest of the proof is the same.

REMARK 4.5. (a) *Let $n \geq 2$ and let F be a tame endomorphism of $\mathcal{C}^n(\mathbb{T}^2)$. Then a morphism $T : \mathcal{C}^n(\mathbb{T}^2) \to \operatorname{Aut}\mathbb{T}^2$ in the 'tame representation' $F = F_T$ of F is uniquely determined by F.* Indeed, if $F_T = F_{T'}$ for two morphisms T, T', then $T(Q)Q = T'(Q)Q$ and $(*)$ $[T(Q)]^{-1}T'(Q)Q = Q$ for all $Q \in \mathcal{C}^n(\mathbb{T}^2)$. Furthermore, a torus automorphism is uniquely determined by its values at a generic pair of distinct points; since $n \geq 2$, the identity $(*)$ shows that $[T(Q)]^{-1}T'(Q) = \mathrm{id}$ for any generic point $Q \in \mathcal{C}^n(\mathbb{T}^2)$ and hence $T = T'$.

(b) In view of Theorem 4.3, (a) shows that *for $n > 4$ any holomorphic non-abelian map $F : \mathcal{C}^n(\mathbb{T}^2) \to \mathcal{C}^n(\mathbb{T}^2)$ admits a unique tame representation $F = F_T$ and T is regular whenever F is*. By Remark 4.4, the same statement still holds true whenever $n = 3, 4$ and F is a (biregular) automorphism.

DEFINITION 4.6. *The map $s : \mathcal{C}^n(\mathbb{T}^2) \ni Q = \{q_1, \ldots, q_n\} \mapsto s(Q) = (q_1 + \cdots + q_n) \in \mathbb{T}^2$ is a locally trivial holomorphic fibring whose fibre $M_0 = s^{-1}(0)$ is an irreducible quasiprojective variety. The presentation of $\pi_1(M_0)$, found by O. Zariski [21], shows that $H_1(M_0, \mathbb{Z}) = \mathbb{Z}_{2n}$.*

Let $\gamma : \mathbb{C} \to \mathbb{T}^2$ be the universal covering; then there exists a holomorphic covering $h : M_0 \times \mathbb{C} \ni (Q, \zeta) \mapsto h(Q, \zeta) = \{q_1 + \gamma(\zeta), \ldots, q_n + \gamma(\zeta)\} \in \mathcal{C}^n(\mathbb{T}^2)$.

The following theorem completes the classification of self-maps of $\mathcal{C}^n(\mathbb{T}^2)$.

THEOREM 4.7. *If $m > 2$, then a holomorphic map $F : \mathcal{C}^n(\mathbb{T}^2) \to \mathcal{C}^m(\mathbb{T}^2)$ is orbit-like if and only if it is abelian.*

PROOF. Let F be abelian. Clearly, $H_1(\mathcal{C}^n(\mathbb{T}^2), \mathbb{Z}) = B_n(\mathbb{T}^2)/B_n'(\mathbb{T}^2) = \mathbb{Z}_2 \oplus \mathbb{Z}^2$. As $B_m(\mathbb{T}^2)$ has no elements of a finite order, the image $\operatorname{Im} F_*$ of the induced homomorphism $F_* : B_n(\mathbb{T}^2) \to B_m(\mathbb{T}^2)$ is a free abelian group. Since $\pi_1(M_0)/(\pi_1(M_0))' = \mathbb{Z}_{2n}$, any homomorphism $\pi_1(M_0) \to \operatorname{Im} F_*$ is trivial; in particular, the homomorphism $(F \circ h)_*$ is trivial, where $h : M_0 \times \mathbb{C} \to \mathcal{C}^n(\mathbb{T}^2)$ is the above-defined covering. This implies that there is a holomorphic map $f = (f_1, \dots, f_m)\colon M_0 \times \mathbb{C} \to \mathcal{E}^m(\mathbb{T}^2)$ such that $F \circ h = p \circ f$, where $p\colon \mathcal{E}^m(\mathbb{T}^2) \to \mathcal{C}^m(\mathbb{T}^2)$ is the standard projection. The induced homomorphism $f_*\colon \pi_1(M_0) \to P_n(\mathbb{T}^2)$ is trivial; thus, for any j, the map $(q_j - q_1) \circ f\colon M_0 \times \mathbb{C} \xrightarrow{f} \mathcal{E}^m(\mathbb{T}^2) \xrightarrow{q_j - q_1} \mathbb{T}^2 \setminus \{0\}$ is contractible and lifts to a holomorphic map $g_j\colon M_0 \times \mathbb{C} \to \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Since $M_0 \times \mathbb{C}$ is algebraic and irreducible, Liouville's theorem shows that $g_j =$ const and, therefore, $f_j - f_1 = (q_j - q_1) \circ f = c_j \in \mathbb{T}^2 \setminus \{0\}$. Thus, $f(q) = (0 + f_1(q), c_2 + f_1(q), \dots, c_m + f_1(q))$ and F is orbit-like.

Suppose now that F is orbit-like. To prove that F is abelian, it suffices to show that for any point $q \in \mathcal{C}^m(\mathbb{T}^2)$, the fundamental group of any connected component of the $(\operatorname{Aut}\mathbb{T}^2)$-orbit $\mathcal{O}_q = (\operatorname{Aut}\mathbb{T}^2)(q)$ is abelian. For $m > 2$, any component of $\mathcal{O}_q$ is diffeomorphic to the orbit $\mathcal{O}_q^*$ of the action of $\mathbb{T}^2$ in $\mathcal{C}^m(\mathbb{T}^2)$ by translations. The latter orbit $\mathcal{O}_q^*$ is a quotient group of $\mathbb{T}^2$ by a finite subgroup and hence is homeomorphic to $\mathbb{T}^2$. Thus, $\pi_1(\mathcal{O}_q^*) = \mathbb{Z}^2$. □

We skip the proof of the next result about abelian maps.

PROPOSITION 4.8. (a) *Any abelian map $f\colon \mathcal{C}^n(\mathbb{T}^2) \to \mathcal{C}^m(\mathbb{T}^2)$ is homotopically equivalent to a composition $g \circ s$ of the standard map $s\colon \mathcal{C}^n(\mathbb{T}^2) \to \mathbb{T}^2$ and an appropriate continuous map $g\colon \mathbb{T}^2 \to \mathcal{C}^m(\mathbb{T}^2)$.*

(b) *Any holomorphic map $F\colon \mathbb{T}^2 \to \mathcal{C}^n(\mathbb{T}^2)$ is orbit-like.*

5. BIREGULAR AUTOMORPHISMS

Here we describe all biregular automorphisms of the algebraic variety $\mathcal{C}^n(\mathbb{T}^2)$.

LEMMA 5.1. *Any regular map $R\colon \mathcal{C}^n(\mathbb{T}^2) \to \mathbb{T}^2$ is of the form*

$$R(\{q_1, \dots, q_n\}) = \sum_{\mathfrak{m} \in \mathfrak{N}} k_{\mathfrak{m}} \mathfrak{m}(q_1 + \dots + q_n) + c,$$

where $k_{\mathfrak{m}} \in \mathbb{Z}$ and $c \in \mathbb{T}^2$.

PROOF. Consider the map $r = R \circ p$, where $p\colon \mathcal{E}^n(\mathbb{T}^2) \to \mathcal{C}^n(\mathbb{T}^2)$ is the standard projection. By Lemma 3.12, $r(q_1, \dots, q_n) \equiv \sum_{i=1}^n \sum_{\mathfrak{m} \in \mathfrak{N}} k_{i,\mathfrak{m}} \mathfrak{m} q_i + c$. Since r is $\mathbf{S}(n)$-invariant, it follows that $k_{1,\mathfrak{m}} = \dots = k_{n,\mathfrak{m}} = k_{\mathfrak{m}}$. Thus, $r(q_1, \dots, q_n) \equiv \sum_{\mathfrak{m} \in \mathfrak{N}} k_{\mathfrak{m}} \mathfrak{m}(q_1 + \dots + q_n) + c$ and $R(\{q_1, \dots, q_n\}) \equiv \sum_{\mathfrak{m} \in \mathfrak{N}} k_{\mathfrak{m}} \mathfrak{m}(q_1 + \dots + q_n) + c$. □

THEOREM 5.2. *For $n > 2$, any biregular automorphism F of $\mathcal{C}^n(\mathbb{T}^2)$ is of the form $F(Q) = AQ$, where $A \in \operatorname{Aut}\mathbb{T}^2$.*

PROOF. By Theorem 1.2 and Remarks 4.4 and 4.5, there is a unique regular map $T\colon \mathcal{C}^n(\mathbb{T}^2) \to \operatorname{Aut}\mathbb{T}^2$ such that $F(Q) = T(Q)Q$ for all $Q = \{q_1, \dots, q_n\} \in \mathcal{C}^n(\mathbb{T}^2)$.

Since $T(Q) \in \operatorname{Aut}\mathbb{T}^2$, there exist a regular map $R\colon \mathcal{C}^n(\mathbb{T}^2) \to \mathbb{T}^2$ and $\mathfrak{m}_0 \in \mathfrak{M}$ such that $T(Q)z = \mathfrak{m}_0 z + R(Q)$ for all $z \in \mathbb{T}^2$ (see [7, Chap. V, Sec. V.4.7]). Together with Lemma 5.1, this implies that for any $z \in \mathbb{T}^2$ we have $T(Q)z = \mathfrak{m}_0 z + \sum_{\mathfrak{m} \in \mathfrak{N}} k_{\mathfrak{m}} \mathfrak{m}(q_1 + \dots + q_n) + c$, where $\mathfrak{m}_0 \in \mathfrak{M}$, $k_{\mathfrak{m}} \in \mathbb{Z}$, and $c \in \mathbb{T}^2$ do not depend on z and Q. Recall that $s(Q) = q_1 + \dots + q_n$ for $Q = \{q_1, \dots, q_n\}$ and set $s_1 = s \circ F$, i.e. $s_1(Q) = (s \circ F)(Q) = s(T(Q)Q) = T(Q)q_1 + \dots + T(Q)q_n$. Using the explicit formula for $T(Q)z$ for

$z = q_1, \dots, q_n$, we see that $s_1(Q) = \mathfrak{m}_0 s(Q) + n(\sum_{\mathfrak{m}\in\mathfrak{N}} k_{\mathfrak{m}} \mathfrak{m} s(Q) + c)$. The latter implies $(*)$ $s_1(Q) = (\mathfrak{m}_0 + n\sum_{\mathfrak{m}\in\mathfrak{N}} k_{\mathfrak{m}}\mathfrak{m})s(Q) + nc$. On the other hand, F^{-1} is a regular self-map as well. Similarly, there is a unique regular $T'\colon \mathcal{C}^n(\mathbb{T}^2) \to \operatorname{Aut}\mathbb{T}^2$ such that $F^{-1}(Q) = T'(Q)Q$ for $Q \in \mathcal{C}^n(\mathbb{T}^2)$; since $s_1 \circ F^{-1} = s$, $s(Q) = s_1(\{T'(Q)q_1, \dots, T'(Q)q_n\})$ and $(*)$ implies $(**)$ $s(Q) = (\mathfrak{m}_0 + n\sum_{\mathfrak{m}\in\mathfrak{N}} k_{\mathfrak{m}}\mathfrak{m})(T'(Q)q_1 + \cdots + T'(Q)q_n) + nc$. As above, we conclude that $T'(Q)z = \mathfrak{m}'_0 z + \sum_{\mathfrak{m}\in\mathfrak{N}} k'_{\mathfrak{m}}\mathfrak{m}s(Q) + c'$ for any $z \in \mathbb{T}^2$, where $\mathfrak{m}'_0 \in \mathfrak{M}$, $k'_{\mathfrak{m}} \in \mathbb{Z}$ and $c' \in \mathbb{T}^2$ do not depend on z and Q. Thus, $(**)$ is equivalent to $s(Q) = (\mathfrak{m}_0 + n\sum_{\mathfrak{m}\in\mathfrak{N}} k_{\mathfrak{m}}\mathfrak{m})(\mathfrak{m}'_0 + n\sum_{\mathfrak{m}\in\mathfrak{N}} k'_{\mathfrak{m}}\mathfrak{m})s(Q) + \text{const}$. Since $\operatorname{Im} s = \mathbb{T}^2$, the latter shows that the composition $\lambda = \mu \circ \nu = \nu \circ \mu$ of the endomorphisms $\mu\colon z \mapsto (\mathfrak{m}_0 + n\sum_{\mathfrak{m}\in\mathfrak{N}} k_{\mathfrak{m}}\mathfrak{m})z$ and $\nu\colon z \mapsto (\mathfrak{m}'_0 + n\sum_{\mathfrak{m}\in\mathfrak{N}} k'_{\mathfrak{m}}\mathfrak{m})z$ is the identity. Hence μ and ν are group automorphisms and $\mu(z) \equiv \mathfrak{m}_1 z$ with $\mathfrak{m}_1 \in \mathfrak{M}$; clearly, $(\mathfrak{m}_0 - \mathfrak{m}_1 + n\sum_{\mathfrak{m}\in\mathfrak{N}} k_{\mathfrak{m}}\mathfrak{m})z \equiv 0$, i.e. $\mathfrak{m}_0 - \mathfrak{m}_1 + n\sum_{\mathfrak{m}\in\mathfrak{N}} k_{\mathfrak{m}}\mathfrak{m} = 0$. Since $n > 2$ and elements of $\mathfrak{N}$ are linearly independent over $\mathbb{Q}$, the latter implies $k_{\mathfrak{m}} = 0$ for all $\mathfrak{m} \in \mathfrak{N}$. Thus $T(Q)z = \mathfrak{m}_0 z + c$ for all Q and z. $\square$

6. Configuration spaces of universal families

Here we construct configuration spaces of the universal Teichmüller family of tori and describe their holomorphic self-maps.

The Teichmüller space $T(1,1)$ of tori with one marked point is isomorphic to the upper half plane $\mathbb{H}^+$. The group $H = \mathbb{Z} \times \mathbb{Z}$ acts discontinuously and freely in the space $\mathcal{V} = T(1,1) \times \mathbb{C} = \mathbb{H}^+ \times \mathbb{C}$ by weighted translations $(\tau, z) \mapsto (\tau, z + l + m\tau)$, $(l, m) \in H$. Let $V(1,1) = \mathcal{V}/H$; the map $\psi\colon \mathcal{V} \to V(1,1)$ is a covering, and the holomorphic projection $\pi\colon V(1,1) \to \mathbb{H}^+ = T(1,1)$ is called the *universal Teichmüller family* of tori with one marked point (see [4, Sec. 4.11]). All fibres $\pi^{-1}(\tau)$ are tori; each of them carries a natural group structure, marked points are neutral elements and they form a holomorphic section of π.

DEFINITION 6.1. *Let $\mathcal{C}^n_\pi(V(1,1))$ be the complex subspace of the configuration space $\mathcal{C}^n(V(1,1))$ of $V(1,1)$ consisting of all $Q = \{q_1, \dots, q_n\} \in \mathcal{C}^n(V(1,1))$ such that $\pi(q_1) = \cdots = \pi(q_n)$. Define the holomorphic projection $\rho\colon \mathcal{C}^n_\pi(V(1,1)) \to T(1,1)$ by $\rho(Q) = \pi(q_1) = \cdots = \pi(q_n)$, $Q = \{q_1, \dots, q_n\} \in \mathcal{C}^n_\pi(V(1,1))$; the triple $\{\rho, \mathcal{C}^n_\pi(V(1,1)), T(1,1)\}$, or simply $\rho\colon \mathcal{C}^n_\pi(V(1,1)) \to T(1,1)$, is called the* fibred configuration space of the universal Teichmüller family $\pi\colon V(1,1) \to T(1,1)$ (cf. M. Engber [5]). *A* fibred morphism *of fibred configuration spaces is a holomorphic map $F\colon \mathcal{C}^n_\pi(V(1,1)) \to \mathcal{C}^m_\pi(V(1,1))$ which respects the projection ρ, that is, $\rho \circ F = \rho$. One can easily check that $\mathcal{C}^n_\pi(V(1,1))$ is a connected complex manifold.*

DEFINITION 6.2. *Let $g\colon \mathcal{C}^n_\pi(V(1,1)) \to V(1,1)$ be a fibred morphism. Any point $Q \in \mathcal{C}^n_\pi(V(1,1))$ belongs to a certain fibre $\rho^{-1}(\tau)$, which is the configuration space $\mathcal{C}^n(\pi^{-1}(\tau))$ of the torus $\mathbb{T}^2_\tau = \pi^{-1}(\tau)$; so Q may be viewed as an n-point subset of $\mathbb{T}^2_\tau$. Since g is a fibred morphism, $g(Q)$ is a point of the same torus $\mathbb{T}^2_\tau$; thus, $Q + g(Q)$ and $-Q + g(Q)$ are well-defined n-point subsets of $\mathbb{T}^2_\tau$, or, which is the same, points of $\mathcal{C}^n(\mathbb{T}^2_\tau) \subset \mathcal{C}^n_\pi(V(1,1))$. This provides us with two fibred maps $G_\pm = \pm\operatorname{Id} + g\colon \mathcal{C}^n_\pi(V(1,1)) \to \mathcal{C}^n_\pi(V(1,1))$ defined by $Q \mapsto \pm Q + g(Q)$. It can be easily shown that the fibred maps $G_\pm$ are holomorphic.*

One can prove statements analogous to Theorem 1.2 for the case of fibred morphisms. For instance, we sketch the proof of the following theorem.

THEOREM 6.3. *Let* $n > 4$ *and* $F : \mathcal{C}^n_\pi(V(1,1)) \to \mathcal{C}^n_\pi(V(1,1))$ *be a fibred non-abelian morphism. There exists a fibred morphism* $g : \mathcal{C}^n_\pi(V(1,1)) \to V(1,1)$ *such that* F *is either* $\mathrm{Id} + g$ *or* $-\mathrm{Id} + g$.

SKETCH OF PROOF. According to Theorem 1.2, for any $\tau \in T(1,1)$ there exists a unique holomorphic map $T_\tau : \rho^{-1}(\tau) \to \mathrm{Aut}\,\pi^{-1}(\tau)$ such that $F(Q) = T_\tau(Q)Q$ for any $Q \in \rho^{-1}(\tau) \subset \mathcal{C}^n_\pi(V(1,1))$. There is no complex multiplication on a generic torus. Thus, for any generic $\tau \in T(1,1)$ and any $Q \in \rho^{-1}(\tau)$, there exists $c_\tau(Q) \in \pi^{-1}(\tau)$ such that the automorphism $T_\tau(Q)$ maps a point $z \in \pi^{-1}(\tau)$ either to $z + c_\tau(Q)$ or to $-z + c_\tau(Q)$. Since the representation of $T_\tau(Q)$ is unique, F is continuous and the fibred configuration spaces are irreducible, it can be easily seen that for all $Q \in \mathcal{C}^n_\pi(V(1,1))$ only one of the above-mentioned possibilities takes place; moreover, there exists a fibred morphism $g : \mathcal{C}^n_\pi(V(1,1)) \to V(1,1)$ such that $c_\tau(Q) = g(Q)$. $\square$

REMARK 6.4. For an automorphism F the above statement holds true for $n = 3, 4$.

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