Calculus of variations. — *The Grötzsch problem in higher dimensions*, by TOMASZ ADAMOWICZ, communicated by C. Sbordone on 12 January 2007.

ABSTRACT. — An extension of the Grötzsch problem to higher dimensions is considered. The problem is formulated and solved for a subclass of polyconvex energy integrals and counterexamples in the general case are given. A conjecture about generalized distortion functions is stated.

1. INTRODUCTION

The purpose of this paper is to extend the Grötzsch problem in the plane to higher dimensions (see below or [2] for formulation of the problem). This generalization is obtained for a wide class of polyconvex energy integrals under certain conditions imposed on them. The motivation for our work comes from recent developments in the theory of mappings with integrable distortion [3]—a promising, dynamically growing branch of the calculus of variations.

The paper is organized as follows. In Section 2 we briefly recall the classical planar Grötzsch problem of finding a nearly conformal map between two rectangles. This classical framework employs the supremum norm of the distortion function.

Section 3 features new distortion functions in $\mathbb{R}^n$.

Section 4 is an epitomized survey of some of the recent developments in the theory of extremal problems for mappings with integrable distortion. A definition of the Grötzsch property is given. Roughly speaking, an energy integral has the Grötzsch property if its minimum among admissible mappings is attained at a linear one. This property is very much reminiscent of quasiconvexity, introduced by C. B. Morrey in 1952, [8].

Section 5 deals with the notion of polyconvex functions; some basic properties are listed.

The purpose of Section 6 is to formulate and prove the main result of the paper, Theorem 3 in Section 6.2. This is arranged in a chain of auxiliary theorems, corollaries and technical details. It is shown that a large class of polyconvex functionals have the Grötzsch property. However, we give several counterexamples to an analogous theorem in a more general setting.

Section 7 is devoted to formulation and explanation of some open problems. We raise a conjecture about the most general form of distortion functions. Also we pose a question concerning relations between the Grötzsch property, quasiconvexity and rank-one convexity.

2. GRÖTZSCH PROBLEM IN THE PLANE

This introductory section is based on Ahlfors' book [2], so the interested reader should consult this excellent source for a more detailed exposition of the subject.

The following problem often appears in the course of Complex Analysis.

Consider two rectangles R and R′ in the plane. When does there exist a conformal mapping of R onto R′ which takes vertices to vertices?

A necessary and sufficient condition turns out to be that the rectangles are similar. Moreover, the similarity map is the only such conformal equivalence between R and R', modulo orthogonal automorphisms of the rectangles.

In 1928 H. Grötzsch [4] asked about more general homeomorphism (not necessarily conformal) between two given rectangles which is nearly conformal. This led him to the notion of quasiconformality; this name was coined by Ahlfors in 1935 [1]. It is also worth mentioning the pioneering work of Teichmüller, 1937 [9], where this subject was ingeniously explored. In order to be more precise we have to specify what it means for a mapping to be nearly conformal. This is done via the concept of the distortion function.

DEFINITION 1. *Let f be a sense preserving homeomorphism between two regions in $\mathbb{R}^2$ having partial derivatives defined almost everywhere. The following expression is called the* distortion function *of f:*

$$D_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \geq 1.$$

Under suitable regularity assumptions the mapping f is conformal if and only if $D_f \equiv 1$ for almost all points in the domain of f.

In the classical setting the mapping f is considered *nearly conformal* if it minimizes the supremum norm of D_h over all sense preserving mappings h between given two regions. Recall that in the case of the Grötzsch problem the minimizer turns out to be an affine map ([2, Theorem 1, p. 8]).

3. DISTORTION FUNCTIONS AND EXTREMAL MAPPINGS

In higher dimensions we may measure the deviation from conformality in many different ways. To this effect, we introduce various *distortion functions*.

DEFINITION 2. *Consider a matrix $A \in \mathbb{R}^{n \times n}$ with positive determinant. Denote by*

$$A^{l \times l}, \quad l = 1, \ldots, n,$$

any $\binom{n}{l} \times \binom{n}{l}$ matrix whose entries are $l \times l$ subdeterminants of A. The following expressions will serve as building blocks of distortion functions *defined on matrices (with positive determinant):*

$$\mathbb{K}_l(A) = \frac{\|A^{l \times l}\|^{n/(n-l)}}{(\det A)^{l/(n-l)}}, \quad l = 1, \ldots, n-1.$$

REMARK 1.   In what follows we will apply these formulas to the Jacobian matrix of an orientation preserving mapping f. Accordingly, we denote them by

$$\mathbb{K}_l f = \mathbb{K}_l(x, f) = \mathbb{K}_l(Df(x)) = \frac{\|D^{l \times l} f(x)\|^{n/(n-l)}}{J(x, f)^{l/(n-l)}}, \quad l = 1, \ldots, n-1.$$

This formula is well defined at the points where the differential $Df(x)$ exists and has positive determinant.

Let us point out that the distortion functions are here understood in a little bit more general fashion than usually; that is, the symbol $\|\cdot\|$ can be any norm in $\mathbb{R}^{\binom{n}{l} \times \binom{n}{l}}$. However, in what follows we will only consider the Hilbert–Schmidt norm of matrices (for the definition and motivation of such norm see [3, Theorems 6.2, 6.6])[*].

REMARK 2.   For further properties of distortion functions we refer to [5, Section 6.4]. Recently ([3], [6], [7]) there has been an increasing interest and substantial progress made in the theory of extremal quasiconformal mappings. In this new development the proximity to conformal mappings is measured by means of integral averages rather than of supremum norm.

4. A BRIEF SURVEY OF RECENT RESULTS

Following the notation from [3] we consider the minimization problem

$$(4.1) \qquad \min_{f \in \mathscr{F}} \fint_{\mathbb{Q}} \mathbb{K}_l(x, f) \, dx,$$

where $\mathscr{F}$ consists of the homeomorphisms $f : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}'}$ of Sobolev class $\mathbb{W}_{\mathrm{loc}}^{1,p}(\mathbb{Q}, \mathbb{Q}')$, $p > l$, with integrable distortion and positive Jacobian determinant. Here $\mathbb{Q}$ and $\mathbb{Q}'$ are rectangular boxes,

$$\mathbb{Q} = [0, a_1] \times \cdots \times [0, a_n] \subset \mathbb{R}^n, \quad \mathbb{Q}' = [0, a_1'] \times \cdots \times [0, a_n'] \subset \mathbb{R}^n.$$

We will also assume—in analogy to the original Grötzsch problem—that f maps $(n-1)$-dimensional faces of $\mathbb{Q}$ into corresponding faces of $\mathbb{Q}'$. This implies that f also maps every l-dimensional face, $l = 0, 1, \ldots, n-1$, of $\mathbb{Q}$ into the corresponding l-dimensional face of $\mathbb{Q}'$.

The simplest example of a mapping in $\mathscr{F}$ is the linear transformation

$$(4.2) \qquad g(x) = (\lambda_1 x_1, \ldots, \lambda_n x_n) \quad \text{with } \lambda_k = \frac{a_k'}{a_k}.$$

The following result has recently been proven [3].

---

[*] Let $A \in \mathbb{R}^{n \times n}$ be a matrix. Here the Hilbert–Schmidt norm of A is defined as follows:

$$\|A\|^2 = \frac{1}{n} \operatorname{tr}(A^T A).$$

Usually the Hilbert–Schmidt norm is defined without the factor $1/n$, but we introduce it to get simpler formulas and to normalize the Hilbert–Schmidt norm of the identity matrix.

THEOREM 1. *For each $l = 1, \ldots, n - 1$ the minimization problem* (4.1) *has exactly one solution, namely the linear map g.*

REMARK 3. From now on we are going to consider more general energy integrals, so the definition of the class of admissible mappings has to be modified accordingly. First of all the mappings in $\mathscr{F}$ have to possess sufficient degree of integrability of derivatives in order to speak of their energy.

We shall take on stage rather general energy integrals, of the form

$$\mathscr{E}(f) = \fint_{\mathbb{Q}} \mathbb{E}(Df(x))\, dx \quad \text{for } f \in \mathscr{F}.$$

Here $\mathbb{E} : \mathbb{R}^{n \times n} \to [0, \infty)$ is a given stored energy integrand whose regularity will be specified later on.

DEFINITION 3. *We say that the energy integral $\mathscr{E}$ has the* Grötzsch *property if its minimum value is assumed at the linear transformation* (4.2).

5. POLYCONVEX, QUASICONVEX AND RANK-ONE CONVEX FUNCTIONS

One of the main goals of this paper is to investigate the Grötzsch problem for a wide class of energy integrals, a subclass of the so called polyconvex energy integrals.

For a matrix $A \in \mathbb{R}^{n \times n}$ we denote by $A^\sharp$ the list of all $l \times l$ minors of A with $l = 1, \ldots, n$. The order in this list is immaterial for the subsequent discussion as long as it is fixed once for all. We shall view this list as a point in $\mathbb{R}^{\sigma(n)}$, where

$$(5.1) \qquad \sigma(n) = \sum_{i=1}^{n} \binom{n}{i}^2 = \binom{2n}{n} - 1.$$

A matrix function $\Psi = \Psi(A)$ is *polyconvex* if it can be expressed as a convex function of the minors of A; that is,

$$\Psi(A) = \mathcal{M}(A^\sharp) \quad \text{for some convex } \mathcal{M} : \mathbb{R}^{\sigma(n)} \to \mathbb{R}.$$

More precisely:

DEFINITION 4. *A function $\Psi : U \subset \mathbb{R}^{n \times n} \to \mathbb{R}$ is said to be* polyconvex *if there is a measurable map $\mathcal{M}' : U \to \mathbb{R}^{\sigma(n)}$ (a subgradient of $\mathcal{M}$) such that*

$$\Psi(A) - \Psi(B) \geq \langle \mathcal{M}'(B), A^\sharp - B^\sharp \rangle \quad \text{for all } A, B \in U.$$

The notation $\langle\, ,\, \rangle$ stands for the usual inner product in $\mathbb{R}^{\sigma(n)}$.

Polyconvex functions are a special case of null-Lagrangians, one of the fundamental notions in the calculus of variations.

DEFINITION 5. *A function* $\Phi : \mathbb{R}^{n \times n} \to \mathbb{R}$ *is said to be a* null-Lagrangian *if it is a linear function of minors, meaning that*

$$\Phi(X) = \langle C, X^\sharp \rangle$$

for all $X \in \mathbb{R}^{n \times n}$ *and some* $C \in \mathbb{R}^{\sigma(n)}$.

One significant feature of polyconvex functions is that all distortion functions $\mathbb{K}_l$ with $l = 1, \ldots, n - 1$ are polyconvex on the set of matrices with positive determinant (see [5, 8.8]); also convex functions on $\mathbb{R}^{n \times n}$ are polyconvex, since the entries of $\mathbb{R}^{n \times n}$ are none other than the 1×1 minors.

In Section 7 we will raise the question of relations between energy functionals with the Grötzsch property and quasiconvex and rank-one convex functionals, therefore we recall these concepts now.

DEFINITION 6. *Let* $U \subset \mathbb{R}^{n \times n}$ *be open. A function* $\mathbb{E} : U \subset \mathbb{R}^{n \times n} \to \mathbb{R}$ *is said to be* quasiconvex *if for every matrix* $A \in U \subset \mathbb{R}^{n \times n}$ *and for every* $\Phi \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$ *with sufficiently small derivatives, we have*

$$\int_{\mathbb{R}^n} (\mathbb{E}(A + D\Phi) - \mathbb{E}(A)) \geq 0.$$

A necessary condition for a function to be quasiconvex is rank-one convexity.

DEFINITION 7. *A function* $\mathbb{E} : U \subset \mathbb{R}^{n \times n} \to \mathbb{R}$ *is said to be* rank-one convex *if for every matrix* $A \in U$ *and every* $a, b \in \mathbb{R}^n$ *the function*

$$t \mapsto \mathbb{E}(A + ta \otimes b) \quad \text{is convex.}$$

6. RESULTS

We are now ready to formulate numerous generalizations of Theorem 1 of Section 4. We will do it by gradually expanding the class of energy integrals with the Grötzsch property (Definition 3).

First, we state an auxiliary observation. If we add to the definition of the class $\mathscr{F}$ the assumption that $p \geq n$ (cf. Remark 3) then

$$(*) \qquad \fint_{\mathbb{Q}} J(x, f)\, dx = \fint_{\mathbb{Q}} J(x, g)\, dx$$

for any mapping $f \in \mathscr{F}$ and g an affine map (in fact, for any two mappings from $\mathscr{F}$). This is an immediate consequence of the definition of $\mathscr{F}$.

REMARK 4. From now on we will use exchangeably the following notations for the Jacobian determinant: $J_f = J(x, f)$.

LEMMA 1. *The following inequality holds for mappings* f *and* g *as above and* $l = 1, \ldots, n - 1$:

$$\fint_{\mathbb{Q}} \mathbb{K}_l^{2 - 2l/n}(x, f) J^{2l/n}(x, f)\, dx \geq \fint_{\mathbb{Q}} \mathbb{K}_l^{2 - 2l/n}(x, g) J^{2l/n}(x, g)\, dx.$$

Equality occurs if and only if $f \equiv g$.

PROOF. Let us recall that

$$\mathbb{K}_l^{2-2l/n}(x, f) J^{2l/n}(x, f) = \|D^{l \times l} f(x)\|^2.$$

By Theorem 7.5 in [3] we have the so-called L^1-estimate

$$\|D^{l \times l} g\| = \fint_{\mathbb{Q}} \|D^{l \times l} g(x)\| \, dx = \left[\binom{n}{l}^{-1} \sum_{1 \le i_1 < \cdots < i_l \le n} (\lambda_{i_1} \cdots \lambda_{i_l})^2 \right]^{1/2}$$

$$\le \fint_{\mathbb{Q}} \|D^{l \times l} f(x)\| \, dx,$$

where λ_i are the stretching factors as in the definition of the linear mapping g (Section 4). Since $\|D^{l \times l} g(x)\|$ is actually independent of x, with the aid of Hölder's inequality, we easily deduce the L^p-estimate as well, for $p \ge 1$:

$$\fint_{\mathbb{Q}} \|D^{l \times l} g\|^p = \left(\fint_{\mathbb{Q}} \|D^{l \times l} g\| \right)^p = \left[\binom{n}{l}^{-1} \sum_{1 \le i_1 < \cdots < i_l \le n} (\lambda_{i_1} \cdots \lambda_{i_l})^2 \right]^{p/2}$$

$$\le \left(\fint_{\mathbb{Q}} \|D^{l \times l} f(x)\| \, dx \right)^p \le \fint_{\mathbb{Q}} \|D^{l \times l} f(x)\|^p \, dx.$$

Taking $p = 2$ gives the lemma.

The uniqueness part of Lemma 1 follows in much the same way as in [3, Theorem 7.6] and is therefore omitted. $\square$

We are now in a position to formulate and prove:

THEOREM 2. *Let $\Psi : [1, \infty) \to [1, \infty)$ be a strictly increasing convex function. Then the statement of Theorem* 1 *holds for the energy integrals of the form*

$$\mathbb{E}(Df(x)) = \Psi(\mathbb{K}_l(x, f)), \quad l = 1, \dots, n - 1,$$

for $f \in \mathscr{F}$.

REMARK 5. In this paper the term *convex function* will always mean *convex and differentiable function*.

PROOF. Fix l. Theorem 7.6 of [3] provides us with an explicit formula for $\mathbb{K}_l(x, g)$; namely, $\mathbb{K}_l$ is a constant function given by

$$\mathbb{K}_l(x, g) = (\lambda_1 \cdots \lambda_n)^{l/(n-l)} \left[\binom{n}{l}^{-1} \sum_{1 \le i_1 < \cdots < i_l \le n} (\lambda_{i_1} \cdots \lambda_{i_l})^2 \right]^{n/(2n-2l)},$$

where λ_i are the stretching factors as in the definition of g (Section 4).

We emphasize once again that the lth distortion of g does not depend on a point in the domain $\mathbb{Q}$. This observation allows us to write

$$\Psi(\mathbb{K}_l(x, g)) = \fint_{\mathbb{Q}} \Psi(\mathbb{K}_l(x, g)) \, dx.$$

Hence

$$\Psi\left(\fint_{\mathbb{Q}} \mathbb{K}_l(x, g)\, dx\right) = \fint_{\mathbb{Q}} \Psi(\mathbb{K}_l(x, g))\, dx.$$

Therefore, for each $f \in \mathscr{F}$, we obtain

$$\fint_{\mathbb{Q}} \Psi(\mathbb{K}_l(x, f))\, dx \geq \Psi\left(\fint_{\mathbb{Q}} \mathbb{K}_l(x, f)\, dx\right)$$

$$\geq \Psi\left(\fint_{\mathbb{Q}} \mathbb{K}_l(x, g)\, dx\right) = \fint_{\mathbb{Q}} \Psi(\mathbb{K}_l(x, g))\, dx.$$

Here we have applied the monotonicity of $\Psi(x)$, Theorem 1 and Jensen's inequality for convex functions. Obviously, this computation works on the assumption of integrability of $\mathbb{K}_l(x, f)$ in the definition of the class $\mathscr{F}$.

As Ψ is increasing, the last inequality becomes an equality when

$$\fint_{\mathbb{Q}} \mathbb{K}_l(x, f)\, dx = \fint_{\mathbb{Q}} \mathbb{K}_l(x, g)\, dx,$$

which in turn implies that $f = g$ [3, Theorem 7.6].   □

Let us emphasize that the uniqueness statements below follow by the same type of reasoning as we just presented.

As an immediate corollary we obtain the following result.

COROLLARY 1. *The statement of Theorem 2 holds for*

$$\mathbb{E}(Df) = \Psi(\mathbb{K}_1 f) + \cdots + \Psi(\mathbb{K}_{n-1} f) + \Psi(J_f),$$

for Ψ as above and all $f \in \mathscr{F}$.

PROOF. We apply linearity of the integral and Theorem 2 to each term of the sum.   □

A comment on the regularity of f is needed. In the formulation of the minimization problem for $\mathbb{K}_l$ (see formula (4.1) in Section 4) we assume that $p > l$ (Remark 3). In Corollary 1 we use all distortion functions $\mathbb{K}_l, l = 1, \ldots, n - 1$, hence we require $p > n - 1$.

In addition we need integrability of the Jacobian determinant. This validates the assumption that $f \in W^{1,p}_{\text{loc}}(\mathbb{Q}, \mathbb{Q}')$ for all $p \geq n$.

In the same fashion we may prove the following.

COROLLARY 2. *The statement of Theorem 2 holds for*

$$\mathbb{E}(Df) = \Psi_1(\mathbb{K}_1 f) + \cdots + \Psi_{n-1}(\mathbb{K}_{n-1} f) + \Psi_n(J_f),$$

where each Ψ_i, $i = 1, \ldots, n$, satisfies the assumptions of Theorem 2.

Another corollary is a consequence of the several variable variant of Jensen's inequality.

COROLLARY 3. *The statement of Theorem 2 holds for*

$$\mathbb{E}(Df) = \Psi(\mathbb{K}_1 f, \ldots, \mathbb{K}_{n-1} f, J_f).$$

Here the function Ψ is strictly convex and increasing with respect to each of the first $n-1$ variables (that is, with the other variables fixed). The regularity assumption on f is the same as in Corollaries 1 and 2.

PROOF. We will sketch only the proof of uniqueness, the rest is straightforward from the variant of Jensen's inequality and previous discussion. The details are the same as in the proof of Theorem 2, and are therefore omitted.

Analytic formulation of convexity of Ψ reads as

$$\Psi(F) - \Psi(G) \geq \langle \nabla \Psi(G), F - G \rangle \quad \text{for any vectors } F, G \text{ in } \mathbb{R}^n.$$

Set

$$G = (\mathbb{K}_1 g, \ldots, \mathbb{K}_{n-1} g, J_g), \qquad F = (\mathbb{K}_1 f, \ldots, \mathbb{K}_{n-1} f, J_f).$$

Passing to the integral averages we have

$$\fint_{\mathbb{Q}} \Psi(F) - \fint_{\mathbb{Q}} \Psi(G) \geq \frac{\partial \Psi}{\partial x_1}(G) \fint_{\mathbb{Q}} (F_1 - G_1) + \cdots + \frac{\partial \Psi}{\partial x_n}(G) \fint_{\mathbb{Q}} (F_n - G_n) \geq 0.$$

The reader may observe that all terms on the right hand side are nonnegative, by Theorem 1 and monotonicity of Ψ with respect to each variable.

Now assume that the integral averages on the left hand side are equal, to obtain

$$\fint_{\mathbb{Q}} \mathbb{K}_i f = \fint_{\mathbb{Q}} \mathbb{K}_i g, \quad i = 1, \ldots, n-1.$$

Also note that the last term vanishes because of $(*)$. Uniqueness in Theorem 1 completes the proof. □

This corollary suggests that we should look for a similar result when $\mathbb{E}$ is convex in the minors. We are going to prove that under certain conditions on Ψ the Grötzsch property is true for a subclass of polyconvex functionals.

Our first goal is to find algebraic relations between two vectors:

$$(\mathbb{K}_1 f, \mathbb{K}_2 f, \ldots, \mathbb{K}_{n-1} f, J_f) \in \mathbb{R}^n \quad \text{and} \quad (Df, D^{2 \times 2} f, \ldots, D^{n \times n} f) \in \mathbb{R}^{\sigma(n)}.$$

Let us recall that $D^{l \times l} f(x)$ stands for an ordered set of $l \times l$ minors of the Jacobian matrix, and in particular $D^{n \times n} f = \det Df = J_f$.

The dimension of $\mathbb{R}^{\sigma(n)}$ can be explicitly computed (see formula (5.1)).

The definition of $\mathbb{K}_l$ implies that

$$(J_f^{2/n} \mathbb{K}_1^{2-2/n} f, \ldots, J_f^{2l/n} \mathbb{K}_l^{2-2l/n} f, \ldots, J_f^2) = (\|Df\|^2, \ldots, \|D^{l \times l} f\|^2, \ldots, J_f^2).$$

The right hand side consists of the squares of the Hilbert–Schmidt norms of the matrices of minors. Each $\|D^{l \times l} f\|^2, l = 1, \ldots, n$, is the arithmetic average of the squares of the $l \times l$ minors. We are going to relate this vector to $D^{\sharp} f$, the list of all minors.

6.1. *The relation*

Let us introduce the squaring operation $(\cdot)^2 : \mathbb{R}^{\sigma(n)} \to \mathbb{R}^{\sigma(n)}$ defined for vectors $v = (v_1, \ldots, v_{\sigma(n)})$ by the rule $(v)^2 := (v_1^2, \ldots, v_{\sigma(n)}^2)$.

Thus for $D^\sharp f(x)$ our formula reads

$$[D^\sharp f(x)]^2 = ([D^\sharp f(x)]_1^2, \ldots, [D^\sharp f(x)]_{\sigma(n)}^2).$$

Next we compose the squaring operation with a linear map $\Phi : \mathbb{R}^{\sigma(n)} \to \mathbb{R}^n$, whose matrix representation is built of 0 and 1 in the following way:

$$\mathbf{M}(\Phi) = \begin{bmatrix} \overbrace{1 \ldots 1}^{\binom{n}{1}^2} & \overbrace{}^{\binom{n}{2}^2} & \overbrace{}^{\binom{n}{i}^2} & \overbrace{}^{\binom{n}{n}^2} \\ & 1 \ldots 1 & & \mathbf{0} \\ & & \ddots & \\ & & 1 \ldots 1 & \\ & \mathbf{0} & & \ddots \\ & & & 1 \end{bmatrix}$$

The composition gives us a mapping $\Phi \circ (\cdot)^2$, which takes $D^\sharp f$ into the vector of the squares of the Hilbert–Schmidt norms of the lth-cofactor matrices $D^{l \times l} f$, with $l = 1, \ldots, n$,

$$(\|Df\|^2, \|D^{2 \times 2} f\|^2, \ldots, \|D^{n \times n} f\|^2).$$

6.2. *The main theorem*

THEOREM 3. *Let an energy functional be defined on the set of mappings $f \in \mathscr{F}$ by the rule*

$$\mathscr{E}(f) = \fint_{\mathbb{Q}} \mathbb{E}(Df(x)) \, dx.$$

As for the integrand we assume that there exists a strictly convex and increasing (with respect to the first $n-1$ variables) function $\widetilde{E} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathbb{E}(A) = \widetilde{E}(\Phi(A^\sharp)^2).$$

Then $\mathscr{E}$ assumes its minimal value exactly at the linear mapping g.

REMARK 6. Notice that the integrand

$$\mathbb{E} = \widetilde{E} \circ \Phi \circ (\cdot)^2 : \mathbb{R}^{n \times n} \to \mathbb{R}$$

is a convex function of minors of the matrix Df, hence is polyconvex. To see this, let us observe that each coordinate function of $(\cdot)^2$ is convex. Obviously the composition with the linear mapping Φ does not change convexity. These facts together with monotonicity of $\widetilde{E}$ in each variable (when all other variables are held fixed) imply, after lengthy though

elementary computation, that $\mathbb{E}$ is polyconvex. We are now in a position to complete the argument for Theorem 3.

PROOF OF THEOREM 3.   We have

$$
\begin{aligned}
\fint_{\mathbb{Q}} \mathbb{E}(Df(x))\,dx &= \fint_{\mathbb{Q}} \widetilde{E}(\Phi(D^\sharp f)^2) \\
&\geq \widetilde{E}\left(\fint_{\mathbb{Q}} [\Phi(D^\sharp f)^2]_1, \ldots, \fint_{\mathbb{Q}} [\Phi(D^\sharp f)^2]_{n-1}, \fint_{\mathbb{Q}} [\Phi(D^\sharp f)^2]_n \right) \\
&= \widetilde{E}\left(\fint_{\mathbb{Q}} J_f^{2/n} \mathbb{K}_1^{2-2/n} f, \ldots, \fint_{\mathbb{Q}} J_f^{2-2/n} \mathbb{K}_{n-1}^{2/n} f, \fint_{\mathbb{Q}} J_f^2 \right) \\
&\geq \widetilde{E}\left(\fint_{\mathbb{Q}} J_g^{2/n} \mathbb{K}_1^{2-2/n} g, \ldots, \fint_{\mathbb{Q}} J_g^{2-2/n} \mathbb{K}_{n-1}^{2/n} g, \fint_{\mathbb{Q}} J_g^2 \right) \\
&\qquad\text{(here we use Lemma 1 and the comment before Remark 4)} \\
&= \widetilde{E}(J_g^{2/n} \mathbb{K}_1^{2-2/n} g, \ldots, J_g^{2-2/n} \mathbb{K}_{n-1}^{2/n} g, J_g^2) \\
&= \fint_{\mathbb{Q}} \widetilde{E}(J_g^{2/n} \mathbb{K}_1^{2-2/n} g, \ldots, J_g^{2-2/n} \mathbb{K}_{n-1}^{2/n} g, J_g^2) \\
&= \fint_{\mathbb{Q}} \widetilde{E}(\Phi(D^\sharp g)^2) = \fint_{\mathbb{Q}} \mathbb{E}(Dg).
\end{aligned}
$$

All inequalities become equalities for $f = g$. This is due to the fact that all differential expressions above which contain g are constant.

     A similar analysis to that in Corollary 3 along with Lemma 1 results in the uniqueness statement. $\square$

REMARK 7.   It is well known that convexity implies polyconvexity. Let us provide the reader with an example of a polyconvex function which is not convex and still satisfies the hypothesis and assertion of Theorem 3. This emphasizes the novelty of our result.

     Consider $n \geq 2$, thus $\sigma(n) = \binom{2n}{n} - 1$. The integrand in question takes the form

$$
\mathbb{E}(Df) = [D^\sharp f]_1^2 + \cdots + [D^\sharp f]_{\sigma(n)}^2 = [D^\sharp f]_1^2 + \cdots + [D^\sharp f]_{\sigma(n)-1}^2 + J_f^2.
$$

This function, being convex in the minors, is polyconvex. However, $\mathbb{E}$ is not convex, largely because determinant is not a convex function of the matrix. Nonetheless, taking

$$
\widetilde{E}(x_1, \ldots, x_n) = x_1 + \cdots + x_n
$$

we find that Theorem 3 remains valid.

6.2.1. *Examples of energies with the Grötzsch property.*   In this subsection we discuss some examples of classical energy functionals which also share the Grötzsch property.

EXAMPLE 1.   An interesting energy functional arises from considering expressions of type

$$
f(A) = |A|^p + h(\det A), \quad p > 2, h \text{ convex.}
$$

As $f(A)$ is a sum of two convex functions we immediately see from Theorem 3 that f is a function with the Grötzsch property.

EXAMPLE 2. Let us take on stage the functional

$$f(A) = \lambda |A|^p + |A|^p \mathbb{K}_1(A)^{1-n} \quad \text{for } p \geq n \text{ and for all } A \in \mathbb{R}^{n \times n},$$

where $\lambda > 0$. It is worth mentioning that its two-dimensional analog is the subject of intensive studies in harmonic analysis, probability theory, geometric function theory and calculus of variations.

Conspicuously, the first term is a convex function. To see that so is the second, we invoke Lemma 8.8.2 in [5]: a function x^a/y^b of variables $x, y \in (0, \infty)$ is convex provided that $a \geq b + 1 \geq 1$. Apply this fact to

$$x = |A|, \quad y = \mathbb{K}_1(A), \quad a = p, \quad b = n - 1$$

to obtain convexity, completing the argument that f shares the Grötzsch property.

6.2.2. *Examples of nonuniqueness.* In this section we are going to address the following naturally arising question:

Does Theorem 3 hold for all polyconvex functions?

Unfortunately, the answer is no. Below we give examples of energy functionals without the uniqueness property.

REMARK 8. The uniqueness is the real essence of the matter. To see it recall the definition of quasiconvexity (Definition 6, Section 5). It says that affine transformations are minimizers of $\mathbb{E}$ among mappings with the same affine boundary values; whereas in our case we do not impose boundary values and therefore the uniqueness property is delicate and difficult to prove.

EXAMPLE 3. We start with a polyconvex function (in fact linear function of minors, hence a null-Lagrangian) which does not enjoy the uniqueness property. We construct a nonlinear mapping which belongs to $\mathscr{F}$ and has the same energy as the affine mapping g. Let

$$\mathbb{Q} = [0, 1] \times [0, 1] \times [0, 1] \quad \text{and} \quad \mathbb{Q}' = [0, 1] \times [0, 1] \times [0, 2]$$

be rectangles in $\mathbb{R}^3$. Then $g(x, y, z) = (x, y, 2z)$ is the unique affine map in the family $\mathscr{F}$. We then consider the polynomial map defined by

$$f(x, y, z) = (x^2, y^2, 2z).$$

It is easy to verify that f lies in $\mathscr{F}$. To this end we see that f is a homeomorphism and it maps $\mathbb{Q}$ onto $\mathbb{Q}'$ so that the faces of $\mathbb{Q}$ are mapped into the corresponding faces of $\mathbb{Q}'$. Obviously f belongs to $\mathbb{W}_{\text{loc}}^{1,p}(\mathbb{Q}, \mathbb{Q}')$ for all $p \geq 2$.

As $n = 3$, $\sigma(3) = 19$. Take

$$\mathbb{E}(Df) = [D^\sharp f]_1 + \cdots + [D^\sharp f]_{19}.$$

This is a sum of all minors of the Jacobian matrix. Indisputably, $\mathbb{E}(Df)$ is polyconvex (the inequality in Definition 4 is satisfied trivially, in fact it becomes an equality), actually a null-Lagrangian. A straightforward calculation shows that

$$\int_{\mathbb{Q}} \mathbb{E}(Dg) = 11,$$

and also

$$\int_{\mathbb{Q}} \mathbb{E}(Df) = \int_{\mathbb{Q}} (2 + 6x + 6y + 12xy) \, dx \, dy \, dz = 11,$$

so the uniqueness is lost (cf. Theorem 1).

REMARK 9.   This counterexample may be easily generalized to any dimension $n \geq 3$ by taking the rectangles

$$\mathbb{Q} = \underbrace{[0,1] \times \cdots \times [0,1]}_{n} \quad \text{and} \quad \mathbb{Q}' = \underbrace{[0,1] \times \cdots \times [0,1]}_{n-1} \times [0,2]$$

and the mappings

$$f(x_1, \ldots, x_n) = (x_1^2, \ldots, x_{n-1}^2, 2x_n), \quad g(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, 2x_n).$$

Elementary calculation reveals that

$$\int_{\mathbb{Q}} \mathbb{E}(Df) = \int_{\mathbb{Q}} \mathbb{E}(Dg) = 3 \cdot 2^{n-1} - 1.$$

The same holds for

$$f(x_1, \ldots, x_n) = (x_1^{\alpha_1}, \ldots, x_{n-1}^{\alpha_{n-1}}, 2x_n), \quad \text{where } \alpha_i > 1, i = 1, \ldots, n - 1.$$

EXAMPLE 4.   The next example shows that the energy functional has to depend on all squares of 1×1 minors, otherwise uniqueness is lost.

Consider $\mathbb{Q} = \mathbb{Q}' = [0,1] \times \cdots \times [0,1] \subset \mathbb{R}^n$, and so $g = \text{Id}$. Define $f(x_1, \ldots, x_n) = (x_1, \ldots, x_i^2, \ldots, x_n) \in \mathscr{F}$. Let now

$$\mathbb{E}(Df) = \sum_{\substack{k=1 \\ k \neq i}}^{n} \left(\frac{\partial f_k}{\partial x_k} \right)^2.$$

Then the mappings f and g have the same energy. By permuting coordinates of f we may obtain examples of nonuniqueness for any i.

EXAMPLE 5.   Example below explains that dependence on squares is vital for uniqueness. For the sake of simplicity we will restrict ourselves to the three-dimensional case.

Consider $\mathbb{Q} = [0,1] \times [0,1] \times [0,1]$ and $\mathbb{Q}' = [0,1] \times [0,1] \times [0,2]$, and thus $g(x, y, z) = (x, y, 2z)$. Define a mapping

$$f(x, y, z) = (x, y^\beta, 2z^\gamma),$$

for positive numbers β, γ to be found. Let the energy integrand be defined via the following formula:

$$\mathbb{E}(Df) = \left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2 + \left(\frac{\partial f_3}{\partial z}\right)^\alpha, \quad 1 < \alpha < 2,$$

for α close enough to 2 (see below). Then

$$\fint_\mathbb{Q} \mathbb{E}(Dg) = 2 + 2^\alpha \quad \text{and} \quad \fint_\mathbb{Q} \mathbb{E}(Df) = 1 + \frac{\beta^2}{2\beta - 1} + \frac{2^\alpha \gamma^\alpha}{\alpha(\gamma - 1) + 1}.$$

In order to show the loss of uniqueness we need to find $\beta \neq 1, 1/2$, $\gamma \neq 1$ and α such that the above integral mean values are equal, i.e.

$$2 + 2^\alpha = 1 + \frac{\beta^2}{2\beta - 1} + \frac{2^\alpha \gamma^\alpha}{\alpha(\gamma - 1) + 1}.$$

For $\gamma = 2$ the last equation reduces to

$$(*) \qquad\qquad h(\alpha) := 1 + 2^\alpha - \frac{2^{2\alpha}}{\alpha + 1} = \frac{\beta^2}{2\beta - 1}.$$

As $1 < \alpha < 2$, $h(1) = 1$, $h(2) = -1/3$. Using the intermediate value theorem we solve the equation $(*)$ for $1 < \alpha < 2$.

For instance, to obtain $h(\alpha) = (1 - \sqrt{2})/2$ we take $\beta = 1 - 1/\sqrt{2}$. The mapping $f(x, y, z) = (x, y^{1-1/\sqrt{2}}, 2z^2)$ has the same energy as g.

7. Open problems

To proceed further we will consider a function Ψ similar to one considered in Corollary 3 but with one distinction; namely, not depending explicitly on the Jacobian determinant of f:

$$\mathbb{E}(Df(x)) = \Psi(\mathbb{K}_1(x, f), \dots, \mathbb{K}_{n-1}(x, f)).$$

Conspicuously, this energy also satisfies the statement of Theorem 2.

We conveniently normalize Ψ as follows:

$$\Psi(1, \dots, 1) = 1.$$

Notice that $\mathbb{E}$ has the following properties:

(1) $\mathbb{E}(A) = 1$ for any conformal matrix A, that is, a multiple of an orthogonal matrix having positive determinant (see e.g. [5] for details concerning conformal matrices). The same property is shared by the distortion functions defined in Section 3.

(2) $\mathbb{E}(A)$ is 0-homogeneous, i.e. $\mathbb{E}(\lambda A) = \mathbb{E}(A)$ for any $\lambda > 0$. Moreover, $\mathbb{E}$ is invariant with respect to a conformal change of variables; precisely $\mathbb{E}(UAV) = \mathbb{E}$ for conformal matrices U and V. This follows from the same property for distortion functions and the normalization we imposed on Ψ.

(3) $\mathbb{E}(A)$ is polyconvex, just like the basic distortion functions $\mathbb{K}_1, \dots, \mathbb{K}_{n-1}$.

These properties justify calling

$$\mathbb{K}(f) = \Psi(\mathbb{K}_1 f, \ldots, \mathbb{K}_{n-1} f)$$

a *generalized distortion function*. A natural question arises as to whether the conditions (1)–(3) are enough for $\mathbb{E}$ to become a generalized distortion function. We rephrase this question as a conjecture.

CONJECTURE 1. *Let a function $\mathbb{E} : \mathbb{R}^{n \times n} \to [1, \infty]$ be strictly convex and satisfy the conditions:*

(1) $\mathbb{E}(A) = 1 \Leftrightarrow A$ *is a conformal matrix, i.e. $A \in \lambda O(n)$ for some $\lambda > 0$.*
(2) $\mathbb{E}(\lambda O \cdot A) = \mathbb{E}(A \cdot \lambda O) = \mathbb{E}(A)$ *for any $\lambda > 0$ and $O \in O(n)$ (i.e. E is conformally invariant in the domain and range of A).*
(3) $\mathbb{E}$ *is polyconvex.*

Does there exist a convex function Ψ which is non-decreasing with respect to each variable such that

$$\mathbb{E}(A) = \Psi(\mathbb{K}_1(A), \ldots, \mathbb{K}_{n-1}(A)) ?$$

The affirmative answer to this conjecture would provide us with a large class of distortion functions and in this way we would gain a deeper insight into the geometric function theory.

Another interesting question concerns the relations between the Grötzsch property and quasiconvexity and rank-one convexity:

Assume that a functional has the Grötzsch property. Does it imply other convexity properties?

We will state the problem in an abstract setting. Let $\mathscr{F}$ be a mapping family consisting of homeomorphisms between rectangular boxes $\mathbb{Q}$ and $\mathbb{Q}'$ in $\mathbb{R}^n$. We assume the mappings in $\mathscr{F}$ possess suitable degree of integrability of derivatives in order to be able to speak of their energy. Denote by $\mathscr{A} = \{Df : f \in \mathscr{F}\}$ the family of the Jacobian matrices of $\mathscr{F}$.

Let $\mathbb{E} : \mathscr{A} \subset \mathbb{R}^{n \times n} \to [1, \infty)$ satisfy conditions (1) and (2) of Conjecture 1, and assume it has the Grötzsch property.

Under what additional assumptions imposed on $\mathbb{E}$ is this energy integrand quasiconvex (rank-one convex)?

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