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**Probability theory.** — *Modified log-Sobolev inequalities and isoperimetry*, by ALEXANDER V. KOLESNIKOV, communicated by G. Da Prato on 9 February 2007.

ABSTRACT. — We find sufficient conditions for a probability measure $\mu$ to satisfy an inequality of the type

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\right) d\mu \le C\int_{\mathbb{R}^d} f^2 c^*\left(\frac{|\nabla f|}{|f|}\right) d\mu + B\int_{\mathbb{R}^d} f^2\,d\mu,$$

where $F$ is concave and $c$ (a cost function) is convex. We show that under broad assumptions on $c$ and $F$ the above inequality holds if for some $\delta > 0$ and $\varepsilon > 0$ one has

$$\int_0^{\varepsilon} \Phi\left(\delta c\left[\frac{tF(1/t)}{\mathcal{I}_\mu(t)}\right]\right) dt < \infty,$$

where $\mathcal{I}_\mu$ is the isoperimetric function of $\mu$ and $\Phi = (yF(y) - y)^*$. In the particular case when

$$\mathcal{I}_\mu(t) \ge kt\varphi^{1-1/\alpha}(1/t),$$

where $\varphi$ is a concave function growing not faster than log, $k > 0$, $1 < \alpha \le 2$ and $t \le 1/2$, we establish a family of tight inequalities interpolating between the $F$-Sobolev and modified inequalities of log-Sobolev type. A basic example is given by convex measures satisfying certain integrability assumptions.

KEY WORDS: Modified log-Sobolev inequalities; isoperimetric inequalities; convex measures.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 60E15; Secondary 26D10.

## 1. INTRODUCTION

The celebrated *logarithmic Sobolev inequality*

$$\operatorname{Ent}_\mu f^2 := \int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\right) d\mu \le 2C\int_{\mathbb{R}^d} |\nabla f|^2\,d\mu, \tag{1}$$

where $\mu = e^{-V}\,dx$ is a probability measure, has numerous applications in probability theory, mathematical physics, and geometry. It appeared first in the work of Gross [19], where he established (1) for the standard Gaussian measure. Gross discovered that (1) implies hypercontractivity of the semigroup $e^{tL}$ generated by $L = \Delta - \langle \nabla V, \nabla\rangle$.

Necessary and sufficient conditions for (1) have been intensively studied by many authors (see [1]). It is well-known that for every probability measure satisfying (1) there exists $\varepsilon > 0$ such that

$$e^{\varepsilon|x|^2} \in L^1(\mu). \tag{2}$$

It has been shown by Wang [26] that this assumption is sufficient provided $\mu$ is convex, i.e., has the form $\mu = e^{-V}\,dx$, where $V$ is a convex function (in the literature convex measures

are also called log-concave). Wang's proof employs the associated diffusion semigroup. Bobkov [6] gave another proof of this result by applying the Prékopa–Leindler theorem and isoperimetric inequalities. There exist non-convex measures satisfying (1). For example, according to a result of Holley and Stroock, if $\mu$ satisfies (1), every probability measure $e^{\varphi} \cdot \mu$ with $a \le \varphi \le b$ satisfies the logarithmic Sobolev inequality with $C' = e^{2(b-a)}C$.

Recall that (1) implies the *Poincaré inequality*

$$\operatorname{Var}_\mu f := \int_{\mathbb{R}^d} f^2\,d\mu - \left(\int_{\mathbb{R}^d} f\,d\mu\right)^2 \le C\int_{\mathbb{R}^d} |\nabla f|^2\,d\mu. \tag{3}$$

The log-Sobolev inequality can be considered as a Poincaré-type inequality for the $L^2 \log L$-Orlicz norm. By using this observation and some classical results on Hardy's inequality with weights, Bobkov and Götze [7] established necessary and sufficient conditions for (1) on the real line. Namely, $\mu = \rho\,dx$ satisfies (1) if and only if

$$\sup_{x<m} F(x)\log\left(\frac{1}{F(x)}\right)\int_x^m \frac{dx}{\rho(x)} < \infty,$$
$$\sup_{x>m} (1-F(x))\log\left(\frac{1}{1-F(x)}\right)\int_m^x \frac{dx}{\rho(x)} < \infty,$$

where $F(x) = \mu((-\infty, x])$ and $m$ is the median of $\mu$.

It is well-known that (1) (as well as the classical Sobolev inequalities) is closely related to the isoperimetric inequalities. For every Borel $A \subset \mathbb{R}^d$ we denote by $\mu^+(A)$ the surface measure of the boundary $\partial A$:

$$\mu^+(A) = \varliminf_{h\to 0} \frac{\mu(A^h) - \mu(A)}{h},$$

where $A^h = \{x : \operatorname{dist}(x, A) \le h\}$ is the $h$-neighborhood of $A$. It was proved by Ledoux [23] that the isoperimetric inequality of the Gaussian type

$$\mu^+(A) \ge c\varphi(\Phi^{-1}(\mu(A)))$$

implies (1). Here

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \qquad \Phi(x) = \int_{-\infty}^x \varphi(s)\,ds.$$

Some sufficient conditions for (1) can be obtained by perturbation methods. For example, Carlen and Loss applied in [12] the log-Sobolev inequality

$$\int_{\mathbb{R}^d} f^2 \log f^2\,dx \le \frac{1}{\pi e^2}\int_{\mathbb{R}^d} |\nabla f|^2\,dx, \qquad \int_{\mathbb{R}^d} f^2\,dx = 1,$$

for Lebesgue measure. In particular, they proved that $\mu = e^{-V}\,dx$ satisfies (1) provided that

$$\frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V - \pi e^2 V$$

is bounded from below and $\mu$ satisfies (3) (see also [3] and [13]).

It follows from (2) that $\mu$ has a very fast decay. However, many distributions exhibit some weaker, yet useful properties. Below we consider the following generalizations of (1):

1) The *defective log-Sobolev inequality*

$$\operatorname{Ent}_\mu f^2 \le 2C \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu + B \int_{\mathbb{R}^d} f^2 \, d\mu.$$

2) The *F-Sobolev inequality*

$$\int_{\mathbb{R}^d} f^2 F\left( \frac{f^2}{\int_{\mathbb{R}^d} f^2 \, d\mu} \right) d\mu \le 2C \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu + B \int_{\mathbb{R}^d} f^2 \, d\mu,$$

where $F$ is a concave function.

3) The *modified log-Sobolev inequality*

$$\operatorname{Ent}_\mu f^2 \le C \int_{\mathbb{R}^d} f^2 c^* \left( \frac{|\nabla f|}{|f|} \right) d\mu \tag{4}$$

for some convex $c : \mathbb{R}^+ \to \mathbb{R}^+$. Here $c^*(x) = \sup_{y \in \mathbb{R}^+} (\langle x, y \rangle - c(y))$.

Inequality of type 1) implies the hyperboundedness of the associated semigroups (see [15]). A basic example for 2) and 3) is given by the following measure on the real line:

$$\mu_\alpha = Z_\alpha e^{-|x|^\alpha} \, dx,$$

where $1 < \alpha \le 2$. It was proved in [16] that $\mu_\alpha$ satisfies (4) with

$$c(x) = c_{A,\alpha}(x) = \begin{cases} \dfrac{x^2}{2} & \text{if } |x| \le A, \\ A^{2-\alpha} \dfrac{|x|^\alpha}{\alpha} + A^2 \dfrac{\alpha - 2}{2\alpha} & \text{if } |x| \ge A, \end{cases} \tag{5}$$

for every $A > 0$. By the tensorization argument the result holds also in the multidimensional case for the product measure $\prod_{i=1}^d \mu_\alpha(dx_i)$ and the cost function $c_{d,A,\alpha}(x) = \sum_{i=1}^d c_{A,\alpha}(x_i)$. On the other hand, by a result from [3], $\mu_\alpha$ satisfies

$$\int f^2 \log^{2/\beta}(1 + f^2) \, d\mu - \left( \int f^2 \, d\mu \right) \log^{2/\beta} \left( 1 + \int f^2 \, d\mu \right) \le C \int |\nabla f|^2 \, d\mu,$$

where $1/\alpha + 1/\beta = 1$. One can easily verify that $c^*_{A,\alpha} = c_{A,\beta}$.

The case $\alpha \ge 2$ has been considered in [9]. In this case the measure

$$\mu = Z_{\alpha,d} \prod_{i=1}^d e^{-|x_i|^\alpha} \, dx_i$$

on $\mathbb{R}^d$ satisfies the inequality

$$\operatorname{Ent}_\mu |f|^\beta \le C \int_{\mathbb{R}^d} \sum_i |\partial_{x_i} f|^\beta \, d\mu. \tag{6}$$

Among other generalizations of (1) let us mention an important result from [22] on a family of inequalities interpolating between log-Sobolev and Poincaré. If $1 < \alpha \le 2$, $1 \le p \le 2$, then for every smooth $f$ one has

$$\int_{\mathbb{R}^d} f^2\,d\mu_\alpha - \left(\int_{\mathbb{R}^d} |f|^p\,d\mu_\alpha\right)^{2/p} \le C(2-p)^{2(1-1/\alpha)} \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu_\alpha.$$

Inequalities of this type were first proved by Beckner in [5] for Gaussian measures. For further development and connections with the $F$-Sobolev inequality, see [3], [4], and [27].

Inequality (4) is closely related to the *Talagrand transportation inequality*

$$W_c(\mu, f\cdot\mu) \le \mathrm{Ent}_\mu\, f, \tag{7}$$

where $f\cdot\mu$ is another probability measure and $W_c$ is the minimum of the Kantorovich functional for the cost function $c$ (see [25] for details). In fact, under broad assumptions on $c$, inequality (4) is stronger than (7). This was shown in [24] for the case of quadratic cost function. It was proved in [14] by the optimal transportation method that (4) holds for measures of the type $\mu = e^{-V}\,dx$, where $V$ satisfies

$$V(b) - V(a) \ge \langle \nabla V(a), b-a\rangle + \alpha c(b-a)$$

for some $\alpha > 0$ and a proper choice of $c$. For recent progress in transportation inequalities, including some exponential- and power-type estimates, see [10], [11], [17], [18], [21], and the references therein.

In this paper we obtain sufficient conditions which guarantee inequalities of the following type:

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\right) d\mu \le C\int_{\mathbb{R}^d} f^2 c^*\left(\frac{|\nabla f|}{|f|}\right) d\mu + B\int_{\mathbb{R}^d} f^2\,d\mu, \tag{8}$$

where $F$ is concave and $c:\mathbb{R}^+\to\mathbb{R}^+$ is convex (Theorem 2.1). This inequality unifies the defective modified log-Sobolev inequalities and the $F$-Sobolev inequalities. Obviously, the tight $F$-Sobolev inequality corresponds to the case $c = |x|^2$, $B = 0$, and the modified Sobolev inequality corresponds to the case $F = \log$, $B = 0$.

An important assumption on $c$ which we use below (though not everywhere) is the following:

(H) for any $k > 0$ there is $n(k) > 0$ such that

$$c(kx) \le n(k)c(x), \qquad c^*(kx) \le n(k)c^*(x).$$

Our estimate is based on the use of a special isoperimetric function

$$I_F(r) = \sup_{A\in\mathcal{M}_r} \frac{\mu(A)F(1/\mu(A))}{\mu^+(A)}.$$

Here $\mathcal{M}_r = \{A : \mu(A) = \mu(\{x : |x| > r\})\}$. Assume that (H) holds. The main result (Theorem 2.1, Remark 2.4) can be roughly formulated in the following way:

*Integrability of $\Phi(\delta c(I_F))$ for some $\delta > 0$, where $\Phi = (yF(y) - y)^*$, implies* (8).

Let us give some important examples of the function $I_F$. In the case of a convex measure $\mu$ and $F = \log$, the function $I_F(r)$ can be estimated for large values of $r$ by $Cr$ with some $C > 0$. This follows from an estimate obtained in [8] (see Lemma 4.1). In the case of an entropy functional $F$ growing as $\log^\tau(x)$, $\tau \le 1$, and under the additional assumption that $\exp(|x|^\alpha) \in L^1(\mu)$, this result combined with Chebyshev's inequality yields $I_F(r) \le Cr^{1-\alpha(1-\tau)}$ (see Lemma 4.2 for a precise result).

The integrability assumption can be rewritten in an even more elegant way if we employ the classical isoperimetric function $\mathcal{I}_\mu$ of $\mu$ defined by

$$\mathcal{I}_\mu(t) = \inf_{A \subset \mathbb{R}^d : \mu(A) = t} \mu^+(A). \tag{9}$$

Assume that $c$ satisfies (H). It turns out that (8) holds for a broad class of $F$ and $c$ if for some $\delta > 0$, $K > 1$ one has

$$\int_0^{1/K} \Phi\left(\delta c\left[\frac{tF(1/t)}{\mathcal{I}_\mu(t)}\right]\right) dt < \infty \tag{10}$$

(see Theorem 2.3 and Remark 2.4).

Let us list our main assumptions on the entropy function $F$ which will be used below. A typical example is given by $F = \log$.

**A1.** $F$ is concave, increasing and $F(1) = 0$.
**A2.** $\lim_{y\to 0} yF(y) = 0$, $\lim_{y\to\infty} F(y) = \infty$.
**A3.** $yF(y)$ is convex on $[0, 1 + \Delta]$ for some $\Delta > 0$.
**A4.** There exists $y_0 \ge 1$ such that $yF'(y)$ is non-increasing and $yF'(y) \le 1$ on $[y_0, \infty)$.

REMARK. Assumptions A1 and A2 will be used throughout the paper. Assumptions A3 and A4 will be used for tight estimates.

In Section 3 we obtain sufficient conditions for the related tight inequalities. The case of the $F$-inequality follows immediately from the main result (Theorem 2.5) without any further assumptions. In the case of modified log-Sobolev inequalities we restrict ourselves to a special choice of a cost function. Namely, we consider for every $1 < \alpha \le 2$ the corresponding family of cost functions $c_{A,\alpha}$ given by (5). Under some additional assumptions on the entropy, we prove a modification of (8), where $\int_{\mathbb{R}^d} f^2\, d\mu$ is replaced by $\mathrm{Var}_\mu f$ (Theorem 3.6). In the proof we use techniques developed in [16].

Before we give the precise formulation of the main result of Sections 3 and 4, let us briefly explain the relationships between the functions $F$, $c$, and $\mathcal{I}_\mu$ leading to tight inequalities. We want to prove (10). It turns out that under assumptions A1–A4 on $\varphi$ every entropy function $F$ such that $F \sim A\varphi^\tau$, $\tau \le 1$, satisfies

$$\Phi(x) \le F^{-1}(1 + x) \sim \varphi^{-1}\left(\left[\frac{x+1}{A}\right]^{1/\tau}\right). \tag{11}$$

Assume, in addition, that $\mathcal{I}_\mu(t) \ge kt\varphi^{1-1/\alpha}(t)$ for some $1 < \alpha \le 2$. Now take a cost function $c$ such that $c \sim B|x|^q$. We set

$$q = \frac{\tau}{\tau - 1 + 1/\alpha}.$$

Then

$$\Phi\left(\delta c\left[\frac{tF(1/t)}{\mathcal{I}_\mu(t)}\right]\right) \le F^{-1}(1+\varepsilon(\delta)F(1/t)),$$

where $\lim_{\delta\to 0}\varepsilon(\delta) = 0$. Taking into account property A4, one can easily show that $F^{-1}(1+\varepsilon F(1/t)) \le at^{-p}$ for some $p < 1$ and sufficiently small $\varepsilon$. Hence (10) holds.

We consider the generalized entropies defined by

$$f \mapsto \int_{\mathbb{R}^d} f F_\tau\left(\frac{f}{\mu(f)}\right) d\mu,$$

where

$$F_\tau(x) = \begin{cases} \varphi(x) & \text{if } 0 < x \le x_0, \\ \dfrac{1}{\tau}(\varphi^\tau(x) - 1) + 1 & \text{if } x \ge x_0, \end{cases}$$

$\varphi$ satisfies A1–A4, $\tau \le 1$ and $x_0$ is chosen in such a way that $\varphi(x_0) = 1$.

Recall that $m_f = \inf\{t : \mu(f > t) \le 1/2\}$ is called the *median* of $f$. Throughout the paper we assume that $\mu$ has convex support.

THEOREM 1.1. *Let $\varphi$ satisfy* A1–A4 *and let $\mathcal{I}_\mu$ satisfy*

$$\mathcal{I}_\mu(t) \ge kt\varphi\left(\frac{1}{t}\right)^{1-1/\alpha}$$

*for some $k > 0$, $1 < \alpha \le 2$ and all $t \le 1/2$. Then for every $2(1-1/\alpha) \le \tau \le 1$ there exists $C_\tau > 0$ depending on $\tau, \alpha, k, \lambda_2, \Delta$ such that for every smooth $f$ one has*

$$\int_{\mathbb{R}^d} f^2 F_\tau\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\right) d\mu \le C_\tau \int_{\mathbb{R}^d} f^2 c_{A,\alpha\tau/(\alpha-1)}\left(\frac{|\nabla f|}{|f|}\right) d\mu.$$

*In particular,*

$$\int_{\mathbb{R}^d} f^2 F_{2(1-1/\alpha)}\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\right) d\mu \le C_{2(1-1/\alpha)} \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu,$$

$$\begin{aligned}\int_{\mathbb{R}^d} f^2 \varphi\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\right) d\mu &\le C_1 \int_{\mathbb{R}^d} f^2 c^*_{A,\alpha}\left(\frac{|\nabla f|}{|f|}\right) d\mu \\ &= C_1 \int_{\mathbb{R}^d} f^2 c_{A,\alpha/(\alpha-1)}\left(\frac{|\nabla f|}{|f|}\right) d\mu.\end{aligned}$$

*In particular, the result holds if $\mu$ is convex and $g : \mathbb{R}^+ \to \mathbb{R}$ is increasing such that $\int_{\mathbb{R}^d} e^{g(r)}\, d\mu = 1$ and for some $C > 0$ one has*

$$\frac{g(r)}{\varphi^{1-1/\alpha}(e^{g(r)})} \ge Cr.$$

Obviously, if $\mu$ is convex, $\varphi = \log$ and

$$\int_{\mathbb{R}^d} e^{\varepsilon|x|^\alpha}\, d\mu < \infty \tag{12}$$

for some $\varepsilon > 0$, we obtain

$$\operatorname{Ent}_\mu f^2 \le C_1 \int_{\mathbb{R}^d} f^2 c_{A,\alpha/(\alpha-1)}\left(\frac{|\nabla f|}{|f|}\right) d\mu.$$

In particular, we generalize Wang's criterion for convex measures as well as the result of [16]. Note that unlike [16] we deal directly with multidimensional distributions and use a slightly different cost function for $d \ge 2$. We also apply the method developed in Theorem 2.1 to establish the following result (Theorem 4.4): let $\mu$ be a convex measure satisfying (12) for some $\alpha > 1$. Then

$$\operatorname{Ent}_\mu |f|^\beta \le C\left[\int_{\mathbb{R}^d} |\nabla f|^\beta \, d\mu + \operatorname{Var}_\mu |f|^{\beta/2}\right].$$

This inequality is weaker than (6) but unlike (6) it is established for an arbitrary convex measure.

During the preparation of the paper the author learned from Franck Barthe that modified Sobolev inequalities for convex measures can be obtained by using the transfer principle method (see [2]) and the results from [16]. However, this requires proving first inequalities on the real line by different methods. Another achievement in this direction has been obtained by Nathael Golzan [18], who proved a criterion for transportation inequalities of Talagrand type for the real line. In particular, his result implies modified Sobolev inequalities for convex measures on the real line, since they are known to be equivalent to transportation inequalities in the log-concave case.

## 2. Main result

Consider a probability measure $\mu = \rho\, dx$ on $\mathbb{R}^d$. We assume throughout that $X := \operatorname{supp}(\mu)$ is convex. In addition, without loss of generality we assume that $0 \in X$. Set

$$B_r = \{x : |x| \le r\}.$$

We denote by $R(X) \in (0, \infty]$ the smallest number such that $X \subset B_{R(X)}$. Recall that for every measurable mapping $F : X \to Y$ the image measure $\mu_F$ on $Y$ is defined by

$$\mu_F(A) = \mu(\{x : F(x) \in A\})$$

for every Borel set $A \subset Y$. For every non-negative function $f$ we denote by $\tilde{f}$ the corresponding spherical rearrangement, i.e., the function of the form $\tilde{f}(x) = g(|x|)$ such that $g$ is increasing and

$$\mu \circ f^{-1} = \mu \circ \tilde{f}^{-1}.$$

This can be rewritten as

$$\mu_f = \mu_r \circ g^{-1}$$

where $\mu_f = \mu \circ f^{-1}$ and $\mu_r$ is the image of $\mu$ under $x \mapsto |x|$. For a probability measure $\nu$ on $\mathbb{R}^+$ set

$$F_\nu(t) = \nu([0, t)) \quad \text{and} \quad G_\nu(u) = \{\inf s : F_\nu(s) \ge u\}.$$

Then $g$ has the form

$$g = G_{\mu_f} \circ F_{\mu_r}. \tag{13}$$

We denote by $B_r^c$ the complement of $B_r$ and by $R_t > 0$ the number such that

$$\mu(|x| \le R_t) = t, \qquad R_1 = R(X).$$

Since $X$ is convex and $0 \in X$, $R_t$ is well-defined.

For every $F : \mathbb{R}^+ \to \mathbb{R}$ we define the corresponding isoperimetric function $I_F$. First we set

$$J_F(s) = \frac{sF(1/s)}{\mathcal{I}_\mu(s)}.$$

Equivalently,

$$J_F(s) = \sup_{A\,:\,A\subset\mathbb{R}^d,\ \mu(A)=s} \left[\frac{sF(1/s)}{\mu^+(A)}\right].$$

Then we define

$$I_F(r) = J_F(1 - \mu(B_r)).$$

This is equivalent to

$$I_F(r) = \sup_{A\in\mathcal{M}_r} \frac{\mu(A)F(1/\mu(A))}{\mu^+(A)}, \quad \text{where} \quad \mathcal{M}_r = \{A : \mu(A) = 1 - \mu(B_r)\}.$$

We follow the convention that $I_F(R(X)) = 0$.

In what follows we consider a convex cost function $c : \mathbb{R}^+ \to \mathbb{R}^+$. Let

$$c^*(x) = \sup_{y\in\mathbb{R}^+} (\langle x, y\rangle - c(y)).$$

We recall that $c$ is called *superlinear* if $\lim_{x\to\infty} c(|x|)/|x| = \infty$. In what follows, for simplicity we set $\mu(f^2) = \int_{\mathbb{R}^d} f^2\,d\mu$.

THEOREM 2.1. *Let $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex superlinear function such that $c(0) = 0$ and let $F$ be a function on $\mathbb{R}^+$ satisfying assumptions* A1 *and* A2. *Let $K > 1$. Assume that for $R = R_{(K-1)/K}$ one has*

$$\int_{B_R^c} \Phi(4c \circ I_F(|x|))\,d\mu < \infty, \tag{14}$$

*where*

$$\Phi(x) = \sup_{y\in\mathbb{R}^+} (\langle x, y\rangle - yF(y) + y) = (yF(y) - y)^*(x).$$

*Then there exist $B, C > 0$ such that for every smooth $f$ the following estimates hold:*

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\right) d\mu \le 4 \int_{\{f^2 \ge K \int f^2\,d\mu\}} f^2 c^*\left(\frac{|\nabla f|}{|f|}\right) d\mu + B\int_{\mathbb{R}^d} f^2\,d\mu, \tag{15}$$

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\right) d\mu \le C \int_{\mathbb{R}^d} (f - \mu(f))^2 c^*\left(\frac{|\nabla f|}{|f - \mu(f)|}\right) d\mu + B\,\mathrm{Var}_\mu\, f. \tag{16}$$

PROOF. Let us fix some Lipschitz function $f$. Without loss of generality we may assume that $f \geq \varepsilon > 0$. Set $\nu := g \cdot \mu$, where $g = F(f^2/\int f^2\,d\mu)$. By a well-known result from measure theory one has

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\right) d\mu = \int_{\mathbb{R}^d} f^2 g\,d\mu = \int_{\mathbb{R}^d} f^2\,d\nu = \int_0^\infty \nu(f^2(x) > t)\,dt$$
$$= \int_0^\infty \left(\int_{\{x\,:\,f^2(x)>t\}} g\,d\mu\right) dt.$$

We split this integral into the following two parts:

$$I_1 = \int_0^{K\mu(f^2)} \left(\int_{\{x\,:\,f^2(x)>t\}} g\,d\mu\right) dt, \quad I_2 = \int_{K\mu(f^2)}^\infty \left(\int_{\{x\,:\,f^2(x)>t\}} g\,d\mu\right) dt.$$

The following proof will be divided into several steps.

STEP 1 (Estimation of $I_1$). We show that for some $C(K) > 0$ one has

$$I_1 \leq C(K)\,\mathrm{Var}_\mu\, f.$$

This part is quite elementary. By the concavity of $F$ one has

$$g \leq F'(1)\left(\frac{f^2}{\mu(f^2)} - 1\right).$$

Hence

$$\frac{I_1}{F'(1)} \leq \frac{1}{\mu(f^2)} \int_{\mathbb{R}^d} (f^2 - \mu(f^2))\left(\int_0^{K\mu(f^2)} I_{\{x\,:\,f^2(x)>t\}}\,dt\right) d\mu.$$
$$= \frac{1}{\mu(f^2)} \int_{\mathbb{R}^d} (f^2 - \mu(f^2)) \min(f^2, K\mu(f^2))\,d\mu$$
$$= \frac{1}{\mu(f^2)} \int_{\mathbb{R}^d} (f^2 - \mu(f^2))[\min(f^2, K\mu(f^2)) - \mu(f^2)]\,d\mu.$$

The latter equals

$$\frac{1}{\mu(f^2)} \int_{\{f^2 \leq K\mu(f^2)\}} (f^2 - \mu(f^2))^2\,d\mu$$
$$+ (K-1) \int_{\{f^2 \geq K\mu(f^2)\}} (f^2 - \mu(f^2))\,d\mu.$$

The first term can be estimated in the following way:

$$\frac{1}{\mu(f^2)} \int_{\{f^2 \leq K\mu(f^2)\}} (f^2 - \mu(f^2))^2\,d\mu$$
$$\leq \frac{2}{\mu(f^2)} \int_{\{f^2 \leq K\mu(f^2)\}} (f^2 - \mu(f)^2)^2\,d\mu + \frac{2}{\mu(f^2)}[\mathrm{Var}_\mu\, f]^2$$
$$\leq 4(K+1) \int_{\mathbb{R}^d} (f - \mu(f))^2\,d\mu + 2\,\mathrm{Var}_\mu\, f$$
$$= (4(K+1) + 2)\,\mathrm{Var}_\mu\, f.$$

Further we get

$$\int_{\{f^2\geq K\mu(f^2)\}}(f^2-\mu(f^2))\,d\mu\leq\int_{\{f^2\geq K\mu(f^2)\}}(f^2-\mu(f)^2)\,d\mu.$$

One can easily check that

$$|f+\mu(f)|\leq\frac{\sqrt{K}+1}{\sqrt{K}-1}|f-\mu(f)|$$

on $\{f^2\geq K\mu(f^2)\}$. Hence

$$\int_{\{f^2\geq K\mu(f^2)\}}(f^2-\mu(f^2))\,d\mu\leq\frac{\sqrt{K}+1}{\sqrt{K}-1}\operatorname{Var}_\mu f. \tag{17}$$

Finally, we obtain

$$I_1\leq[(4K+6)+(\sqrt{K}+1)^2]F'(1)\operatorname{Var}_\mu f.$$

STEP 2. Here we estimate $I_2$ by a quantity depending on the isoperimetric function $I_F$. Set

$$A_t=\{x:f^2(x)>t\}.$$

By the concavity of $F$ one has

$$\begin{aligned}
I_2&=\int_{K\mu(f^2)}^{\infty}\int_{\mathbb{R}^d}I_{A_t}F\left(\frac{f^2}{\int_{\mathbb{R}^d}f^2\,d\mu}\right)d\mu\,dt\\
&\leq\int_{K\mu(f^2)}^{\infty}\mu(A_t)\left[F\left(\int_{A_t}\frac{f^2}{\mu(A_t)\int_{\mathbb{R}^d}f^2\,d\mu}\right)d\mu\right]dt\\
&\leq\int_{K\mu(f^2)}^{\infty}\mu(A_t)F\left(\frac{1}{\mu(A_t)}\right)dt\\
&=\int_{K\mu(f^2)}^{\infty}\mu(\{x:f^2(x)>t\})F\left(\frac{1}{\mu(\{x:f^2(x)>t\})}\right)dt.
\end{aligned}$$

Since $f$ is continuous and $X$ is convex, the function $t\mapsto\mu(A_t)$ is strictly decreasing on

$$[\inf_{x\in X}f^2(x),\sup_{x\in X}f^2(x)].$$

Hence one can find a nondecreasing function $r_{f^2}(s)$ such that

$$\mu(A_s)=\mu(B^c_{r_{f^2}(s)})$$

and $r_{f^2}(0)=0$, $r_{f^2}(s)=R(X)$ if $s\geq\sup f^2$. Set

$$f_h(x)=\sup_{\{|x-y|\leq h\}}f(y).$$

By the definition of $I_F$ we have

$$I_2 \le \int_{K\mu(f^2)}^{\infty} I_F(r_{f^2}(t))\mu^+(A_t)\,dt \le \varliminf_{h\to 0+} \int_{K\mu(f^2)}^{\infty} I_F(r_{f^2}(t))\frac{\mu(A_t^h)-\mu(A_t)}{h}\,dt,$$

where $\{x \in \mathbb{R}^d : f_h^2(x) > t\} = \{x \in \mathbb{R}^d : f^2(x) > t\}^h = A_t^h$. Assume for a while that $s \mapsto I_F(r_{f^2}(s))$ is locally integrable and define

$$Z(t) := \begin{cases} \displaystyle\int_{K\mu(f^2)}^{t} I_F(r_{f^2}(s))\,ds, & t \ge K\mu(f^2) \\ 0, & t \le K\mu(f^2). \end{cases}$$

Applying the formula

$$\int \Phi(f^2)\,d\mu = \int_0^{\infty} \Phi'(t)\mu(A_t)\,dt,$$

which holds for every increasing $\Phi$ such that $\Phi(0) = 0$, we get

$$I_2 \le \varliminf_{h\to 0+} \int_{\mathbb{R}^d} \frac{Z(f_h^2)-Z(f^2)}{h}\,d\mu \le 2\int_{\{f^2\ge K\mu(f^2)\}} I_F(r_{f^2}(f^2))|f|\,|\nabla f|\,d\mu.$$

It remains to note that this estimate still holds even if $I_F(r_{f^2})$ is not locally integrable. Indeed, approximating $I_F$ by $I_F^N = I_F \wedge N$, we find in the same way as above that

$$\begin{aligned}\int_{K\mu(f^2)}^{\infty} I_F^N(r_{f^2}(t))\mu^+(A_t)\,dt &\le 2\int_{\{f^2\ge K\mu(f^2)\}} I_F^N(r_{f^2}(f^2))|f|\,|\nabla f|\,d\mu \\ &\le 2\int_{\{f^2\ge K\mu(f^2)\}} I_F(r_{f^2}(f^2))|f|\,|\nabla f|\,d\mu.\end{aligned}$$

We apply the monotone convergence theorem

$$I_2 \le \int_{K\mu(f^2)}^{\infty} I_F(r_{f^2}(t))\mu^+(A_t)\,dt = \lim_N \int_{K\mu(f^2)}^{\infty} I_F^N(r_{f^2}(t))\mu^+(A_t)\,dt,$$

and obtain the claim.

STEP 3. We now estimate

$$\int_{\{f^2\ge K\mu(f^2)\}} I_F(r_{f^2}(f^2))|f|\,|\nabla f|\,d\mu.$$

We complete the desired estimate by using the Young inequality. In this part rearrangement techniques will be employed. Namely, in the estimate below we replace $I_F(r_{f^2}(f^2))$ by $I_F(r_{f^2}(\tilde{f}^2))$ and take into account that $r_{f^2}(\tilde{f}^2(x)) = |x|$ on the set $\{x : |\nabla f(x)| \ne 0\}$.

Let $\mathbb{R}_\delta = \{t : \mu \circ (f^2)^{-1}(t) > 0\}$ be the set of atoms of the measure $\mu \circ (f^2)^{-1}$. Note that $|\nabla f| = 0$ almost everywhere on $D = \{x : f^2(x) \in \mathbb{R}_\delta\}$. Hence by the Young

inequality we find

$$(18)\quad 2\int_{\{f^2\ge K\mu(f^2)\}} I_F(r_{f^2}(f^2))|f|\,|\nabla f|\,d\mu \le 2\int_{\{f^2\ge K\mu(f^2)\}} f^2 c^*\left(\frac{|\nabla f|}{|f|}\right)d\mu + 2\int_{\{f^2\ge K\mu(f^2)\}\cap D^c} f^2[c\circ I_F(r_{f^2}(f^2))]\,d\mu.$$

Let $O_K=\{x : f^2(x)\ge K\mu(f^2)\}\cap D^c$. One has

$$I_{O_K} = I_{\{f^2\ge K\mu(f^2)\}}\cdot I_{\mathbb{R}^c_\delta}(f^2)$$

and by the Young inequality

$$\begin{aligned}2\int_{O_K} f^2 c(I_F(r_{f^2}(f^2)))\,d\mu &= 2\int_{\mathbb{R}^d} f^2 I_{O_K} c(I_F(r_{f^2}(f^2)))\,d\mu\\ &= \frac{1}{2}\mu(f^2)\int_{\mathbb{R}^d}\left[\frac{f^2}{\mu(f^2)}\right][4I_{O_K}c(I_F(|r_{f^2}(f^2)|))]\,d\mu\\ &\le \frac{1}{2}\int_{\mathbb{R}^d} f^2\left[F\left(\frac{f^2}{\mu(f^2)}\right)-1\right]d\mu + \frac{1}{2}\mu(f^2)\int_{\mathbb{R}^d}\Phi(4I_{O_K}c(I_F(r_{f^2}(f^2))))\,d\mu.\end{aligned}$$

Since $f$ and $\tilde f$ have the same laws considered as random variables on the probability space $(\mathbb{R}^d,\mu)$, one has

$$\int_{\mathbb{R}^d}\Phi(4I_{O_K}c(I_F(r_{f^2}(f^2))))\,d\mu = \int_{\mathbb{R}^d}\Phi(4I_{\tilde O_K}c(I_F(r_{f^2}(\tilde f^2))))\,d\mu$$

where $\tilde O_K=\{x : \tilde f^2(x)\ge K\mu(f^2)\}\cap\{x : \tilde f^2(x)\in\mathbb{R}^c_\delta\}$. Let $x\in\widetilde{O}_K$. By the definition of $\tilde f$ we have

$$\mu(\{y : f^2(y)>\tilde f^2(x)\}) = \mu(\{y : \tilde f^2(y)>\tilde f^2(x)\}).$$

Then for every such $x$ by the definition of $r_{f^2}$ we have

$$\mu(B^c_{r_{f^2}(\tilde f^2)}) = \mu(\{y : \tilde f^2(y)>\tilde f^2(x)\}) = \mu(\{y : |y|>|x|\}).$$

Indeed, otherwise there exist $r_1<r_2$ such that $\tilde f(z)=\tilde f(x)$ for every $z$ with $r_1\le|z|\le r_2$. But this implies that $\mu(\{y : f(y)=\tilde f(x)\})>0$. This contradicts the choice of $x$. Hence $r_{f^2}(\tilde f^2)(x)=|x|$ on $\tilde O_K$. Moreover, if $x\in\{\tilde f^2\ge K\mu(\tilde f^2)\}$, then by the Chebyshev inequality

$$\mu(B^c_{|x|}) = \mu(B^c_{r_{f^2}(\tilde f^2)})\le\mu(\{\tilde f^2\ge K\mu(\tilde f^2)\})\le 1/K.$$

Hence $|x|=r_{f^2}(\tilde f^2(x))\ge R_{(K-1)/K}$ if $x\in\{\tilde f^2\ge K\mu(\tilde f^2)\}$. Thus

$$\tilde O_K\subset\{x : |x|\ge R_{(K-1)/K}\}.$$

Hence

$$\int_{\mathbb{R}^d}\Phi(4I_{\tilde O_K}c(I_F(r_{f^2}(\tilde f^2))))\,d\mu\le\Phi(0)+\int_{B^c_{R_{(K-1)/K}}}\Phi(4c(I_F(|x|)))\,d\mu =: \tilde B<\infty.$$

Finally,

$$\frac{1}{2}\int_{\mathbb{R}^d} f^2\left[F\left(\frac{f^2}{\mu(f^2)}\right)-1\right]d\mu+\frac{\mu(f^2)}{2}\int_{\mathbb{R}^d}\Phi(4I_{O_K}c(I_F(r_{f^2}(f^2))))\,d\mu \le \frac{1}{2}\int_{\mathbb{R}^d} f^2F\left(\frac{f^2}{\mu(f^2)}\right)d\mu+\frac{\tilde{B}-1}{2}\int_{\mathbb{R}^d} f^2\,d\mu$$

and

$$I_2 \le \frac{\tilde{B}-1}{2}\int_{\mathbb{R}^d} f^2\,d\mu+\frac{1}{2}\int_{\mathbb{R}^d} f^2F\left(\frac{f^2}{\mu(f^2)}\right)d\mu+2\int_{\{f^2\ge K\mu(f^2)\}} f^2c^*\left(\frac{|\nabla f|}{|f|}\right)d\mu.$$

Combining all the inequalities obtained above, we get (15).

The proof of (16) is similar and we just briefly describe the main difference. Instead of (18) we use

$$2\int_{\{f^2\ge K\mu(f^2)\}} I_F(r_{f^2}(f^2))|f|\,|\nabla f|\,d\mu \le C'\int_{\{f^2\ge K\mu(f^2)\}}(f-\mu(f))^2c^*\left(\frac{|\nabla f|}{|f-\mu(f)|}\right)d\mu + C'\int_{O_K}(f-\mu(f))^2[c\circ I_F(r_{f^2}(f^2))]\,d\mu.$$

This follows from the Young inequality and the observation that

$$f^2 \le \frac{K}{(\sqrt{K}-1)^2}(f-\mu(f))^2$$

on $\{f^2 \ge K\mu(f^2)\}$. In the same way as above we estimate the second term by $\operatorname{Var}_\mu f$ and $\int_{\mathbb{R}^d}\tilde{f}^2F(\tilde{f}^2/\int_{\mathbb{R}^d}\tilde{f}^2\,d\mu)\,d\mu$, where $\tilde{f}=f-\mu(f)$. Finally, by (15) one has

$$\int_{\mathbb{R}^d}\tilde{f}^2F\left(\frac{\tilde{f}^2}{\int_{\mathbb{R}^d}\tilde{f}^2\,d\mu}\right)d\mu \le 4\int_{\mathbb{R}^d}(f-\mu(f))^2c^*\left(\frac{|\nabla f|}{|f-\mu(f)|}\right)d\mu+B\operatorname{Var}_\mu f.$$

The proof is complete. □

EXAMPLE 2.2. *Assume that $c$ is a convex superlinear function satisfying* (H). *Let $\mu$ be a convex measure such that $\int_{\mathbb{R}^d} e^{\varepsilon c(r)}\,d\mu<\infty$ for some $\varepsilon>0$. Then for every $K$ there exist $B, C>0$ such that*

$$\operatorname{Ent}_\mu f^2 \le C\int_{\{f^2\ge K\int f^2\,d\mu\}} f^2c^*\left(\frac{|\nabla f|}{|f|}\right)d\mu+B\int_{\mathbb{R}^d} f^2\,d\mu. \tag{19}$$

PROOF. Let $F=\log$. It will be shown below that $\sup_{r\ge R_{1/2}} I_{\log}(r)/r<\infty$ for every convex $\mu$ (Lemma 4.1). The result then follows immediately from Theorem 2.1. □

THEOREM 2.3. *Let $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex superlinear function such that $c(0) = 0$. Assume that $F$ satisfies assumptions* A1–A2 *and there exists $K > 1$ such that*

$$\int_0^{1/K} \Phi\left(4c\left[\frac{tF(1/t)}{\mathcal{I}_\mu(t)}\right]\right) dt < \infty. \tag{20}$$

*Then inequalities* (15) *and* (16) *hold.*

PROOF. By the definition of $I_F$ one has

$$I_F(r) = \frac{(1-\mu(B_r))F\left(\frac{1}{1-\mu(B_r)}\right)}{\mathcal{I}_\mu(1-\mu(B_r))}.$$

It suffices to show that

$$\int_{B^c_{R_{(K-1)/K}}} \Phi(4c \circ I_F(|x|))\, d\mu < \infty.$$

The mapping $\mathbb{R}^d \ni x \mapsto 1 - \mu(y : |y| \le |x|) = t \in [0,1]$ transforms $\mu$ into Lebesgue measure on $[0,1]$. Hence the integrability of $\Phi(4c(I_F))$ is equivalent to (20) for some $\varepsilon > 0$. □

REMARK 2.4. Note that the constant 4 in (14) and (20) yields the term

$$4\int_{\{f^2 \ge K \int f^2\, d\mu\}} f^2 c^*\left(\frac{|\nabla f|}{|f|}\right) d\mu$$

in (15). However, if $c$ satisfies (H), it is more convenient to assume that

$$\int_0^{1/K} \Phi\left(\delta c\left[\frac{tF(1/t)}{\mathcal{I}_\mu(t)}\right]\right) dt < \infty$$

for some $\delta > 0$ and $K > 1$. It is easy to check (just apply Theorems 2.1 and 2.3 to $\tilde{c} = \varepsilon c$ with appropriate $\varepsilon$) that (15), (16) still hold (possibly with some other constant in place of 4).

The following theorem is a direct corollary of (16).

THEOREM 2.5. *Let $F$ and $\mu$ satisfy the assumptions of Theorem* 2.1 *with $c = \delta|x|^2$ and some $\delta > 0$. Then for every smooth $f$ one has*

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\right) d\mu \le C\int_{\mathbb{R}^d} |\nabla f|^2\, d\mu + B\,\mathrm{Var}_\mu\, f.$$

*In particular, the result holds if assumptions* A1–A2 *are satisfied and there exist $K > 1$, $\delta > 0$ such that*

$$\int_0^{1/K} \Phi\left(\delta\left[\frac{tF(1/t)}{\mathcal{I}_\mu(t)}\right]^2\right) dt < \infty. \tag{21}$$

EXAMPLE 2.6 ($d = 1$). Consider a probability measure on the real line $\mu = e^{-V(t)}\,dt$. In the one-dimensional case the proof can be simplified. We omit the details and just briefly explain the main ideas. Instead of using the coarea inequality one can apply the Newton–Leibniz formula

$$f(x) = f(m) + \int_m^x f'(s)\,ds,$$

where $m \in \mathbb{R}$. It is convenient to take for $m$ the median of $\mu$. The use of the Newton–Leibniz formula allows one to apply the simplified analog of the isoperimetric function $\tilde{\mathcal{I}}_\mu$. Let $0 \le t \le 1/2$. Define $u(t) \le m$ and $v(t) \ge m$ as follows:

$$\mu((-\infty, u(t)]) = \mu([v(t), \infty) = t.$$

Then

$$\tilde{\mathcal{I}}_\mu(t) = \min\{e^{V(u(t))}, e^{V(v(t))}\}t.$$

One can get the following analog of Theorem 2.5:

*Let assumptions* A1–A2 *be satisfied and let* $K > 2$ *and* $\delta > 0$ *be such that*

$$\int_0^{1/K} \Phi\left(\delta\left[\frac{tF(1/t)}{\tilde{\mathcal{I}}_\mu(t)}\right]^2\right) dt < \infty. \tag{22}$$

*Then*

$$\int_{\mathbb{R}} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}} f^2\,d\mu}\right) d\mu \le C\int_{\mathbb{R}} |f'|^2\,d\mu + B\cdot\int_{\mathbb{R}} (f - f(m))^2\,d\mu \tag{23}$$

*for some* $B, C > 0$ *and every smooth* $f$.

*If, in addition,* $\mu$ *satisfies the Poincaré inequality, then the term* $\int_{\mathbb{R}}(f - f(m))^2\,d\mu$ *can be estimated by* $C'\int_{\mathbb{R}} |f'|^2\,d\mu$ (see [9]) *and be omitted in* (23)*:*

$$\int_{\mathbb{R}} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}} f^2\,d\mu}\right) d\mu \le C\int_{\mathbb{R}} |f'|^2\,d\mu. \tag{24}$$

As an example consider the following measure on the line:

$$\mu = Ze^{-|x|\log(1+x^2)}\,dx.$$

It can be easily verified that as $s \to \infty$ one has

$$\mu((-\infty, -s]) = \mu([s, \infty)) \sim \frac{Ze^{-|s|\log(1+s^2)}}{\log(1+s^2)}.$$

Since $\mu^+([s, \infty)) = Ze^{-|s|\log(1+s^2)}$, we get

$$\tilde{\mathcal{I}}_\mu(t) \ge C't\log(\log(1/t))$$

for some $C'$ and every $t < \varepsilon$ with some sufficiently small $\varepsilon$. Let us choose a function $F$ satisfying assumptions A1–A2 of Theorem 2.1 such that

$$F(x) \sim \log^2(\log x)$$

for large values of $x$. In this case

$$\Phi(y) \sim \exp(e^{\sqrt{y}})$$

for large $y$. Hence for any sufficiently small $\delta$ and all $t \in [0, 1/2]$ one has

$$\Phi\left(\delta\left[\frac{tF(1/t)}{\tilde{\mathcal{I}}_\mu(t)}\right]^2\right) \le \exp(\log^p(1/t)),$$

where $p$ can be taken arbitrarily small. Since

$$\int_0^{1/2} \exp(\log^p(1/t))\,dt < \infty$$

for $p < 1$, we obtain (24).

## 3. Tight estimates

In this section we establish some *tight estimates*, i.e., estimates whose right-hand sides vanish on constant functions. The case of the $F$-Sobolev inequality has already been considered in Theorem 2.5. Unlike the $F$-Sobolev inequality, the case of tight modified log-Sobolev inequalities is more difficult. We use an idea from [16] and consider two cases: of large and small entropy. The large entropy case follows immediately from our main result. In the case of small entropy we reduce the problem to the $F$-inequality.

In what follows we assume that there exists $\lambda_2 > 0$ such that for every smooth $f$ one has

$$\int_{\mathbb{R}^d} (f - m_f)^2\,d\mu \le \lambda_2 \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu. \tag{25}$$

Since $\int_{\mathbb{R}^d}(f - \int_{\mathbb{R}^d} f\,d\mu)^2\,d\mu \le \int_{\mathbb{R}^d}(f - m_f)^2\,d\mu$, this inequality is stronger than the classical $L^2$-Poincaré inequality.

Definition 3.1. *We say that a probability measure $\mu$ satisfies the* Cheeger isoperimetric inequality *if there exists $\lambda_1 > 0$ such that for every Borel set $A$ one has*

$$\min(\mu(A), 1 - \mu(A)) \le \lambda_1 \mu^+(A). \tag{26}$$

Inequality (26) is equivalent to the following $L^1$-Poincaré-type inequality:

$$\int_{\mathbb{R}^d} \left| f - \int_{\mathbb{R}^d} f\,d\mu \right| d\mu \le \lambda_1 \int_{\mathbb{R}^d} |\nabla f|\,d\mu. \tag{27}$$

It was shown in [8] that (26) implies (25). It is known that every convex measure satisfies (27) with some $\lambda_1$ (see [20] and [8]).

We start this section with several lemmas.

Lemma 3.2. *Let $F$ satisfy assumptions* A1, A2 *and* A4. *Then for every $\delta \in (0, 1/2]$, there exists $T$ depending on $\delta$ and $y_0$ such that for any $y \ge T$ one has*

$$\Phi(\delta F(y)) \le y^{2\delta}.$$

PROOF. Since $F$ is increasing and $\lim_{y\to\infty} F(y) = \infty$, the supremum of $xy - yF(y) + y$ is attained at some $y^*$. Moreover, there exists $x_0$ such that $y^* \geq y_0$ if $x \geq x_0$. In this case one has

$$x = F(y^*) + y^* F'(y^*) - 1 \tag{28}$$

and by the properties of $F$,

$$F(y^*) - 1 \leq x \leq F(y^*).$$

Consequently, $y^* \leq F^{-1}(1+x)$ and by (28) we find

$$\Phi(x) = xy^* - y^* F(y^*) + y^* = (y^*)^2 F'(y^*).$$

Hence for any $x \geq x_0$ one has

$$\Phi(x) \leq y^* \leq F^{-1}(1+x). \tag{29}$$

Next, for any $y \geq y_0$, we have

$$F(y^{2\delta}) - F(y_0^{2\delta}) = 2\delta \int_{y_0}^{y} s^{2\delta-1} F'(s^{2\delta})\, ds.$$

Taking into account that $s^{2\delta} \leq s$, by A4 we get

$$F(y^{2\delta}) - F(y_0^{2\delta}) = 2\delta \int_{y_0}^{y} \frac{s^{2\delta} F'(s^{2\delta})}{s}\, ds \geq 2\delta \int_{y_0}^{y} \frac{sF'(s)}{s}\, ds = 2\delta(F(y) - F(y_0)).$$

Finally,

$$\delta F(y) \leq \delta F(y_0) - \frac{1}{2} F(y_0^{2\delta}) + \frac{1}{2} F(y^{2\delta}).$$

Thus, if $F(y) \geq x_0/\delta$, by (29) we obtain

$$\Phi(\delta F(y)) \leq F^{-1}\left(1 + \delta F(y_0) - \frac{1}{2} F(y_0^{2\delta}) + \frac{1}{2} F(y^{2\delta})\right).$$

Choosing $T \geq F^{-1}(x_0/\delta)$ such that $\frac{1}{2} F(y^{2\delta}) \geq 1 + \delta F(y_0) - \frac{1}{2} F(y_0^{2\delta})$ for $y \geq T$, we obtain

$$\Phi(\delta F(y)) \leq F^{-1}\left(1 + \delta F(y_0) - \frac{1}{2} F(y_0^{2\delta}) + \frac{1}{2} F(y^{2\delta})\right) \leq F^{-1}(F(y^{2\delta})) \leq y^{2\delta}. \qquad \square$$

LEMMA 3.3. *Let $\mu$ be a probability measure and let $F$ satisfy assumptions* A1, A2, *and* A4. *Then there exists $C > 0$ such that for all $f, g \in L^2(\mu)$ one has*

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{g^2}{\int_{\mathbb{R}^d} g^2\, d\mu}\right) d\mu \leq 2 \int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\right) d\mu + C \int_{\mathbb{R}^d} f^2\, d\mu.$$

PROOF. Set

$$u = F\left(\frac{g^2}{\mu(g^2)}\right) + \frac{g^2}{\mu(g^2)} F'\left(\frac{g^2}{\mu(g^2)}\right) - 1.$$

Since $F' > 0$, one has

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{g^2}{\mu(g^2)}\right) d\mu \le \int_{\mathbb{R}^d} f^2 u \, d\mu + \int_{\mathbb{R}^d} f^2 \, d\mu.$$

By the Young inequality

$$\begin{aligned}\int_{\mathbb{R}^d} f^2 u \, d\mu &= \int_{\mathbb{R}^d} f^2 \, d\mu \int_{\mathbb{R}^d} \frac{f^2}{\mu(f^2)} u \, d\mu \\ &\le 2 \int_{\mathbb{R}^d} f^2 \, d\mu \left( \int_{\mathbb{R}^d} \left[ \frac{f^2}{\mu(f^2)} F\left(\frac{f^2}{\mu(f^2)}\right) - \frac{f^2}{\mu(f^2)} \right] d\mu \right) \\ &\quad + 2 \int_{\mathbb{R}^d} f^2 \, d\mu \int_{\mathbb{R}^d} \Phi(u/2) \, d\mu.\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^d} f^2 F\left(\frac{g^2}{\mu(g^2)}\right) d\mu &\le 2 \int_{\mathbb{R}^d} f^2 \, d\mu \left( \int_{\mathbb{R}^d} \left[ \frac{f^2}{\mu(f^2)} F\left(\frac{f^2}{\mu(f^2)}\right) - \frac{f^2}{\mu(f^2)} \right] d\mu \right) \\ &\quad + 2 \int_{\mathbb{R}^d} f^2 \, d\mu \int_{\mathbb{R}^d} \Phi(u/2) \, d\mu + \int_{\mathbb{R}^d} f^2 \, d\mu \\ &= 2 \int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu + \int_{\mathbb{R}^d} f^2 \, d\mu \left( \int_{\mathbb{R}^d} (2\Phi(u/2) - 1) \, d\mu \right).\end{aligned}$$

Using the estimate $\Phi(x) \le F^{-1}(1 + x)$ obtained in the proof of Lemma 3.2 for large values of $x$, we find that for sufficiently large values of $g^2/\mu(g^2)$,

$$\Phi(u/2) \le F^{-1}\left(1 + \frac{u}{2}\right) \le F^{-1}\left(1 + \frac{1}{2} F\left(\frac{g^2}{\mu(g^2)}\right)\right) \le F^{-1}\left(F\left(\frac{g^2}{\mu(g^2)}\right)\right) = \frac{g^2}{\mu(g^2)}.$$

Hence $\Phi(u/2)$ is bounded by $g^2/\mu(g^2) + B$ for a sufficiently large number $B$ depending only on $F$ and $\int_{\mathbb{R}^d} (2\Phi(u/2) - 1) \, d\mu \le 2B + 1$. This completes the proof. □

In the following lemma we prove some simple estimates which will be used below.

LEMMA 3.4. *Suppose that $F$ satisfies assumptions* A1–A3. *For every $K > 1$ there exist a number $B$ depending on $K$ and a number $C$ depending on $K$ and $\Delta$ such that for every $f \in L^2(\mu)$ one has*

$$\int_{\{f^2 \ge K\mu(f^2)\}} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \le C \operatorname{Var}_\mu f + \int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu,$$

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \le B \operatorname{Var}_\mu f + 2 \int_{\mathbb{R}^d} \left(f(x) - \sqrt{K\mu(f^2)}\right)_+^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu.$$

PROOF. To prove the first estimate let $\tilde{K} = \min(K, 1 + \Delta)$. Since $F(y) \geq 0$ for $y \geq 1$, one has

$$-\int_{\{f^2 \leq K\mu(f^2)\}} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \leq -\int_{\{f^2 \leq \tilde{K}\mu(f^2)\}} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu.$$

By the concavity of $-yF(y)$ on $[0, 1 + \Delta]$ one has

$$-yF(y) \leq (-yF(y))'_{y=1}(y - 1) = F'(1)(1 - y).$$

Hence

$$\begin{aligned} -\int_{\{f^2 \leq \tilde{K}\mu(f^2)\}} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu &\leq F'(1)\mu(f^2) \int_{\{f^2 \leq \tilde{K}\mu(f^2)\}} \left(1 - \frac{f^2}{\mu(f^2)}\right) d\mu \\ &= F'(1)\mu(f^2) \int_{\{f^2 \geq \tilde{K}\mu(f^2)\}} \left(\frac{f^2}{\mu(f^2)} - 1\right) d\mu. \end{aligned}$$

The desired estimate now follows from (17).

Let us prove the second estimate. Since $F(y) \geq 0$ for $y \geq K > 1$ and

$$f^2 \leq 2K\mu(f^2) + 2\left(f - \sqrt{K\mu(f^2)}\right)_+^2,$$

one has

$$\begin{aligned} &\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \\ &\quad \leq \int_{\{f^2 \leq K\mu(f^2)\}} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu + \int_{\{f^2 \geq K\mu(f^2)\}} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu. \\ &\quad \leq \int_{\mathbb{R}^d} \min(f^2, K\mu(f^2)) F\left(\frac{f^2}{\mu(f^2)}\right) d\mu + 2K\mu(f^2) \int_{\{f^2 \geq K\mu(f^2)\}} F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \\ &\quad + 2\int_{\mathbb{R}^d} \left(f - \sqrt{K\mu(f^2)}\right)_+^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu. \end{aligned}$$

The first term on the right-hand side does not exceed

$$F'(1) \int_{\mathbb{R}^d} \min(f^2, K\mu(f^2)) \left(\frac{f^2}{\mu(f^2)} - 1\right) d\mu.$$

This can be estimated by $\tilde{C}(K) \operatorname{Var}_\mu f$ (see Step 1 in the proof of Theorem 2.1). Applying (17) and concavity of $F$ we get a similar estimate of the second term. The proof is complete. □

Now we are ready to prove the main result on the tight inequalities. Following an idea from [16] we reduce the problem to $F$-Sobolev inequalities. Set $\beta = \alpha/(\alpha - 1)$. For every $\tau \geq 2/\beta$, we consider the following perturbation of $F$:

$$F_{\tau,\beta} = \psi_{\tau,\beta}(F),$$

where

$$\psi_{\tau,\beta}(x) = \begin{cases} x, & x \le 1, \\ \frac{1}{2}\beta([1+\tau(x-1)]^{2/\tau\beta} - 1) + 1, & x \ge 1. \end{cases}$$

Note that $\psi_{\tau,\beta}(x)$ is a concave increasing function such that $\psi_{\tau,\beta}(x) \le x$. Obviously, $\psi_{2/\beta,\beta}(x) = x$.

REMARK 3.5. It can be easily verified that this perturbation preserves functions satisfying assumptions A1–A4.

THEOREM 3.6. *Let $\alpha > 1$ and $1 \ge \tau \ge 2/\beta$. Consider the cost function $c = c^*_{A,\alpha\tau/(\alpha-1)}$, where $A, \varepsilon > 0$. Assume that $F$, $c$, $\mu$, and $K$ satisfy the assumptions of Theorem* 2.1 *for some $K \ge 2$. Assume in addition that*

1) *$F$ satisfies assumptions* A3–A4,
2) *there exists $\delta > 0$ such that for $R = R_{(K-1)/K}$ one has*

$$\int_{B_R^c} \Phi_{\tau,\beta}(\delta |I_{F_{\tau,\beta}}|^2)\, d\mu < \infty,$$

*where*

$$\Phi_{\tau,\beta}(x) = \sup_{\{y>0\}} (\langle x, y\rangle - yF_{\tau,\beta}(y) + y) = (yF_{\tau,\beta}(y) - y)^*(x),$$

3) *$\mu$ satisfies* (25) *for some $\lambda_2$.*

*Then there exist $B, C > 0$ such that the following modified $F$-Sobolev inequality holds:*

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\right) d\mu \le C \int_{\{f^2 \ge K \int_{\mathbb{R}^d} f^2\, d\mu\}} f^2 c^*\left(\frac{|\nabla f|}{|f|}\right) d\mu + B \operatorname{Var}_\mu f. \tag{30}$$

PROOF. We follow the arguments from [16]. The case $\tau = 2/\beta$ follows from Theorem 2.5 and Remark 2.4. Let $\tau > 2/\beta$. Consider a smooth function $f$. Without loss of generality one can assume that $\inf_{x\in X} f(x) > 0$. If $f$ satisfies the inequality

$$\int_{\mathbb{R}^d} f^2\, d\mu \le \frac{1}{2B} \int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu,$$

where $B = B(K)$ is as in (15), then (30) follows directly from Theorem 2.1. Hence one can assume that

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\right) d\mu \le 2B \int_{\mathbb{R}^d} f^2\, d\mu. \tag{31}$$

Note that if $\sup_{x\in X} f^2 \le K\mu(f^2)$, then by the concavity of $F$,

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\right) d\mu \le C(K) \operatorname{Var}_\mu f$$

(see the reasoning of Theorem 2.1, Step 1). Hence without loss of generality one can assume that there exists $x_0$ such that $f(x_0) = \sqrt{K\mu(f^2)}$. Set

$$g(x) = f(x_0) + (f(x) - f(x_0))_+ P\left(\frac{f^2}{\mu(f^2)}\right)/P(K),$$

where

$$P(x) = \sqrt{\frac{F(x)}{F_{\tau,\beta}(x)}} = \sqrt{\frac{F(x)}{\psi_{\tau,\beta}(F(x))}}.$$

Obviously, $g \geq f$, since $x \mapsto x/\psi_{\tau,\beta}(x)$ is increasing. In addition, since $\psi_{\tau,\beta}$ is increasing, we get

$$\psi_{\tau,\beta}\left(F\left(\frac{f^2}{\mu(f^2)}\right)\right) \geq \psi_{\tau,\beta}(F(K))$$

if $f(x) \geq f(x_0)$. Hence by the Cauchy inequality we get

$$\int_{\mathbb{R}^d} g^2\,d\mu \leq C_1(K)\left(\int_{\mathbb{R}^d} f^2\,d\mu + \int_{\{f^2 \geq K\mu(f^2)\}} f^2 F\left(\frac{f^2}{\mu(f^2)}\right)d\mu\right)$$

for some $C_1(K)$. By Lemma 3.4,

$$\int_{\{f^2 \geq K\mu(f^2)\}} f^2 F\left(\frac{f^2}{\mu(f^2)}\right)d\mu \leq \int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\mu(f^2)}\right)d\mu + C_2(K)\operatorname{Var}_\mu f. \tag{32}$$

Hence by (31) there exists $M = M(K)$ such that

$$\int_{\mathbb{R}^d} g^2\,d\mu \leq M\int_{\mathbb{R}^d} f^2\,d\mu.$$

Taking into account that $g \geq f$, one gets

$$\begin{aligned}\int_{\mathbb{R}^d} ([g - g(x_0)]_+)^2 F_{\tau,\beta}\left(\frac{g^2}{\mu(g^2)}\right)d\mu &\geq \int_{\mathbb{R}^d} ([g - g(x_0)]_+)^2 F_{\tau,\beta}\left(\frac{f^2}{M\mu(f^2)}\right)d\mu \\ &= \frac{1}{P^2(K)}\int_{\mathbb{R}^d} ([f - f(x_0)]_+)^2 F\left(\frac{f^2}{\mu(f^2)}\right)\frac{F_{\tau,\beta}(f^2/M\mu(f^2))}{F_{\tau,\beta}(f^2/\mu(f^2))}\,d\mu.\end{aligned}$$

By the concavity of $F_{\tau,\beta}$ one has $\inf_{x\geq 2M} F_{\tau,\beta}(x/M)/F_{\tau,\beta}(x) = a > 0$. Hence

$$\begin{aligned}&\int_{\mathbb{R}^d} ([g - g(x_0)]_+)^2 F_{\tau,\beta}\left(\frac{g^2}{\mu(g^2)}\right)d\mu \\ &\qquad \geq \frac{a}{P^2(K)}\int_{\{f^2 \geq 2M\mu(f^2)\}} ([f - f(x_0)]_+)^2 F\left(\frac{f^2}{\mu(f^2)}\right)d\mu \\ &\qquad\quad - \sup_{K\leq t\leq 2M}\left|\frac{F_{\tau,\beta}(t/M)}{F_{\tau,\beta}(t)}F(t)\right|\int_{\mathbb{R}^d} ([f - f(x_0)]_+)^2\,d\mu.\end{aligned}$$

Thus for some $A_1 = A_1(K) > 0$ one has

$$\int_{\mathbb{R}^d} ([f - f(x_0)]_+)^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \leq A_1 \int_{\mathbb{R}^d} ([g - g(x_0)]_+)^2 F_{\tau,\beta}\left(\frac{g^2}{\mu(g^2)}\right) d\mu + A_1 \int_{\mathbb{R}^d} ([f - f(x_0)]_+)^2 d\mu.$$

We observe that the second term on the right-hand side can be estimated by $\mathrm{Var}_\mu f$, since $(f - f(x_0))_+ \leq |f - \mu(f)|$. By Lemma 3.3 we obtain

$$\int_{\mathbb{R}^d} ([g - g(x_0)]_+)^2 F_{\tau,\beta}\left(\frac{g^2}{\mu(g^2)}\right) d\mu \leq 2 \int_{\mathbb{R}^d} ([g - g(x_0)]_+)^2 F_{\tau,\beta}\left(\frac{([g - g(x_0)]_+)^2}{\mu([g - g(x_0)]_+)^2)}\right) d\mu + c' \int_{\mathbb{R}^d} ([g - g(x_0]_+)^2 d\mu.$$

Since

$$\mu(\{x : g(x) > g(x_0)\}) = \mu(\{x : f(x) > f(x_0)\}) \leq \frac{1}{K} \leq \frac{1}{2},$$

0 is the median of $(g - g(x_0))_+$. Hence by (25),

$$\int_{\mathbb{R}^d} ([g - g(x_0)]_+)^2 d\mu \leq \lambda_2 \int_{\mathbb{R}^d} |\nabla g|^2 d\mu.$$

By assumption 2) and Theorem 2.5, $\mu$ satisfies the $F_{\tau,\beta}$-Sobolev inequality, hence

$$\int_{\mathbb{R}^d} ([g - g(x_0)]_+)^2 F_{\tau,\beta}\left(\frac{([g - g(x_0)]_+)^2}{\mu([g - g(x_0)]_+)^2}\right) d\mu \leq A_2 \int_{\mathbb{R}^d} |\nabla g|^2 d\mu.$$

Combining the estimates obtained above, we get

$$\int_{\mathbb{R}^d} ([f - f(x_0)]_+)^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \leq C'\left(\int_{\mathbb{R}^d} |\nabla g|^2 d\mu + \mathrm{Var}_\mu f\right).$$

Let us estimate $\nabla g$. Set $h = f^2/\mu(f^2)$. One has

$$\nabla g = \left[\frac{(f - f(x_0))_+}{P(h) \cdot P(K)}\left(\frac{F'}{\psi_{\tau,\beta}(F)} - \frac{F\psi'_{\tau,\beta}(F)F'}{\psi^2_{\tau,\beta}(F)}\right)(h)\frac{f}{\mu(f^2)}\right]\nabla f + \left[I_{\{f \geq f(x_0)\}}\frac{P(h)}{P(K)}\right]\nabla f.$$

Let us show that for some $B_1 = B_1(K) > 0$ one has

$$|\nabla g|^2 \leq B_1 P^2(h)|\nabla f|^2.$$

It is sufficient to verify that

$$\frac{(f - f(x_0))_+}{P^2(h)}\left(\frac{F'}{\psi_{\tau,\beta}(F)} - \frac{F\psi'_{\tau,\beta}(F)F'}{\psi^2_{\tau,\beta}(F)}\right)(h)\frac{f}{\mu(f^2)}$$

is bounded. Since $f(f - f(x_0))_+/\mu(f^2) \le h$ and $P^2 = F/\psi_{\tau,\beta}(F)$, we have to show that

$$\frac{x\psi_{\tau,\beta}(F)}{F}\left(\frac{F'}{\psi_{\tau,\beta}(F)} - \frac{F\psi'_{\tau,\beta}(F)F'}{\psi^2_{\tau,\beta}(F)}\right) = \frac{xF'}{F} - \frac{x\psi'_{\tau,\beta}(F)F'}{\psi_{\tau,\beta}(F)} = \frac{xF'}{F}\left(1 - \frac{F\psi'_{\tau,\beta}(F)}{\psi_{\tau,\beta}(F)}\right)$$

is uniformly bounded on $[K, \infty)$. Indeed, it can be verified directly that

$$0 \le \frac{x\psi'_{\tau,\beta}(x)}{\psi_{\tau,\beta}(x)} \le 1.$$

The boundedness of $xF'/F$ is obvious. Finally, we obtain

$$\int_{\mathbb{R}^d} ([f - f(x_0)]_+)^2 F(h)\, d\mu \le C \int_{\{f^2 \ge K\mu(f^2)\}} |\nabla f|^2 \frac{F(h)}{\psi_{\tau,\beta}(F(h))}\, d\mu.$$

The right-hand side can be estimated by

$$CN^{\beta_\tau} \int_{\{f^2 \ge K\mu(f^2)\}} f^2 \left|\frac{\nabla f}{f}\right|^{\beta_\tau} d\mu + \frac{C}{N^{(\beta_\tau/2)^*}} \int_{\{f^2 \ge K\mu(f^2)\}} f^2 \left|\frac{F(h)}{\psi_{\tau,\beta}(F(h))}\right|^{(\beta_\tau/2)^*} d\mu$$

for arbitrary $N$. Here

$$\beta_\tau = \frac{\alpha\tau}{\alpha - 1}, \qquad \left(\frac{\beta_\tau}{2}\right)^* = \frac{\alpha\tau}{2 + \alpha(\tau - 2)}.$$

We note that there exists $C' = C'(K)$ such that for $x \ge K$ one has

$$\left(\frac{x}{\psi_{\tau,\beta}(x)}\right)^{\frac{\alpha\tau}{2+\alpha(\tau-2)}} \le C' x^{(1 - \frac{2(\alpha-1)}{\tau\alpha})(\frac{\alpha\tau}{2+\alpha(\tau-2)})} = C'x.$$

Hence for arbitrary $\varepsilon > 0$ and all sufficiently large $N$, $\int_{\mathbb{R}^d} ([f - f(x_0)]_+)^2 F(h)\, d\mu$ does not exceed

$$CN^{\beta_\tau} \int_{\{f^2 \ge K\mu(f^2)\}} f^2 \left|\frac{\nabla f}{f}\right|^{\beta_\tau} d\mu + \frac{CC'}{N^{(\beta_\tau/2)^*}} \int_{\{f^2 \ge K\mu(f^2)\}} f^2 F(h)\, d\mu.$$

We recall that

$$c^*(x) = c_{A,\alpha\tau/(\alpha-1)}(x) \le \lambda |x|^{\alpha\tau/(\alpha-1)}.$$

for $|x| > 1$ and some $\lambda = \lambda(A, \alpha, \tau)$. Obviously, there exists a number $a(A, \alpha, K) > 0$ such that

$$|x|^{\beta_\tau} \le a(A, \alpha, K) c^*(x)$$

for $x \ge K$. Hence by (32) there exists $C = C(\alpha, A, K)$ such that

$$\begin{aligned}\int_{\mathbb{R}^d} (f - f(x_0))_+^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \\ \le C \int_{\{f^2 \ge K\mu(f^2)\}} f^2 c^*\left(\frac{|\nabla f|}{|f|}\right) d\mu + C \operatorname{Var}_\mu f + \varepsilon \int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu,\end{aligned}$$

where $\varepsilon$ can be chosen arbitrarily small. It remains to estimate the last term on the right-hand side by Lemma 3.4:

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \le B(K)\cdot \mathrm{Var}_\mu f + 2\int_{\mathbb{R}^d} (f-\sqrt{K\mu(f^2)})_+^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu$$

and choose a sufficiently small $\varepsilon$. The proof is complete. □

Now let us apply this result in the case of a special lower bound for the isoperimetric function.

THEOREM 3.7. *Let $\varphi$ be a function satisfying assumptions* A1–A4 *such that $\varphi(x_0)=1$. For every $\tau \le 1$ define the corresponding generalized entropy*

$$F(x) = F_\tau(x) := \begin{cases} \varphi(x) & \text{if } 0 < x \le x_0, \\ \dfrac{1}{\tau}(\varphi^\tau(x)-1)+1 & \text{if } x \ge x_0. \end{cases}$$

*Assume that*

$$\mathcal{I}_\mu(t) \ge Ct\varphi\left(\frac{1}{t}\right)^{1-1/\alpha} \tag{33}$$

*for some $1<\alpha\le 2$ and $t \le 1/2$. Then, whenever $1 \ge \tau \ge 2/\beta = 2(1-1/\alpha)$, there exists $C_\tau > 0$ such that for every smooth $f$ one has*

$$\int_{\mathbb{R}^d} f^2 F_\tau\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\right) d\mu \le C_\tau \int_{\mathbb{R}^d} f^2 c_{A,\tau\alpha/(\alpha-1)}\left(\frac{|\nabla f|}{|f|}\right) d\mu.$$

PROOF. The result follows from Theorem 3.6. Obviously, $F_\tau$ satisfies A1–A4. Let us show that $\mu$ satisfies (25). Indeed, it suffices to show that $\mu$ satisfies (26). But (26) easily follows from (33), since $\varphi$ is increasing. Note that

$$F_{\tau,\beta} = \psi_{\tau,\beta}(F_\tau) = F_{2/\beta}.$$

So it suffices to check that

$$\int_{B^c_{R_{(K-1)/K}}} \Phi_{\tau,\beta}(\delta|I_{F_{\tau,\beta}}|^2)\,d\mu < \infty, \qquad \int_{B^c_{R_{(K-1)/K}}} \Phi_\tau(\delta c(I_{F_\tau}))\,d\mu < \infty$$

for $\delta$ sufficiently small and $K$ sufficiently large. Here

$$\Phi_\tau = (yF_\tau(y)-y)^*, \qquad \Phi_{\tau,\beta} = (yF_{\tau,\beta}(y)-y)^* = (yF_{1/\beta}(y)-y)^*.$$

Recall that the cost function is given by

$$c = c^*_{A,\frac{\tau\alpha}{\tau-1}} = c_{A,\frac{\tau\alpha}{\alpha(\tau-1)+1}}.$$

By the definition of $I_{F_\tau}$ and $\mathcal{I}_\mu$ for all $r \ge R_{1/2}$ we have

$$I_{F_\tau}(r) \le \frac{F_\tau\left(\frac{1}{1-\mu(B_r)}\right)}{C\varphi^{1-1/\alpha}\left(\frac{1}{1-\mu(B_r)}\right)}.$$

Hence

$$I_{F_\tau}(r) \le c_1 \varphi^{\tau-1+1/\alpha}\left(\frac{1}{1-\mu(B_r)}\right) \tag{34}$$

and

$$c(I_{F_\tau}(r)) \le \tilde{c_1}\varphi^{\tau}\left(\frac{1}{1-\mu(B_r)}\right).$$

Analogously,

$$I_{F_{\tau,\beta}}(r) \le \frac{F_{2/\beta}\left(\frac{1}{1-\mu(B_r)}\right)}{C\varphi^{1-1/\alpha}\left(\frac{1}{1-\mu(B_r)}\right)} \le c_2\varphi^{1/\beta}\left(\frac{1}{1-\mu(B_r)}\right)$$

and

$$I^2_{F_{\tau,\beta}}(r) \le \frac{F_{2/\beta}\left(\frac{1}{1-\mu(B_r)}\right)}{C\varphi^{1-1/\alpha}\left(\frac{1}{1-\mu(B_r)}\right)} \le c_2^2\varphi^{2/\beta}\left(\frac{1}{1-\mu(B_r)}\right).$$

Hence by Lemma 3.2 for some $C_1$, $R_0 > 0$ and sufficiently small $\delta$ one has

$$\Phi_\tau(\delta c(I_{F_\tau}(r))) \le \frac{C_1}{(1-\mu(B_r))^{2c_1\delta}} \quad \text{if } r \ge R_0.$$

In the same way we obtain

$$\Phi_{\tau,\beta}(\delta I^2_{F_{\tau,\beta}}(r))) \le \frac{C_2}{(1-\mu(B_r))^{2c_2\delta}} \quad \text{if } r \ge R_0.$$

Hence for $\delta$ sufficiently small and $K$ large the functions $\Phi_\tau(\delta c(I_{F_\tau}))$ and $\Phi_{\tau,\beta}(\delta I^2_{F_{\tau,\beta}}))$ are dominated by $N/(1-\mu(B_r))^p$ with some $p < 1$ and $N > 0$. Since the mapping

$$x \mapsto 1 - \mu(\{y : y \le |x|\})$$

transforms $\mu$ into Lebesgue measure on $[0, 1]$, we obtain

$$\int_{B_{R_{(K-1)/K}}} \Phi_\tau(\delta c(I_{F_\tau}(r)))\, d\mu \le N \int_0^{1/K} \frac{dt}{t^p} < \infty.$$

The same estimate holds for $\Phi_{\tau,\beta}(\delta I^2_{F_{\tau,\beta}})$. Hence the assumptions of Theorem 3.6 are satisfied. Thus, by Theorem 3.6 we have

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\right) d\mu \le C \int_{\{f^2 \ge K \int_{\mathbb{R}^d} f^2\, d\mu\}} f^2 c^*\left(\frac{|\nabla f|}{|f|}\right) d\mu + B \operatorname{Var}_\mu f.$$

By the Poincaré inequality $\operatorname{Var}_\mu f \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu$ (note that the Poincaré inequality is valid since (25) holds). One can easily verify that $|x|^2 \le \tilde{B} c^*(x)$ for some $\tilde{B} > 0$. Hence

$$|\nabla f|^2 \le \tilde{B} f^2 c^*\left(\frac{|\nabla f|}{|f|}\right)$$

and

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\right) d\mu \le (C + \tilde{B}B) \int_{\mathbb{R}^d} f^2 c^*\left(\frac{|\nabla f|}{|f|}\right) d\mu.$$

The proof is complete. □

## 4. Application to convex measures

LEMMA 4.1. *Let $\mu$ be a convex measure. Then* $\sup_{r\geq R_{1/2}} I_{\log}(r)/r < \infty$.

PROOF. We apply the following estimate from [6]:

$$2r\mu^+(A) \geq \mu(A)\log\frac{1}{\mu(A)} + (1-\mu(A))\log\frac{1}{1-\mu(A)} + \log\mu\{|x-x_0|\leq r\}, \tag{35}$$

which holds for every convex measure $\mu$, every set $A$, every point $x_0$, and any $r>0$. Let $\mu(A)\leq 1/2-\varepsilon$, where $\varepsilon>0$. Choose $r$ in such a way that $\mu(A)=\mu(B_r^c)$. Then

$$(1-\mu(A))\log\frac{1}{1-\mu(A)} + \log\mu(B_r) = \mu(B_r^c)\log\mu(B_r). \tag{36}$$

Pick $\delta=\delta(\varepsilon)$ such that

$$\left(\frac{1}{2}+\varepsilon\right)^{1/(1-\delta)} \geq \frac{1}{2}-\varepsilon.$$

Then

$$\mu(B_r) \geq \left(\frac{1}{2}+\varepsilon\right) \geq \left(\frac{1}{2}-\varepsilon\right)^{1-\delta} \geq \mu^{1-\delta}(B_r^c).$$

Therefore,

$$(1-\delta)\mu(A)\log\frac{1}{\mu(A)} + \mu(B_r^c)\log\mu(B_r) = \mu(A)\left(\log\frac{\mu(B_r)}{\mu^{1-\delta}(B_r^c)}\right) \geq 0.$$

Hence by (36) we obtain

$$(1-\delta)\mu(A)\log\frac{1}{\mu(A)} + (1-\mu(A))\log\frac{1}{1-\mu(A)} + \log\mu\{|x-x_0|\leq r\} \geq 0$$

and $\frac{2r}{\delta}\mu^+(A) \geq \mu(A)\log\frac{1}{\mu(A)}$. It remains to show that

$$\sup_{R_{1/2}\leq r\leq R_{1/2+\varepsilon}} \frac{I_{\log}(r)}{r} < \infty.$$

But this follows easily from (35). One has to choose a sufficiently large number $\tilde{R}$ such that

$$\inf_{0\leq\tau\leq\varepsilon} (1/2+\tau)\log\frac{1}{1/2+\tau} + \log\mu(\{x : |x|\leq\tilde{R}\}) \geq 0.$$

Then $I_{\log}(r)\leq\tilde{R}$. The proof is complete. □

COROLLARY 4.2. *Let $\mu$ be a convex measure and let $\varphi$ satisfy assumptions* A1–A4. *Suppose that $g:\mathbb{R}^+\to\mathbb{R}$ is increasing and*

$$\int_{\mathbb{R}^d} e^{g(r)}\,d\mu = 1.$$

*If for some $C > 0$ and $1 < \alpha \le 2$ one has*

$$\frac{g(r)}{\varphi^{1-1/\alpha}(e^{g(r)})} \ge Cr, \tag{37}$$

*then*

$$\mathcal{I}_\mu(t) \ge kt\varphi\left(\frac{1}{t}\right)^{1-1/\alpha}$$

*with some $k > 0$ and $t \le 1/2$.*

PROOF. By the previous lemma

$$\mu^+(A) \ge k_0 \frac{\mu(A)\log(1/\mu(A))}{r}$$

if $\mu(A) = 1 - \mu(B_r)$ and $r \ge R_{1/2}$. By the Chebyshev inequality

$$\mu(B_r^c) \le \frac{\int_{\mathbb{R}^d} e^{g(|x|)}\,d\mu(x)}{e^{g(r)}} = \frac{1}{e^{g(r)}}.$$

Hence by (37) one has

$$\frac{\log}{\varphi^{1-1/\alpha}}\left(\frac{1}{\mu(B_r^c)}\right) \ge Cr$$

for any $r \ge R_{1/2}$. Consequently,

$$\mu^+(A) \ge k_0 \frac{\mu(A)\log(1/\mu(A))}{r} \ge Ck_0\mu(A)\varphi^{1-1/\alpha}\left(\frac{1}{\mu(A)}\right).$$

The proof is complete. □

PROOF OF THEOREM 1.1. Follows from Theorem 3.7 and Corollary 4.2. □

EXAMPLE 4.3. Let $\mu = Ze^{-V}\,dx$ be a convex probability measure on $\mathbb{R}^d$ such that $V(x) \sim |x|\log^p|x|$ with $p > 0$ as $|x| \to \infty$. Suppose that $F$ satisfies A1–A4 and $F \sim \log^{\alpha p/(\alpha-1)}\log|x|$ as $|x| \to \infty$. Applying Theorem 1.1 one sees that for every $A > 0$ there exists $C > 0$ such that for every smooth function $f$ one has

$$\int_{\mathbb{R}^d} f^2 F\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\right)d\mu \le C\int_{\mathbb{R}^d} f^2 c_{A,\alpha/(\alpha-1)}\left(\frac{|\nabla f|}{f}\right)d\mu.$$

Finally, we prove an inequality of the type (6).

THEOREM 4.4. *Let $\mu$ be a convex measure such that $\int_{\mathbb{R}^d} e^{\varepsilon|x|^\alpha}\,d\mu < \infty$ for some $\alpha > 1$ and $\varepsilon > 0$. Then*

$$\operatorname{Ent}_\mu |f|^\beta \le C\left[\int_{\mathbb{R}^d} |\nabla f|^\beta\,d\mu + \operatorname{Var}_\mu |f|^{\beta/2}\right].$$

PROOF. Set $g^2 = |f|^\beta$. Apply Theorem 2.1 to $g^2$ in place of $f^2$. Following the proof of that theorem we get

$$\operatorname{Ent}_\mu |f|^\beta \le C \operatorname{Var}_\mu |f|^\beta + C \int_{|f|^\beta \ge K\mu(|f|^\beta)} I_{\log}(r_{|f|^\beta}(f^\beta))|f|^{\beta-1}|\nabla f|\, d\mu$$

with some $K > 1$. By the Hölder inequality for every $\delta > 0$ there exists $N(C, \delta) > 0$ such that

$$C \int_{|f|^\beta \ge K\mu(|f|^\beta)} I_{\log}(r_{|f|^\beta}(f^\beta))|f|^{\beta-1}|\nabla f|\, d\mu \le N \int_{\mathbb{R}^d} |\nabla f|^\beta\, d\mu + \delta \int_{|f|^\beta \ge K\mu(|f|^\beta)} I_{\log}^{\beta/(\beta-1)}(r_{|f|^\beta}(f^\beta))|f|^\beta\, d\mu.$$

Since $|f| \le C(K, \beta)|f - \mu(f)|$ on $\{|f|^\beta \ge K\mu(|f|^\beta)\}$, we get, by the same arguments as in Theorem 2.1,

$$\begin{aligned}\delta \int_{|f|^\beta \ge K\mu(|f|^\beta)} I_{\log}^{\beta/(\beta-1)}(r_{|f|^\beta}(f^\beta))|f|^\beta\, d\mu &\le \delta C(K, \beta) \int_{\mathbb{R}^d} I_{\log}^{\beta/(\beta-1)}(r_{|f|^\beta}(f^\beta))|f - \mu(f)|^\beta\, d\mu \\ &\le C \int_{\mathbb{R}^d} |f - \mu(f)|^\beta\, d\mu + \frac{1}{2} \operatorname{Ent}_\mu |f - \mu(f)|^\beta,\end{aligned}$$

where $C < \infty$ whenever

$$\int_{B_R^c} \exp(\delta I_{\log}^{\beta/(\beta-1)}(|x|)))\, d\mu < \infty$$

with $R = R_{(K-1)/K}$. By Corollary 4.2 one has $I_{\log}^{\beta/(\beta-1)}(|x|) \le C'|x|^{\beta/(\beta-1)} = C'|x|^\alpha$. Hence, choosing $\delta$ sufficiently small, we obtain

$$\int_{B_R^c} \exp(\delta I_{\log}^{\beta/(\beta-1)}(|x|))\, d\mu < \infty.$$

Since $\mu$ is convex, it satisfies the Cheeger inequality, hence there exists $C(\beta)$ such that for every $f$ one has

$$\int_{\mathbb{R}^d} |f - \mu(f)|^\beta\, d\mu \le C(\beta) \int_{\mathbb{R}^d} |\nabla f|^\beta\, d\mu$$

(see [8] for the proof). Finally, we arrive at the estimate

$$\operatorname{Ent}_\mu |f|^\beta \le C \operatorname{Var}_\mu |f|^{\beta/2} + N' \int_{\mathbb{R}^d} |\nabla f|^\beta\, d\mu + \frac{1}{2} \operatorname{Ent}_\mu |f - \mu(f)|^\beta. \tag{38}$$

In particular, applying (38) to $f - \mu(f)$, we get

$$\begin{aligned}\operatorname{Ent}_\mu |f - \mu(f)|^\beta &\le 2C \int_{\mathbb{R}^d} |f - \mu(f)|^\beta + 2N' \int_{\mathbb{R}^d} |\nabla f|^\beta\, d\mu \\ &\le (2CC(\beta) + 2N') \int_{\mathbb{R}^d} |\nabla f|^\beta\, d\mu.\end{aligned}$$

Combining this estimate again with (38) we get the claim. □

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