Rend. Lincei Mat. Appl. 18 (2007), 209–219

Partial differential equations. — *On the regularity of weak solutions to H-systems*, by ROBERTA MUSINA.

ABSTRACT. — We prove that every weak solution to the H-surface equation is locally bounded, provided the prescribed mean curvature H satisfies a suitable condition at infinity. No smoothness assumption is required on H. We also consider the Dirichlet problem for the H-surface equation on a bounded regular domain with L^∞ boundary data and the H-bubble problem. Under the same assumptions on H, we prove that every weak solution is globally bounded.

KEY WORDS: H-surface equation; prescribed mean curvature; regularity theory.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 53A10; Secondary 49J10.

1. INTRODUCTION

In this paper we are concerned with the *H-surface equation*

(1.1) $$\Delta u = 2H(u)u_x \wedge u_y \quad \text{in } \Omega,$$

where $H : \mathbb{R}^3 \to \mathbb{R}$ is a given map and Ω is a domain in $\mathbb{R}^2$, and with the Dirichlet problem

(1.2) $$\begin{cases} \Delta u = 2H(u)u_x \wedge u_y & \text{in } \Omega, \\ u = \gamma & \text{on } \partial\Omega, \end{cases}$$

where γ is a bounded regular datum.

Every smooth conformal solution to (1.1) parameterizes a surface that has prescribed mean curvature H at each regular point (that is, apart from branch points). We refer to [16], [17] and [13] for a detailed discussion of the main features of (1.1) and its applications to capillarity theory.

We are mainly concerned with L^∞_{loc}-regularity of weak solutions to (1.1) and with L^∞ regularity of weak solutions to (1.2). Let us recall that $u \in H^1_{\text{loc}}(\Omega, \mathbb{R}^3)$ is a *weak solution* to (1.1) if the map $(H \circ u)u_x \wedge u_y$ is locally integrable on Ω, and if equation (1.1) is satisfied in the sense of distributions, that is,

$$-\int_\Omega \nabla u \cdot \nabla v \, dx\, dy = 2\int_\Omega H(u)v \cdot u_x \wedge u_y \, dx\, dy \qquad \forall v \in C^\infty_0(\Omega, \mathbb{R}^3).$$

As concerns the results available in the literature, we mention the pioneering papers by Tomi [22], [23] and, among the most recent contributions, the paper [1], where Bethuel takes advantage of some properties of Lorentz spaces in order to get the regularity under the assumption that H is C^1, bounded and globally Lipschitz on $\mathbb{R}^3$. The same result is proved with a method based on the Hodge decomposition theorem and on some Morrey-type estimates in the recent paper [21] by Strzelecki. We also mention the papers [2] and [3] by Bethuel and Ghidaglia, in which the smooth and bounded curvature H depends only on two variables, or, more generally, it satisfies a suitable decay condition at infinity along a direction in $\mathbb{R}^3$. The main tools in [2], [3] are the co-area formula and the theory of Hardy spaces. The duality between the Hardy space $\mathcal{H}^1$ and the space BMO of functions having bounded mean oscillation is used in an essential way also in [21]. We also refer to this last paper for a complete list of references.

With regard to boundary regularity for the Dirichlet problem, the only general result known to us in the literature is due to Strzelecki, who proves in [21] that every weak solution to (1.2) is continuous up to the boundary, provided γ is continuous on $\partial\Omega$ and $H \in L^\infty(\mathbb{R}^3, \mathbb{R})$ is globally Lipschitz continuous on $\mathbb{R}^3$.

Finally, let us mention [14], where C^0-regularity for conformal solutions to (1.1) is proved without assuming any regularity on H.

We emphasize that, with the exception of [14], in all the above mentioned results some smoothness assumptions on H are always required.

In this paper we prove L^∞_{loc}-regularity results without any smoothness and boundedness hypothesis on the curvature function. On the other hand, we show that the behavior of H at infinity may play a fundamental role.

Our assumptions on the curvature are the following.

ASSUMPTIONS ON H. *There exist measurable maps H_0, H_1 and K on $\mathbb{R}^3$ such that*

$$H = H_0 + H_1 + K$$

and such that the following assumptions are satisfied:

(H_0) *H_0 is a continuous angular map, that is, $H_0 \in C^0(\mathbb{R}^3 \setminus \{0\})$ and*

$$H_0(sp) = H_0(p) \qquad \text{for } p \in \mathbb{R}^3 \setminus \{0\},\ s > 0;$$

(H_1) *H_1 is continuously differentiable in the complement of a ball $B_{R_{H_1}}$, and*

$$C(H_1) := \sup_{|p| \geq R_{H_1}} (|H_1(p)| + |\nabla H_1(p)|\,|p|) < +\infty;$$

(K) *K satisfies*

$$\bar{M}_K := \limsup_{|p| \to +\infty} |K(p)p| < 1.$$

The main results in this paper are the following.

THEOREM 1.1. *Assume that $H = H_0 + H_1 + K$ satisfies the assumptions (H_0), (H_1) and (K). Then every weak solution to* (1.1) *is locally bounded in Ω.*

THEOREM 1.2. *Assume that $H = H_0 + H_1 + K$ satisfies the assumptions (H_0), (H_1) and (K). Let Ω be a bounded domain with Lipschitz boundary, and let $u \in H^1(\Omega, \mathbb{R}^3)$ be a weak solution to the Dirichlet problem* (1.2) *such that $(H \circ u)u_x \wedge u_y \in L^1(\Omega, \mathbb{R}^3)$. If $\gamma \in L^\infty(\partial\Omega, \mathbb{R}^3)$, then $u \in L^\infty(\Omega, \mathbb{R}^3)$.*

As concerns the regularity of weak H-bubbles (see Section 4 for the definition), our result is the following.

THEOREM 1.3. *Assume that* $H = H_0 + H_1 + K$ *satisfies the assumptions* (H_0), (H_1) *and* (K). *Then every weak* H*-bubble is bounded.*

The main step in the proof of our regularity results is the ε-regularity Lemma 3.3, whose proof was inspired by [2]. However, our arguments are surprisingly simple; we do not use any approximation argument, neither Hardy spaces, neither the co-area formula, that were the key arguments in [2]. In essence, the idea is that certain (weighted) volumes "enclosed" by u, in the region where u is big, cannot be too large, thanks to the isoperimetric inequalities proved by Wente in [24] and by Steffen in [20].

Notice that the L^∞-regularity results stated above seem to be new also in the case of a smooth curvature H. Indeed, for $H \equiv K$ satisfying (K) and Lipschitz continuous the regularity of weak solutions to (1.1) has been known since the paper by Tomi [23]; for $H \equiv H_1 \in C^1(\mathbb{R}^3, \mathbb{R})$ satisfying (H_1), the regularity is due to Heinz ([15]; see also [2] and [3] for an alternative proof). However, these two results cannot be interpolated in order to cover (at least) the case $H = H_1 + K$. Actually, unlike Theorems 1.1 and 1.2, the results that have been available up to now in the literature are not satisfied in the settings of the papers [7] and [9], which are concerned with the existence of H-bubbles (that is, with problem (1.1) for $\Omega = \mathbb{R}^2$), and in [10]–[12], which deal with the Dirichlet problem for the H-surface equation.

Indeed, the crucial hypothesis in the above mentioned papers was $H \in C^1$ and

$$M_H = \sup_{p \in \mathbb{R}^3} |(\nabla H(p) \cdot p)p| < 1. \tag{1.3}$$

Notice that (1.3) is incomparable to Heinz's assumption in [15], and to Bethuel's assumption in [1]. On the other hand, it is easy to prove that (1.3) implies that H has a finite limit along radial directions as $|p| \to \infty$, that is, for every $p \in \mathbb{R}^3 \setminus \{0\}$ the limit

$$\lim_{s \to +\infty} H(sp) =: H_0(p)$$

exists. Clearly, H_0 is an angular map, in the sense described above. Now, assume that $H_0 \in C^0(\mathbb{R}^3 \setminus \{0\})$, and notice that (1.3) implies also

$$|(H(p) - H_0(p))p| \le M_H < 1 \quad \forall p \in \mathbb{R}^3.$$

Thus, the above assumptions on H are satisfied, with $H_1 \equiv 0$ and $K = H - H_0$. Assumption (1.3) is far from being purely technical; on the contrary, it prevents several remarkable phenomena, like nonexistence results and blow-up of approximate solutions. In particular, condition (1.3) provides a positive lower bound for the energy of H-bubbles (compare with [7]), affects the geometry of the energy sublevels ([7], [10]), plays a crucial role in existence-nonexistence phenomena ([7], [10]), in the behavior of Palais–Smale sequences for the Dirichlet problem ([11], [12]), and in several other questions related to the H-surface equation.

In case $H_0 = H_1 = 0$, that is, when H satisfies

$$\limsup_{|p|\to+\infty} |H(p)p| < 1, \tag{1.4}$$

the regularity result we can prove is a little stronger.

THEOREM 1.4. *Assume that $H : \mathbb{R}^3 \to \mathbb{R}$ is a mesurable map satisfying* (1.4).

(i) *If $u \in H^1_{\mathrm{loc}}(\Omega, \mathbb{R}^3)$ is a weak solution to* (1.1), *then $u \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^3)$.*
(ii) *Assume that Ω is a bounded domain, and let $u \in H^1(\Omega, \mathbb{R}^3)$ be a weak solution to the Dirichlet problem* (1.2) *such that $(H \circ u)u_x \wedge u_y \in L^1(\Omega, \mathbb{R}^3)$. If $\gamma \in L^\infty(\partial\Omega, \mathbb{R}^3)$, then u is bounded in Ω. If in addition*

$$\sup_{p\in\mathbb{R}^3} |H(p)p| < 1, \tag{1.5}$$

then

$$\|u\|_{L^\infty(\Omega,\mathbb{R}^3)} \le \|\gamma\|_{L^\infty(\partial\Omega,\mathbb{R}^3)}.$$

Assertion (i) was already proved by Tomi in [23, Satz 2], under the additional assumption that H is Lipschitz continuous.

Since the proofs of Theorems 1.1 and 1.4 are based on the same careful choice of test functions, we first prove the simpler Theorem 1.4.

NOTATION. Throughout this work, $D_r(z)$ denotes the open disk in $\mathbb{R}^2$ centered at $z \in \mathbb{R}^2$ and with radius $r > 0$, while B_R is the ball in $\mathbb{R}^3$ centered at 0 and with radius $R > 0$.

We denote by $L^\sigma(A)$, $H^1(A)$ and $H^1_0(A)$ the usual Lebesgue and Sobolev spaces of vector-valued functions $u : A \to \mathbb{R}^3$, while the notations $L^\sigma(A, \mathbb{R})$, $H^1(A, \mathbb{R})$ and $H^1_0(A, \mathbb{R})$ will be used for scalar-valued functions. The norms in $L^\sigma(A)$ and in $L^\sigma(A, \mathbb{R})$ will be denoted with the same notation $\|\cdot\|_\sigma$ if no confusion can arise.

2. PROOF OF THEOREM 1.4

First we prove statement (ii) in Theorem 1.4. Then a standard localization argument will lead to the proof of (i).

Let $H : \mathbb{R}^3 \to \mathbb{R}$ be a curvature satisfying (1.4), and let $u \in H^1(\Omega)$ be as in (ii). Let $R_0 \ge \|\gamma\|_\infty$ be any positive number such that

$$M_{R_0} := \sup_{|p|\ge R_0} |H(p)p| < 1. \tag{2.1}$$

Notice that we can take $R_0 = \|\gamma\|_{L^\infty(\partial\Omega)}$ if the stronger assumption (1.5) is satisfied. Assertion (ii) will be completely proved if we show that $\|u\|_\infty \le R_0$. This is equivalent to proving that $\Lambda_R = 0$ for every $R > R_0$, where we have set

$$\Lambda_R := \int_{\{z\in\Omega\,:\,|u(z)|\ge R\}} |\nabla u|^2.$$

To this end, fix any increasing map $\Phi \in C^\infty([0, +\infty[, \mathbb{R})$ satisfying $0 \le \Phi \le 1$, $\Phi(s) = 0$ if $s \le R_0$ and $\Phi(s) = 1$ if $s \ge R$, and consider the composite map $\Phi(|u(\cdot)|)u(\cdot)$. Since Φ' has compact support, $\nabla(\Phi(|u|)) = \Phi'(|u|)\nabla|u|$ a.e. in Ω, and since $\Phi(|u|) \equiv 0$ on $\partial\Omega$, we can conclude that $\Phi(|u|)u$ belongs to $H_0^1(\Omega)$. Notice also that $|H(u)\Phi(|u|)u| \le 1$ a.e. in Ω, by (2.1), and therefore $H(u)\Phi(|u|)u \cdot u_x \wedge u_y \in L^1(\Omega)$. At the end of this proof we will check that the map $\Phi(|u|)u$ can be used as a test function in (1.1) to obtain

$$-\int_\Omega \nabla u \cdot \nabla(\Phi(|u|)u) = 2\int_\Omega H(u)\Phi(|u|)u \cdot u_x \wedge u_y \, dx\, dy. \tag{2.2}$$

We point out the identity $u \cdot \nabla u := (u \cdot u_x, u \cdot u_y) = |u|\nabla|u|$, which in particular implies

$$(u \cdot \nabla u) \cdot \nabla(\Phi(|u|)) = \Phi'(|u|)|u|\,|\nabla|u|\,|^2 \ge 0,$$

and we compute

$$\int_\Omega \nabla u \cdot \nabla(\Phi(|u|)u) = \int_\Omega \Phi(|u|)|\nabla u|^2 + \int_\Omega \Phi'(|u|)|u||\nabla|u||^2 \ge \int_\Omega \Phi(|u|)|\nabla u|^2.$$

Finally, from (2.1) we estimate the right hand side in (2.2) by

$$\left|2\int_\Omega H(u)\Phi(|u|)u \cdot u_x \wedge u_y \, dx\, dy\right| \le M_{R_0}\int_\Omega \Phi(|u|)|\nabla u|^2,$$

and since $M_{R_0} < 1$ we infer that

$$\Lambda_R \le \int_\Omega \Phi(|u|)|\nabla u|^2 = 0.$$

Statement (ii) is completely proved.

PROOF OF (2.2). First notice that, by a standard density argument,

$$-\int_\Omega \nabla u \nabla w = 2\int_\Omega H(u) w \cdot u_x \wedge u_y \quad \forall\, w \in H_0^1 \cap L^\infty(\Omega). \tag{2.3}$$

For every n large enough set $\rho_n = \min\{1, n|u|^{-1}\}$ and $w^n = \rho_n \Phi(|u|)u$. Using the standard chain rule we first infer that $w^n \in H_0^1 \cap L^\infty(\Omega)$. Thus w^n is an admissible test function for (2.3). To conclude, it suffices to notice that $\nabla w^n \to \nabla(\Phi(|u|)u)$ in L^2 and $H(u)w^n \to H(u)\Phi(|u|)u$ weak* in L^∞, because of (2.1).

PROOF OF (i). It suffices to show that u is bounded in every small disk $D_{r_0}(z_0) \subset\subset \Omega$. By Fubini's theorem, for a.e. small $r > r_0$ the trace of u on $\partial D_r(z_0)$ is bounded in $H^1(\partial D_r(z_0))$, hence in $L^\infty(\partial D_r(z_0))$. Thus we can apply (ii) to conclude. □

REMARK 2.1. Notice that Theorem 1.4 can be easily combined with Strzelecki's theorem ([21, Theorem 1.3]) to get $C^0(\bar{\Omega}, \mathbb{R}^3)$-regularity, provided γ is continuous and H is a locally Lipschitz map satisfying (1.4).

3. Proof of Theorems 1.1 and 1.2

In this section we first prove Theorem 1.1. Then we indicate how our arguments can be modified in order to prove Theorem 1.2.

Let us start with some preliminaries about isoperimetric inequalities. The first result is needed in order to handle the term involving H_1; it was proved by Wente in [24] (see also [4, Lemma A.10]).

LEMMA 3.1. *Let D be a disk in $\mathbb{R}^2$. There is a unique continuous map*

$$R : H_0^1(D) \times H^1(D) \to \mathbb{R}$$

such that

$$R(\psi, w) := \int_D \psi \cdot w_x \wedge w_y \, dx \, dy$$

for every $\psi \in H_0^1 \cap L^\infty(D)$, $w \in H^1(D)$. Moreover

$$|R(\psi, w)| \le \frac{1}{2\sqrt{8\pi}} \|\nabla \psi\|_{L^2(D)} \|\nabla w\|^2_{L^2(D)} \quad \forall \psi \in H_0^1(D),\ \forall w \in H^1(D).$$

In order to handle the term involving the homogeneous part H_0, we have to recall some results by Steffen [20].

Let D be a disk in $\mathbb{R}^3$, and let H_0 be a continuous angular map, as in assumption (H_0). Notice that $H_0 \in L^\infty(\mathbb{R}^2, \mathbb{R})$. Moreover, the vector field $Q : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$Q(p) := \frac{1}{3} H_0(p) p$$

is continuous on $\mathbb{R}^3$ and it satisfies div $Q = H_0$ in the sense of distributions. Set

$$\mathcal{V}_{H_0}(w) := \frac{1}{3} \int_D H_0(w) w \cdot w_x \wedge w_y \, dx \, dy \tag{3.1}$$

for $w \in H_0^1 \cap L^\infty(D)$. It turns out that $\mathcal{V}_{H_0}(w) = \bar{\mathcal{V}}_{H_0}(w, 0)$, where $\bar{\mathcal{V}}_{H_0}(w, 0)$ is the H_0-volume functional introduced by Steffen in [20, Section 3]. If w is regular enough, it measures the algebraic volume enclosed by the closed surface parameterized by w with respect to the weight H_0.

By [20, Proposition 3.3], the functional $\mathcal{V}_{H_0}$ has a unique continuous extension to $H_0^1(D)$, and the following isoperimetric inequality holds true ([20, Theorem 2.10]):

$$|\mathcal{V}_{H_0}(w)| \le \frac{\|H_0\|_\infty}{6\sqrt{8\pi}} \|\nabla w\|^3_{L^2(D)} \quad \forall w \in H_0^1(D). \tag{3.2}$$

Finally, we need a technical lemma. The proof is standard, and it is based on the general chain rule for partial derivatives.

LEMMA 3.2. *Assume that H_1 satisfies (H_1). Let D be a bounded domain, and let $u \in H^1(D)$ be a given map, with a bounded trace on ∂D. Let $\Phi \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $\Phi(s) = 0$ for $s \le \max\{R_{H_1}, \|u\|_{L^\infty(\partial D)}\}$, and assume that the maps $s \mapsto \Phi(s)$ and $s \mapsto \Phi'(s)s$ are bounded on $\mathbb{R}$. Then $\psi := (H_1 \circ u)\Phi(|u|)u \in H_0^1(D)$ and*

$$\|\nabla \psi\|_2 \le C(H_1)(\|\nabla(\Phi(|u|)u)\|_2 + \|\Phi(|u|)\nabla u\|_2).$$

The main step in the proof of Theorem 1.1 is the following “ε-regularity lemma”.

LEMMA 3.3. *Assume that $H = H_0 + H_1 + K$ satisfies assumptions (H_0), (H_1) and (K). Then there exists $\varepsilon_0 > 0$ (depending only on $\|H_0\|_\infty$, $C(H_1)$ and $\bar{M}_K$) such that if $u \in H^1(D)$ is a weak solution to* (1.1) *on a disk D, with $(H \circ u)u_x \wedge u_y \in L^1(D)$ and*

$$\int_D |\nabla u|^2 < \varepsilon_0, \qquad u_{|\partial D} \in L^\infty(\partial D),$$

then $u \in L^\infty(D)$.

PROOF. Fix a large $R_0 \geq \|u\|_{L^\infty(\partial D)}$, $R_0 \geq R_{H_1}$, such that

$$M_{R_0} = \sup_{|p| \geq R_0} |K(p)p| < 1, \tag{3.3}$$

and for $R \geq R_0$ set, as in Section 2,

$$\Lambda_R = \int_{\{z\,:\,|u(z)| \geq R\}} |\nabla u|^2.$$

We have to prove that there exists a large R_∞ such that $\Lambda_{R_\infty} = 0$. To this end, fix $R \geq R_0$ and $\alpha \in (0, 1)$ small, and choose a map $\Phi \in C^1(\mathbb{R}, \mathbb{R})$ such that $\Phi(s) = 0$ for $s \leq R$, $\Phi(s) = 1$ for $s \geq R(1 + \alpha)$, and

$$0 \leq \Phi \leq 1, \qquad \Phi' \geq 0, \tag{3.4}$$

$$\Phi(s) + \Phi'(s)s \leq 2/\alpha. \tag{3.5}$$

As in Section 2 we see that the maps $\Phi(|u|)^3 u$, $\Phi(|u|)u$ are of class $H^1_0(D)$. At the end of this proof we will show that (in essence) we can use $\Phi(|u|)^3 u$ as a test function in

$$\Delta u = 2H(u)u_x \wedge u_y \quad \text{ on } D \tag{3.6}$$

to obtain

$$-\int_D \nabla u \cdot \nabla(\Phi(|u|)^3 u) = 6\mathcal{V}_{H_0}(\Phi(|u|)u) + 2R(\psi, \Phi(|u|)u) + 2\int K(u)\Phi(|u|)^3 u \cdot u_x \wedge u_y, \tag{3.7}$$

where we have set

$$\psi := (H_1 \circ u)\Phi(|u|)u.$$

Notice that $\psi \in H^1_0(D)$ by Lemma 3.2. We can readily estimate, as in Section 2,

$$\int_D \nabla u \nabla(\Phi(|u|)^3 u) \geq \int_D \Phi(|u|)^3 |\nabla u|^2,$$

$$\left| 2\int_D K(u)\Phi(|u|)^3 u \cdot u_x \wedge u_y \, dx\, dy \right| \leq M_{R_0} \int_D \Phi(|u|)^3 |\nabla u|^2,$$

and therefore, using also the isoperimetric inequality (3.2) and Lemma 3.1,

$$(1 - M_{R_0})\Lambda_{R(1+\alpha)} \le \frac{\|H_0\|_\infty}{\sqrt{8\pi}} \|\nabla(\Phi(|u|)u)\|_2^3 + \frac{1}{\sqrt{8\pi}} \|\nabla\psi\|_2 \|\nabla(\Phi(|u|)u)\|_2^2.$$

It remains to compute

$$\int_D |\nabla(\Phi(|u|)u)|^2 \le \int_D (\Phi'(|u|)|u| + \Phi(|u|))^2 |\nabla u|^2 \le \frac{4}{\alpha^2}\Lambda_R \tag{3.8}$$

by (3.5). Also, from Lemma 3.2 and (3.8) we get

$$\|\nabla\psi\|_2 \le C(H_1)(\|\nabla(\Phi(|u|)u)\|_2 + \|\Phi(|u|)\nabla u\|_2) \le 3C(H_1)\frac{1}{\alpha}\Lambda_R^{1/2}.$$

Thus

$$\Lambda_{R(1+\alpha)} \le C_1 \frac{1}{\alpha^3} \Lambda_R^{3/2}, \tag{3.9}$$

where C_1 depends only on $\|H_0\|_\infty$, $C(H_1)$ and on M_{R_0} (that is, on $\bar{M}_K$). Now we assume that ε_0 is so small that

$$C_1 \varepsilon_0^{1/2} < 1/4,$$

and we show that (3.9) leads to the conclusion. We define by recurrence a bounded increasing sequence $R_n \to R_\infty < +\infty$, by setting $R_1 = R_0(1+1)$ and

$$R_{n+1} := R_n(1 + 2^{-n/3}).$$

First notice that we can estimate, using (3.9),

$$\Lambda_{R_1} \le C_1 \varepsilon_0^{1/2} \Lambda_{R_0} < \frac{1}{4}\varepsilon_0.$$

Next one proves by induction that $\Lambda_{R_n} < \varepsilon_0 4^{-n}$. Therefore,

$$\Lambda_{R_\infty} \le \lim_{n\to\infty} \Lambda_{R_n} = 0,$$

and the lemma is completely proved. □

PROOF OF (3.7). For every n large enough set, as in Section 2, $\rho_n = \min\{1, n|u|^{-1}\}$ and $v^n = \rho_n^3 \Phi(|u|)^3 u$. As in Section 2 notice that $v^n \in H_0^1 \cap L^\infty(D)$ and $v^n \to \Phi(|u|)^3 u$ in $H_0^1(D)$. Moreover, from (H_0) and (H_1) we infer that the maps $(H_0 \circ u)v^n$ and $(H_1 \circ u)v^n$ are measurable and bounded on D. Therefore, also $(K \circ u)v^n \cdot u_x \wedge u_y \in L^1(\Omega)$. Using a standard density argument one can check that the map v^n can be used as a test function for (3.6) to get

$$-\int_D \nabla u \nabla(\Phi(|u|)^3 u) = 2\int_D H_0(u)v^n \cdot u_x \wedge u_y + 2\int_D H_1(u)v^n \cdot u_x \wedge u_y$$
$$+ 2\int_D K(u)\Phi(|u|)^3 u \cdot u_x \wedge u_y + o(1)$$

as $n \to \infty$. Here we have used the facts that $\nabla v^n \to \nabla(\Phi(|u|)^3 u)$ in L^2 and $K(u)v^n \to K(u)\Phi(|u|)^3 u$ weak* in L^∞, because of (3.3). Next, notice that $w^n := \Phi(|u|)\rho_n u \to \Phi(|u|)u$ in $H_0^1(D)$ and

$$H_0(u)v^n \cdot u_x \wedge u_y = H_0(w^n)w^n \cdot w_x^n \wedge w_y^n$$

a.e. in D, since H_0 is an angular map. Therefore,

$$\int_D H_0(u)v^n \cdot u_x \wedge u_y = 3\mathcal{V}_{H_0}(w^n) = 3\mathcal{V}_{H_0}(\Phi(|u|)u) + o(1)$$

by (3.1) and by the continuity of the volume functional $\mathcal{V}_{H_0}$ on $H_0^1(D)$. Finally,

$$2\int_D H_1(u)v^n \cdot u_x \wedge u_y = 2\int_D \rho_n \psi \cdot w_x^n \wedge w_y^n = 2R(\psi, \Phi(|u|)u) + o(1),$$

since $\rho_n \psi \to \psi$ and $w^n \to \Phi(|u|)u$ in $H_0^1(D)$. □

END OF PROOF OF THEOREM 1.1. The argument is the same as in Section 2. It suffices to show that u is bounded in every small disk $D_{r_0}(z_0) \subset\subset \Omega$. By Fubini's theorem, for a.e. small $r > r_0$ the trace of u on $\partial D_r(z_0)$ is bounded in $H^1(\partial D_r(z_0))$, hence in $L^\infty(\partial D_r(z_0))$. Thus, if r is small enough (in order to have a small L^2 norm of the gradient on $D = D_r(z_0)$), we can apply Lemma 3.3 to conclude. □

PROOF OF THEOREM 1.2. Let $\bar{u} \in H^1(\mathbb{R}^2)$ be any extension of u. Choose a finite covering $D_{r_0}(z_i)$, $i = 1, \ldots, k$, of $\partial\Omega$ by small disks such that

$$\int_{D_{2r_0}(z_i)} |\nabla \bar{u}|^2 < \varepsilon_0$$

where ε_0 is as in Lemma 3.3. Since $u \in L^\infty_{\mathrm{loc}}(\Omega)$ by Theorem 1.1, we just have to check that $u \in L^\infty(D_r(z_i) \cap \Omega)$ for each $i = 1, \ldots, k$. Fix an index i and choose a radius $r \in (r_0, 2r_0)$ such that the trace of $\bar{u}$ on $\partial D_r(z_i)$ is bounded in L^∞. Define, as in the proof of Lemma 3.3, the map $\Phi(\bar{u})$ for large R, α. Finally, repeat the arguments in the proof of Lemma 3.3 with $D = D_r(z_i)$, replacing $\Phi(|u|)$ with the function

$$\bar{\Phi}(|u|) = \begin{cases} \Phi(u) & \text{in } D_r(z_i) \cap \Omega, \\ 0 & \text{in } D_r(z_i) \setminus \Omega. \end{cases}$$

The conclusion can be achieved as in Lemma 3.3. □

REMARK 3.4. The condition $H_1 \in C^1$ can be relaxed, since we just need that H_1 is a Lipschitz function on the complement of a ball in $\mathbb{R}^3$, that H_1 satisfies (H_1), and that the Nemytskiĭ operator $u \mapsto H_1 \circ u$ maps continuously $H^1(D)$ into $H^1(D)$. However, this is not generally true if H_1 is only Lipschitz continuous (cf. [18]).

4. H-BUBBLES AND PROOF OF THEOREM 1.3

In this section we deal with solutions to (1.1) on the whole plane $\mathbb{R}^2$. For a discussion of this problem we refer to the Appendix in [5] for the case of H constant, and to [7], [8], [9], [6], [19] for H variable.

Let us start by introducing a notation. We set

$$\hat{H}^1(\mathbb{R}^2) = \{U \in H^1_{\text{loc}}(\mathbb{R}^2) : |\nabla U|, |U|/(1+|z|^2) \in L^2(\mathbb{R}^2)\}.$$

Notice that $\hat{H}^1(\mathbb{R}^2)$ can be identified with $H^1(\mathbb{S}^2, \mathbb{R}^3)$ via composition with the stereographic projection from the north pole. It is often convenient to identify a map $U \in \hat{H}^1(\mathbb{R}^2)$ with its composition with the stereographic projection, which is a map defined on $\mathbb{S}^2$.

For $U : \mathbb{R}^2 \to \mathbb{R}^3$ set

$$\hat{U}(z) = U(z/|z|^2).$$

Then $U \in \hat{H}^1(\mathbb{R}^2)$ if and only if $\hat{U} \in \hat{H}^1(\mathbb{R}^2)$. Let us say that U is a *weak H-bubble* if $U \in \hat{H}^1(\mathbb{R}^2)$, $(H \circ U)U_x \wedge U_y \in L^1(\mathbb{R}^2)$ and U is a weak solution to

$$\Delta U = 2H(U)U_x \wedge U_y \quad \text{on } \mathbb{R}^2.$$

It turns out that every smooth H-bubble is indeed a conformal map on the sphere that parameterizes an $\mathbb{S}^2$-type surface having mean curvature H at each regular point (cf. for example [7]).

Theorem 1.3 is a simple corollary of 1.1, which follows from the invariance of the H-surface equation with respect to composition with the Kelvin transform in $\mathbb{R}^2$. We omit the simple proofs. $\square$

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Received 28 August 2006,
and in revised form 5 September 2006.

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