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Partial differential equations. — *Viscosity solutions of the Monge–Ampère equation with the right hand side in L^p*, by ANNA LISA AMADORI, BARBARA BRANDOLINI and CRISTINA TROMBETTI, communicated by C. Sbordone on 12 January 2007.

ABSTRACT. — We compare various notions of solutions of Monge–Ampère equations with discontinuous functions on the right hand side. Precisely, we show that the weak solutions defined by Trudinger can be obtained by the vanishing viscosity approximation method. Moreover, we investigate existence and uniqueness of L^p-viscosity solutions.

KEY WORDS: Viscosity solutions; weak solutions; Monge–Ampère equations.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35J65, 49L25.

1. INTRODUCTION

Let Ω be a uniformly convex bounded domain of $\mathbb{R}^{\mathrm{N}}$ ($\mathrm{N} \geq 2$) and let f be a nonnegative function in Ω. It is well-known that the homogeneous Dirichlet problem for the Monge–Ampère equation

$$\det(D^2u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

has a classical convex solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ when f is strictly positive and sufficiently smooth, that is, $f \in \mathcal{C}^\alpha(\Omega)$ for some $0 < \alpha < 1$ (see [4, 5, 6]). When these assumptions fail, various notions of solutions have been proposed and related results have been obtained. If f is continuous the problem can be settled in the framework of $\mathcal{C}$-viscosity solutions [12]. If f is not continuous, however, the problem should be treated within the theory of L^p-viscosity solutions (see [7, 8]). Unfortunately, the present stage of this theory requires strong ellipticity and, therefore, it does not apply to the Monge–Ampère equation (1.1). On the other hand, one can deal with a broader set of data on the right hand side, namely bounded measures. This is easily done by taking into account weak solutions in the sense of measures, for which we refer to [19, 20]. Moreover Trudinger [18] has given an *ad hoc* notion of weak solutions for L^p data. As far as we know, the relation between weak solutions and those obtained by the vanishing viscosity approximation method has not been investigated. Both the viscosity and weak solutions rely on approximation arguments. While the viscosity method is based on strong ellipticity of $\det(D^2u)$ on a given solution, the approximation method is focused on the regularity of f.

In this paper we show that the weak solution according to [18] is actually an L^p-viscosity solution, and it is the limit of the vanishing viscosity method. We also deal with uniqueness questions in the class of L^p-viscosity solutions. To the authors' knowledge, the question of uniqueness of L^p-viscosity solutions has not found an ultimate answer. In [8] Caffarelli, Crandall, Kocan and Święch have shown that the existence of an a.e. solution with $W^{2,p}$ regularity guarantees uniqueness of L^p-viscosity solutions. The paper [14] gives a remarkable contribution to the uniqueness issue. The authors prove $\mathcal{C}^{1,\alpha}$ regularity of the viscosity solutions under a certain structure condition. Seemingly, all these available techniques cannot be used for the Monge–Ampère equation because of a lack of regularity: it is well-known that in general equation (1.1) cannot be solved in $W^{2,p}$ nor in $\mathcal{C}^{1,\alpha}$ (see [21, 22]). We establish here uniqueness for L^∞-viscosity solutions when $f \in L^\infty$. Our proof makes use of the existence of weak solutions in the sense of [18] but is independent of the uniqueness of weak solutions. In fact, the uniqueness of weak solutions could be inferred at once. Our technique does not apply to the case of unbounded data. Concerning the data in L^q, $q < \infty$, we show that the weak solution (equivalently, the solution produced by vanishing viscosity) is the maximal L^p-viscosity solution for $p = \mathrm{N}q$.

Let us say a few words on the vanishing viscosity approximation method. We approximate equation (1.1) by adding a vanishing viscosity term to gain strong ellipticity. In view of homogeneity, we actually study the following equation:

$$(\det(D^2u))^{1/\mathrm{N}} + \varepsilon\Delta u = f^{1/\mathrm{N}} \quad \text{in } \Omega, \tag{1.1.ε}$$

where $f^{1/\mathrm{N}} \in L^p(\Omega)$, $p = \mathrm{N}q$. For every $\varepsilon > 0$ equation (1.1.ε) has a strong solution $u_\varepsilon \in W^{2,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfying the boundary condition (1.2). Moreover, u_ε turns out to be the unique L^p-viscosity solution. The main purpose of this paper is to pass to the limit in equations (1.1.ε) and show that:

(i) u_ε converges to an L^p-viscosity solution of (1.1)–(1.2),
(ii) such a solution coincides with the weak solution introduced by Trudinger [18],
(iii) possibly, that solution is the unique L^p-viscosity solution, or at least,
(iv) it is a "special" viscosity solution.

Another feasible approximation of (1.1) could be

$$\det(D^2u + \varepsilon\Delta u\, I) = f \quad \text{in } \Omega, \tag{1.3.ε}$$

which was suggested by Trudinger in [16]. Since for any $\mathcal{C}^2$ convex function u we have

$$\begin{aligned}\max\{(\det(D^2u))^{1/\mathrm{N}}, \varepsilon\Delta u\} &\le (\det(D^2u))^{1/\mathrm{N}} + \varepsilon\Delta u \\ &\le (\det(D^2u + \varepsilon\Delta u I))^{1/\mathrm{N}} \le (1/\mathrm{N} + \varepsilon)\Delta u,\end{aligned}$$

it is not hard to believe that (1.1.ε) and (1.3.ε) are very close to each other. Therefore, it seems that our method would equally work for (1.3.ε).

2. Preliminaries

First of all we settle some assumptions and notation that will be in force throughout the paper. The set Ω is a uniformly convex, bounded domain of $\mathbb{R}^\mathrm{N}$ ($\mathrm{N} \ge 2$), f is a nonnegative function belonging to $L^q(\Omega)$ with $1 \le q \le +\infty$ and bounded in some neighbourhood of $\partial\Omega$, say $\mathcal{N}$. We also set $p = \mathrm{N}q$.

It is well-known that the Monge–Ampère operator is elliptic with respect to convex functions. According to [12], viscosity theory recovers ellipticity by introducing the function

$$F(X) := \sup\{-\operatorname{tr}(BX) : B \in \mathbb{M}^{N},\ B \geq 0,\ \det B = 1/N^{N}\} = \begin{cases} -(\det X)^{1/N} & \text{if } X \geq 0, \\ +\infty & \text{otherwise,} \end{cases}$$

for any symmetric $N \times N$ matrix X. Here $\mathbb{M}^N$ denotes the set of $N \times N$ matrices, and from now on $\mathbb{S}^N$ will stand for its subset of symmetric matrices. Thus we read (1.1) as

$$F(D^2u) + f^{1/N} = 0 \quad \text{in } \Omega,$$

with $f^{1/N} \in L^p(\Omega)$. We explicitly observe that, since F is the upper envelope of linear functions, it is subadditive, convex and lower semicontinuous.

The aim of this paper is to compare various notions of solution known in the literature. Thus we briefly review definitions and first properties. We start by recalling the definition of $\mathcal{C}$-viscosity sub/super/solutions due to Crandall, Ishii and Lions. We cite here the paper [10] for a detailed exposition of this theory, and [12] for a particular mention of the Monge–Ampère equation.

DEFINITION 2.1. *Let u be an u.s.c. (respectively, l.s.c.) function; we say that u is a* $\mathcal{C}$-viscosity subsolution *(resp.* supersolution*) of* (1.1) *or, equivalently, that* $F(D^2u) + f^{1/N} \leq 0$ *(resp.,* ≥ 0*) in* $\mathcal{C}$-viscosity sense*, if the following holds. For every test function $\varphi \in \mathcal{C}^2(\Omega)$,*

$$F(D^2\varphi(x)) + (f(x))^{1/N} \leq 0 \qquad (\textit{resp.}, \geq 0)$$

for every $x \in \Omega$ where $u - \varphi$ attains a local maximum (resp., minimum).

A $\mathcal{C}$-viscosity solution *is any continuous function u which is, at the same time, a $\mathcal{C}$-viscosity supersolution and a $\mathcal{C}$-viscosity subsolution. We shall also say that $F(D^2u) + f^{1/N} = 0$ in* $\mathcal{C}$-viscosity sense.

In [21] it has been proved that a classical solution is a $\mathcal{C}$-viscosity solution if and only if it is convex, and that conversely a smooth $\mathcal{C}$-viscosity solution is a classical convex solution.

A more restrictive notion of solution can be given by increasing the set of test functions from $\mathcal{C}^2(\Omega)$ to $W^{2,p}(\Omega)$. We refer the reader to [8] for a detailed account of the L^p-viscosity theory; here we briefly recall the basic definitions.

DEFINITION 2.2. *Let u be an u.s.c. (respectively, l.s.c.) function; we say that u is an* L^p-viscosity subsolution *(resp.,* supersolution*) of* (1.1) *or, equivalently, that $F(D^2u) + f^{1/N} \leq 0$ (resp., ≥ 0)* in L^p-viscosity sense*, if one of the following items holds.*

(i) *If $\varphi \in W^{2,p}(\Omega)$ and $\delta > 0$ are such that $F(D^2\varphi) + f^{1/N} \geq \delta > 0$ (resp., $\leq -\delta < 0$) almost everywhere in an open subset of Ω, then $u - \varphi$ cannot achieve a local maximum (resp., minimum) inside that set.*

(ii) *For every test function* $\varphi \in W^{2,p}(\Omega)$ *and for every* $\bar{x} \in \Omega$ *where* $u - \varphi$ *achieves a local maximum (resp., minimum), we have*

$$\operatorname*{ess\,lim\,inf}_{x\to\bar{x}} F(D^2\varphi(x)) + (f(x))^{1/\mathrm{N}} \le 0 \tag{2.4}$$
$$\textit{(resp., } \operatorname*{ess\,lim\,sup}_{x\to\bar{x}} F(D^2\varphi(x)) + (f(x))^{1/\mathrm{N}} \ge 0\textit{),}$$

where ess lim inf *(*ess lim sup*) means, as usual, the essential inferior (superior) limit.*

An L^p-viscosity solution *is any continuous function* u *which is, at the same time, an* L^p*-viscosity supersolution and an* L^p*-viscosity subsolution. We shall also say that* $F(D^2u) + f^{1/\mathrm{N}} = 0$ in L^p-viscosity sense.

$\mathcal{C}$- and L^p-viscosity sub/super/solutions of equation (1.1.ε) are defined in the same way, after replacing $F(X) + f^{1/\mathrm{N}}$ by $F(X) - \varepsilon \operatorname{tr} X + f^{1/\mathrm{N}}$.

We explicitly mention that an L^p-viscosity sub/super/solution is a $\mathcal{C}$-viscosity sub/super/solution.

REMARK 2.3. Any L^p-viscosity subsolution u of (1.1) satisfies $D^2u \ge 0$ in L^p-viscosity sense. This means that, if $\varphi \in W^{2,p}(\Omega)$ is such that $D^2\varphi$ is not nonnegative almost everywhere in an open subset of Ω, then $u - \varphi$ cannot achieve a maximum inside that set. To prove this, assume towards a contradiction that there exists $\varphi \in W^{2,p}(\Omega)$ such that $D^2\varphi$ is not nonnegative for a.e. x in a ball B contained in Ω and that $u-\varphi$ has a maximum point $\bar{x}$ inside B. Then by definition $F(D^2\varphi) \equiv +\infty$ near $\bar{x}$, contrary to (2.4).

A well-known notion of solution is the one introduced by Trudinger in [18], which coincides with that given by Aleksandrov [1] and Bakelman [2] for measure data.

DEFINITION 2.4. *Let* u *be a continuous function; we say that* u *is a* weak subsolution *of* (1.1) *if there exist sequences* $\{u_m\} \subset \mathcal{C}^2(\Omega)$ *and* $\{f_m\} \subset L^1_{\mathrm{loc}}(\Omega)$ *such that* u_m *is convex,* $u_m \to u$ *uniformly in* Ω*,* $f_m \ge 0$*,* $f_m \to f$ *in* $L^1_{\mathrm{loc}}(\Omega)$*, and* $\det(D^2u_m) \ge f_m$.

Let u *be a continuous function; we say that* u *is a* weak supersolution *of* (1.1) *if there exist sequences* $\{u_m\} \subset \mathcal{C}^2(\Omega)$ *and* $\{f_m\} \subset L^1_{\mathrm{loc}}(\Omega)$ *such that* $u_m \to u$ *uniformly in* Ω*,* $f_m \ge 0$*,* $f_m \to f$ *in* $L^1_{\mathrm{loc}}(\Omega)$*, and* $\det(D^2u_m) \le f_m$ *whenever* u_m *is convex.*

A weak solution *is a continuous function* u *for which there exists a sequence* $\{u_m\} \subset \mathcal{C}^2(\Omega)$ *of convex functions such that* $u_m \to u$ *uniformly in* Ω *and* $\det(D^2u_m) \to f$ *in* $L^1_{\mathrm{loc}}(\Omega)$.

To relate our work to the rest of the literature, we have to mention the notion of *good solution*, which was proposed by Cerutti, Escauriaza and Fabes in [9] for linear equations with discontinuous coefficients and was extended to fully nonlinear, strongly elliptic equations by Jensen, Kocan and Święch in [13]. The obvious adaptation of that definition to the Monge–Ampère equation (which is merely degenerate elliptic) reads as follows.

DEFINITION 2.5. *Let* u *be a continuous function; we say that* u *is a* good solution *of* (1.1) *if there exist a sequence of degenerate elliptic operators* $G_m : \Omega \times \mathbb{S}^{\mathrm{N}} \to \mathbb{R}$*, with*

$$G_m(x, X) \le G_m(x, Y) \quad \textit{whenever } X - Y \ge 0,$$

for all $x \in \Omega$ *and* $X, Y \in \mathbb{S}^N$, *and a sequence of functions* $\{u_m\} \subset W^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathcal{C}(\Omega)$ *with*

$$G_m(x, D^2 u_m) = 0 \quad \text{a.e. in } \Omega$$

so that $u_m \to u$ *uniformly in* Ω *and* $G_m(x, X) \to F(X) + f^{1/N}$ *for a.e.* $x \in \Omega$ *and all* $X \in \mathbb{S}^N$.

It is easily seen that a weak solution according to Definition 2.4 is a good solution according to Definition 2.5. In [13], the authors show that the notions of good and L^p-viscosity solution are equivalent for strongly elliptic equations, but their proof does not carry over to merely degenerate elliptic equations. If we adopt their point of view, our Theorem 4.1 states that any weak (good) solution is an L^p-viscosity solution; moreover, Theorem 4.4 says that the two notions are equivalent when the data f are essentially bounded.

3. L^p-VISCOSITY SOLUTIONS: THE VANISHING VISCOSITY METHOD

In this section we approximate equation (1.1) by means of

$$(\det(D^2u))^{1/N} + \varepsilon \Delta u = f^{1/N} \quad \text{in } \Omega, \tag{1.1.ε}$$

for $\varepsilon > 0$. We observe that the operator $(\det(D^2u))^{1/N} + \varepsilon \Delta u$ is concave with respect to D^2u and satisfies the following structure condition:

$$\varepsilon \Delta u \le (\det(D^2u))^{1/N} + \varepsilon \Delta u \le (\varepsilon + 1/N)\Delta u$$

for any $\mathcal{C}^2$ convex function u.

Let us mention some fundamental facts about the uniformly elliptic equation (1.1.ε), for the reader's convenience. In the following we will denote by C_i positive constants whose values may change from line to line.

PROPOSITION 3.1. *Let* Ω *and* f *satisfy the standing assumptions set in Section 2. Then for every* $\varepsilon > 0$*:*

(i) *there exists a unique* L^p*-viscosity solution* u_ε *of* (1.1.ε)–(1.2)*;*
(ii) u_ε *is a convex function belonging to* $W^{2,p}(\Omega)$ *and satisfying equation* (1.1.ε) *a.e.;*
(iii) $u_\varepsilon \in \mathcal{C}^{0,1}(\Omega)$ *and*

$$\|u_\varepsilon\|_{L^\infty} \le C_1, \qquad \|Du_\varepsilon\|_{L^\infty} \le C_2, \tag{3.5}$$

with C_1, C_2 *independent on* ε.

In order to prove the proposition above we need the following

LEMMA 3.2. *Let* Ω *and* f *satisfy the standing assumptions set in Section* 2*. Let* $u \in \mathcal{C}^{0,1}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ *be a supersolution of the problem*

$$\begin{cases} \det(D^2 v) = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Then

$$\|u\|_{L^\infty} \le \mathrm{C}_3, \quad \|Du\|_{L^\infty} \le \mathrm{C}_4, \tag{3.7}$$

where C_3, C_4 *are positive constants depending on* N, $\mathcal{N}$, Ω, $\|f\|_{L^q}$.

PROOF. Since u is convex in Ω, by the classical comparison principle (see [15], [17]) we have

$$\|u\|_{L^\infty} \le \mathrm{C}_5 \|f\|^{\mathrm{N}}_{L^q}, \quad \mathrm{C}_5 > 0. \tag{3.8}$$

Moreover, since u is a smooth convex function, we get

$$\sup_{\overline{\Omega}} |Du| = \sup_{\partial\Omega} |Du|. \tag{3.9}$$

Combining (3.8) and (3.9), in order to obtain (3.7) it suffices to prove

$$\sup_{\partial\Omega} |Du| \le \mathrm{C}_6, \tag{3.10}$$

with C_6 depending on N, $\mathcal{N}$, Ω and $\|u\|_{L^\infty}$, and this can be done arguing as in [11, Theorem 17.21].

PROOF OF PROPOSITION 3.1. Let us consider a sequence $\{f_n\} \subset \mathcal{C}^\infty(\overline{\Omega})$, $f_n > 0$, with $f_n \to f$ in $L^q(\Omega)$ as $n \to \infty$, $\|f_n\|_q \le \mathrm{C}_7 \|f\|_q$ and $\|f_n\|_{L^\infty} \le \mathrm{C}_8$ in $\mathcal{N}$. By classical results, for every n, the equation

$$(\det(D^2 u))^{1/\mathrm{N}} + \varepsilon \Delta u = (f_n)^{1/\mathrm{N}}$$

has a unique classical convex nonpositive solution $u_{\varepsilon,n} \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}^{0,1}(\overline{\Omega})$ satisfying the homogeneous boundary condition (1.2). Moreover, this $u_{\varepsilon,n}$ turns out to be the unique $\mathcal{C}$-viscosity solution (see [12]) and the unique L^p-viscosity solution (see [8]).

For every n and $\varepsilon > 0$, $u_{\varepsilon,n}$ is a supersolution of

$$\begin{cases} \det(D^2 w) = f_n & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega; \end{cases}$$

hence by Lemma 3.2 we have

$$\|u_{\varepsilon,n}\|_{L^\infty} \le \mathrm{C}_9, \quad \|Du_{\varepsilon,n}\|_{L^\infty} \le \mathrm{C}_{10},$$

for some positive constants $\mathrm{C}_9, \mathrm{C}_{10}$ whose value depends on N, $\mathcal{N}$, Ω, $\|f\|_{L^q}$, but not on n and ε. The Ascoli–Arzelà theorem ensures that, up to subsequences, $u_{\varepsilon,n}$ converges uniformly in $\overline{\Omega}$, as $n \to \infty$, to a function u_ε which satisfies (3.5). By the stability result [8, Theorem 3.8], the limit function u_ε is an L^p-viscosity solution of (1.1.ε)–(1.2), hence by [4], u_ε has $W^{2,p}$ regularity and satisfies equation (1.1.ε) a.e.; finally, [8, Theorem 2.10] applies and consequently u_ε is the unique L^p-viscosity solution.

The main result of this section is the following.

THEOREM 3.3. *Let Ω and f satisfy the standing assumptions set in Section 2. Then the solutions u_ε of* (1.1.ε)–(1.2) *converge in $\mathcal{C}(\overline{\Omega})$ to a Lipschitz continuous function u which is an L^p-viscosity solution of* (1.1)–(1.2).

PROOF. By (3.5) and the Ascoli–Arzelà theorem, u_ε has a subsequence that converges in $\mathcal{C}(\overline{\Omega})$ to a Lipschitz continuous function u. Let us show that u is an L^p-viscosity solution of (1.1). We only check the subsolution part, since the other assertion relies on the same arguments. Suppose by contradiction that u is not an L^p-viscosity subsolution; then, by definition, there exist a point $\bar{x}$ in Ω, a test function φ in $W^{2,p}(\Omega)$ and two positive parameters δ and r such that

- $F(D^2\varphi) + f^{1/\mathrm{N}} \geq \delta$ a.e. in the open ball $B := \{x : |x - \bar{x}| < r\}$,
- the function $u - \varphi$ restricted to $\overline{B} := \{x : |x - \bar{x}| \leq r\}$ has a global strict maximum at $\bar{x}$.

We get a contradiction by building a test function for u_ε, say φ_ε, in such a way that

$$F(D^2\varphi_\varepsilon) - \varepsilon\Delta\varphi_\varepsilon + f^{1/\mathrm{N}} \geq \delta \quad \text{a.e. in } B \quad \text{for all small } \varepsilon > 0, \tag{3.11}$$

$$\varphi_\varepsilon \to \varphi \quad \text{uniformly in } B \quad \text{as } \varepsilon \to 0. \tag{3.12}$$

Indeed, as u_ε is an L^p-viscosity solution of (1.1.ε), the inequality (3.11) prevents $u_\varepsilon - \varphi_\varepsilon$ from achieving a maximum inside the ball B. On the other hand $u_\varepsilon - \varphi_\varepsilon$ does have a maximum point in the closed ball $\overline{B}$, say x_ε, by the Weierstrass theorem. Since $u_\varepsilon - \varphi_\varepsilon \to u - \varphi$ uniformly in Ω, we have $x_\varepsilon \to \bar{x}$ up to a subsequence. In particular x_ε belongs to the open ball B for small ε, a contradiction.

What is left is to construct the auxiliary function φ_ε which does the job; we take it in the form $\varphi_\varepsilon = \varphi - \psi_\varepsilon$, where ψ_ε is a function to be determined. By subadditivity we compute

$$\begin{aligned} F(D^2\varphi_\varepsilon) - \varepsilon\Delta\varphi_\varepsilon + f^{1/\mathrm{N}} &\geq F(D^2\varphi) + f^{1/\mathrm{N}} - F(D^2\psi_\varepsilon) + \varepsilon\Delta\psi_\varepsilon - \varepsilon\Delta\varphi \\ &\geq \delta - F(D^2\psi_\varepsilon) + \varepsilon\Delta\psi_\varepsilon - \varepsilon\Delta\varphi \end{aligned}$$

for a.e. $x \in B$. Hence (3.11) is attained if we choose ψ_ε as the a.e. solution of $\det(D^2\psi)^{1/\mathrm{N}} + \varepsilon\Delta\psi = \varepsilon(\Delta\varphi)_+$ in B, with $\psi = 0$ on ∂B, produced in Proposition 3.1. Eventually φ_ε also satisfies (3.12) by virtue of estimate (3.5).

The proof is completed by checking that the limit function u does not depend on the subsequence. Let $\{\varepsilon'_n\}$ and $\{\varepsilon''_n\}$ be two vanishing sequences of parameters so that $u_{\varepsilon'_n} \to u'$ and $u_{\varepsilon''_n} \to u''$ in $\mathcal{C}(\overline{\Omega})$ as $n \to \infty$. Up to subsequences, we may suppose that the parameters are ordered as follows:

$$\cdots \leq \varepsilon''_{n+1} \leq \varepsilon'_n \leq \varepsilon''_n \leq \varepsilon'_{n-1} \leq \cdots .$$

To prove that $w = u_{\varepsilon'_n} - u_{\varepsilon''_{n+1}} \geq 0$, we show that $F(D^2 w) - \varepsilon'_n \Delta w \geq 0$ in $\mathcal{C}$-viscosity sense and then use a standard comparison principle (see, for instance, [12]). Suppose, contrary to our claim, that there exist a point $\bar{x} \in \Omega$, a test function $\varphi \in \mathcal{C}^2$ and two positive parameters δ and r such that

- $F(D^2\varphi) - \varepsilon'_n\Delta\varphi \leq -\delta$ in $B := \{x : |x - \bar{x}| < r\}$,
- $w - \varphi$ restricted to $\overline{B}$ has a global strict minimum at $\bar{x}$.

Then the function $\psi = \varphi + u_{\varepsilon''_{n+1}}$ has $W^{2,p}$ regularity and touches $u_{\varepsilon'_n}$ from below at the point $\bar{x}$. By the subadditivity of F we get

$$F(D^2\psi) - \varepsilon'_n \Delta\psi + f^{1/\mathrm{N}} \le F(D^2\varphi) + \varepsilon'_n \Delta\varphi + F(D^2 u_{\varepsilon''_{n+1}}) - \varepsilon''_{n+1}\Delta u_{\varepsilon''_{n+1}} + f^{1/\mathrm{N}} - (\varepsilon'_n - \varepsilon''_{n+1})\,\Delta u_{\varepsilon''_{n+1}} \le -\delta$$

since $u_{\varepsilon''_{n+1}}$ is a convex a.e. solution. This contradicts the fact that $u_{\varepsilon'_n}$ is an L^p-viscosity solution, hence $u_{\varepsilon'_n} - u_{\varepsilon''_{n+1}} \ge 0$ and, finally, letting $n \to \infty$ gives $u' \ge u''$.

Completely similar arguments yield $u_{\varepsilon''_n} - u_{\varepsilon'_{n-1}} \ge 0$, and so $u' = u''$.

4. Comparison with weak solutions

This section establishes the relation between weak and L^p-viscosity solutions. The first theorem shows that the limit function obtained with the vanishing viscosity method is actually the weak solution produced by Trudinger in [18]. Moreover, we prove that the weak solution is maximal among L^p-viscosity solutions and that, whenever the datum is bounded, it is, in fact, unique.

THEOREM 4.1. *Let Ω and f satisfy the standing assumptions set in Section 2. The solutions u_ε of* (1.1.ε)–(1.2) *converge in $\mathcal{C}(\overline{\Omega})$ to the weak solution of* (1.1)–(1.2). *In particular, the weak solution is an L^p-viscosity solution.*

PROOF. Let v be the weak solution of (1.1)–(1.2); this means that there exists a sequence v_m of $\mathcal{C}^2$ convex functions such that $v_m \to v$ uniformly in Ω and $\det(D^2 v_m) \to f$ in $L^1_{\mathrm{loc}}(\Omega)$. Let then u be the uniform limit of u_ε obtained in Theorem 3.3; we prove that $u = v$.

To prove that $w = u - v \ge 0$, it suffices to check that $F(D^2 w) \ge 0$ in $\mathcal{C}$-viscosity sense in Ω and then invoke the standard comparison principle. Assume by contradiction that w is not a supersolution; then there exist a point $\bar{x} \in \Omega$, a test function $\varphi \in \mathcal{C}^2$ and two positive parameters δ and r such that

- $F(D^2\varphi) \le -\delta$ in $B := \{x : |x - \bar{x}| < r\}$,
- $w - \varphi$ restricted to $\overline{B}$ has a global strict minimum at $\bar{x}$.

This step of the proof is completed by contradicting the fact that u_ε is an L^p-viscosity supersolution of (1.1.ε). To this end we take an auxiliary function in the form

$$\varphi_{\varepsilon,m} := \varphi + v_m + \psi_{\varepsilon,m},$$

where $\psi_{\varepsilon,m}$ will be chosen later in $W^{2,p}(\Omega)$ in such a way that $\psi_{\varepsilon,m} \to 0$ uniformly in B. The function $u_\varepsilon - \varphi_{\varepsilon,m}$ approaches $w - \varphi$ uniformly, therefore it achieves a minimum inside the ball B, at least for small ε and large m. Moreover, by the subadditivity of F and the convexity of v_m we get

$$\begin{aligned} &F(D^2\varphi_{\varepsilon,m}) - \varepsilon\Delta\varphi_{\varepsilon,m} + f^{1/\mathrm{N}} \\ &\quad \le F(D^2\varphi) - \varepsilon\Delta v_m + F(D^2\psi_{\varepsilon,m}) - \varepsilon\Delta\psi_{\varepsilon,m} - \varepsilon\Delta\varphi + F(D^2 v_m) + f^{1/\mathrm{N}} \\ &\quad \le -\delta + F(D^2\psi_{\varepsilon,m}) - \varepsilon\Delta\psi_{\varepsilon,m} - \varepsilon\Delta\varphi - (\det D^2 v_m)^{1/\mathrm{N}} + f^{1/\mathrm{N}}. \end{aligned}$$

Choosing $\psi_{\varepsilon,m}$ as the convex a.e. solution of

$$\begin{cases} (\det D^2\psi_{\varepsilon,m})^{1/\mathrm{N}} + \varepsilon\Delta\psi_{\varepsilon,m} = (-\varepsilon\Delta\varphi - (\det D^2 v_m)^{1/\mathrm{N}} + f^{1/\mathrm{N}})_+ & \text{in } B, \\ \psi_{\varepsilon,m} = 0 & \text{on } \partial B, \end{cases}$$

produced in Proposition 3.1, ensures that $F(D^2\varphi_{\varepsilon,m}) - \varepsilon\Delta\varphi_{\varepsilon,m} + f^{1/\mathrm{N}} \le -\delta$ a.e. in B. Moreover the estimate (3.5) implies that $\psi_{\varepsilon,m} \to 0$ uniformly in B, as required.

We next apply this argument again, with $w = v - u$, to obtain $v \ge u$ and finally the conclusion.

4.1. *The L^∞ case: comparison and uniqueness*

We now establish at the same time two relevant results concerning bounded data: there exists only one L^∞-viscosity solution, and the solution coincides with the weak solution. It is worth mentioning that our proof is independent of the uniqueness of weak solutions, which was already proved in [18]. Indeed, the uniqueness of weak solutions could be inferred at once.

The core of our argument is that weak and L^∞-viscosity sub/supersolutions compare, in the following sense.

THEOREM 4.2. *Let $\underline{u}$ and $\overline{u}$ be respectively a sub- and supersolution in L^∞-viscosity sense, and $\underline{v}$ and $\overline{v}$ be respectively a sub- and supersolution in weak sense of* (1.1). *Then $\underline{v} \le \overline{u}$ and $\underline{u} \le \overline{v}$ in Ω.*

In order to prove the above theorem, we need the following

LEMMA 4.3. *Let B be an open ball and $h \in L^\infty(B)$, $h \ge 0$ a.e. Then there exists a convex function $\psi \in W^{2,\infty}(B) \cap \mathcal{C}(\overline{B})$ which satisfies in the almost-everywhere sense the following problem:*

$$\begin{cases} \det(D^2\psi) \ge h, & x \in B, \\ \psi = 0, & x \in \partial B. \end{cases} \tag{4.13}$$

Moreover, there exists a constant C *independent of h so that*

$$\|\psi\|_\infty \le \mathrm{C}\|h\|_1. \tag{4.14}$$

PROOF. Let us choose $\bar{h} \in C^\infty(\Omega)$ such that $\bar{h} \ge h$ a.e. in Ω and $\|\bar{h}\|_{L^1} \le 2\|h\|_{L^1}$. Denote by ψ the classical solution of

$$\begin{cases} \det(D^2\psi) = \bar{h}, & x \in B, \\ \psi = 0, & x \in \partial B. \end{cases}$$

Then ψ satisfies (4.13) and (4.14) by classical estimates (see, for example, [15, 17]).

PROOF OF THEOREM 4.2. We first set $w := \overline{u} - \underline{v}$ and prove $w \ge 0$ by checking that $F(D^2 w) \ge 0$ in $\mathcal{C}$-viscosity sense in Ω. Assume by contradiction that there exist a point $\bar{x} \in \Omega$ and a test function $\varphi \in \mathcal{C}^2$ such that

- $F(D^2\varphi) \le -\delta$ in $B := \{x : |x-\bar{x}| < r\}$,
- $w - \varphi$ restricted to $\overline{B}$ has a global strict minimum at $\bar{x}$,

for suitable $\delta, r > 0$. We will follow the line of the proof of Theorem 4.1, actually we will refute that $F(D^2\bar{u}) + f^{1/\mathrm{N}} \ge 0$ in L^p-viscosity sense.

We denote by v_m and f_m the sequences of functions involved in the definition of weak subsolution, namely v_m are convex functions in $\mathcal{C}^2(\Omega)$ which tend to v uniformly in Ω, f_m are nonnegative functions in $L^1_{\mathrm{loc}}(\Omega)$ which tend to f in $L^1_{\mathrm{loc}}(\Omega)$, and $\det(D^2 v_m) \ge f_m$ in Ω. Next, we set

$$h_m = (f^{1/\mathrm{N}} - f_m{}^{1/\mathrm{N}})_+^{\mathrm{N}}, \qquad \varphi_m = \varphi + v_m + \psi_m$$

where ψ_m is the convex function in $W^{2,\infty}(B)$ satisfying (4.13) and (4.14) for $h = h_m$. Note that $h_m \to 0$ in $L^1(B)$, hence (4.14) implies that $\psi_m \to 0$ uniformly in B.

Because $\bar{u} - \varphi_m \to w - \varphi$ uniformly, the function $\bar{u} - \varphi_m$ achieves its minimum inside the ball B, at least for large m. Finally, we compute

$$\begin{aligned} F(D^2\varphi_m) + f^{1/\mathrm{N}} &\le F(D^2\varphi) + F(D^2 v_m) + F(D^2\psi_m) + f^{1/\mathrm{N}} \\ &\le F(D^2\varphi) - f_m^{1/\mathrm{N}} - (\det D^2\psi_m)^{1/\mathrm{N}} + f^{1/\mathrm{N}} \le -\delta, \end{aligned}$$

which is impossible.

The other assertion, i.e. $\overline{v} \ge \underline{u}$, follows similarly by showing that $F(D^2(\overline{v} - \underline{u})) \ge 0$ in $\mathcal{C}$-viscosity sense.

For L^∞ data, our plan is completed.

THEOREM 4.4. *Let $f \in L^\infty(\Omega)$, $f \ge 0$ a.e. Then problem* (1.1)–(1.2) *has a unique L^∞-viscosity solution and a unique weak solution. These two solutions coincide and they are the limit in $\mathcal{C}(\overline{\Omega})$ of the solutions of the vanishing viscosity problems* (1.1.ε)–(1.2).

PROOF. Let u be the L^∞-viscosity solution produced in Theorem 3.3, and v any weak solution. Since solutions are at the same time sub- and supersolutions, by Theorem 4.2 we get $u \le v \le u$. Therefore any weak solution coincides with the limit of the vanishing viscosity procedure, and uniqueness follows. Next, if w is another L^∞-viscosity solution, comparison again shows that $u \le w \le u$. So, also the L^∞-viscosity solution is unique.

4.2. *The general case: unilateral estimate*

We next show that, for general L^q data, the weak solution is the maximal L^p-viscosity solution.

THEOREM 4.5. *The weak solution v of problem* (1.1)–(1.2) *is the maximal L^p-viscosity solution, in the following sense: if u is any L^p-viscosity subsolution, then $u \le v$ pointwise in Ω.*

PROOF. Take two sequences f_m and h_m in $\mathcal{C}^\infty(\Omega)$ with

$$\begin{aligned} f_m \ge 1/m, &\qquad f_m \to f \quad \text{in } L^q \text{ as } m \to \infty, \\ h_m \ge \max\{1/m, f_m - f\}, &\qquad h_m \to 0 \quad \text{in } L^q \text{ as } m \to \infty. \end{aligned}$$

Note that $f + h_m \geq f_m$. Since f_m and h_m are smooth and strictly positive, there exist two classical solutions v_m and w_m of $\det(D^2 v_m) = f_m$ and $\det(D^2 w_m) = h_m$ in Ω, respectively, with homogeneous boundary condition. By [18, Lemma 2.1], v_m and w_m tend uniformly to v and 0, respectively. Therefore the assertion follows by checking that $v_m - w_m \geq u$ in Ω for all m. This, in turns, is implied by

$$F(D^2(v_m - w_m - u)) \geq 0 \quad \text{in } \Omega \tag{4.15}$$

in $\mathcal{C}$-viscosity sense. In order to prove the differential inequality (4.15), suppose that, on the contrary, there are an integer m, a point $\bar{x} \in \Omega$, and a test function $\varphi \in \mathcal{C}^2(\Omega)$ such that

- $F(D^2\varphi) \leq -\delta$ in $B := \{x : |x - \bar{x}| < r\}$,
- $v_m - w_m - u - \varphi$ restricted to $\overline{B}$ has a global strict minimum at $\bar{x}$,

for suitable positive parameters δ and r. In particular, $v_m - w_m - \varphi$ is a test function for u, because it is $\mathcal{C}^2$ and touches the graph of u from above at $\bar{x}$. We thus come to a contradiction by showing that

$$F(D^2(v_m - w_m - \varphi)) + f^{1/\mathrm{N}} \geq \delta \quad \text{a.e. in } B.$$

This is easily seen as follows:

$$\begin{aligned} F(D^2(v_m - w_m - \varphi)) + f^{1/\mathrm{N}} &\geq F(D^2 v_m) - F(D^2 w_m) - F(D^2\varphi) + f^{1/\mathrm{N}} \\ &= -{f_m}^{1/\mathrm{N}} + {h_m}^{1/\mathrm{N}} - F(D^2\varphi) + f^{1/\mathrm{N}} \geq \delta - {f_m}^{1/\mathrm{N}} + (h_m + f)^{1/\mathrm{N}} \geq \delta. \end{aligned}$$

Comments

As a motivation for using L^p-viscosity solutions in this framework, we mention that obtaining stability for a numerical scheme is straightforward, by adapting the reasoning of [3] according to the proofs of Theorems 3.3 and 4.1.

Moreover, the arguments we use here could carry over to more general equations

$$F(D^2 u) + g(x, u, Du) = 0$$

provided that at least the following structure conditions hold true:

a) F is a continuous, degenerate elliptic, subadditive second order operator, satisfying

$$\Lambda\, \mathrm{tr}((D^2u)_-) \geq F(D^2u) \geq -\Lambda\, \mathrm{tr}((D^2u)_+) \quad \text{if } u \in W^{2,p}(\Omega),$$

possibly in a subset of $W^{2,p}(\Omega)$ closed with respect to uniform convergence, as for instance the set of convex $W^{2,p}$ functions; here $(D^2u)_+$ and $(D^2u)_-$ stand for the positive and negative parts of the Hessian matrix D^2u, i.e. $D^2u = (D^2u)_+ - (D^2u)_-$;

b) the function g is jointly measurable in all variables (x, u, Du) with

$$x \mapsto g(x, 0, 0) \in L^p(\Omega) \quad \text{for } p \geq \mathrm{N};$$

moreover g is continuous in u, Du (uniformly with respect to x), “proper”, i.e. nondecreasing with respect to u, and Lipschitz continuous in Du (uniformly with respect to x, u).

In particular various (homogeneous) combinations of Hessian operators, and many Hessian equations with lower order terms can be dealt with by our approach.

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