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Functional analysis. — *Linear isometries of some function algebras*, by EDOARDO VESENTINI, communicated on 9 February 2007.

ABSTRACT. — Linear isometries of a class of logmodular algebras which are generated by unimodular functions are represented by Holsztyński-type weighted composition operators. The description of these operators leads—among other things—to a description of a class of linear isometries of the disc algebra and of the Hardy space of all bounded holomorphic functions on the open unit disc of $\mathbb{C}$. Spectral properties of these isometries are also investigated.

KEY WORDS: Logmodular algebra, Shilov boundary, inner functions.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 30E25.

Linear isometries of Hardy spaces H^p $(1 \leq p < \infty, p \neq 2)$ on the open unit disc Δ of $\mathbb{C}$ have been described by F. Forelli in [8], and surjective linear isometries of H^∞ by N. Nagasawa in [16] and by K. DeLeeuw, W. Rudin and J. Wermer in [6] (see also [11]).

A different approach to the linear isometries of H^∞ and of the disc algebra is motivated by two facts. First of all, they are both logmodular algebras; secondly, their closed unit balls are the closed convex hulls of their inner functions, as was proved in [2] and [3].

Starting from these facts, a theorem established in the first two sections of this article describes the linear isometries of a uniform algebra $\mathcal{A}$ into a uniform function algebra $\mathcal{B}$ under the hypotheses that $\mathcal{A}$ is generated by its unimodular functions and every character of $\mathcal{B}$ has a unique representing measure supported by the Shilov boundary of $\mathcal{B}$.

Among other things, this theorem yields a new proof of Holsztyński's extension of the classical Banach–Stone theorem, a characterization of those self-isometries of the Hardy space H^∞ and of the disc algebra mapping the sets of all inner functions into themselves, showing incidentally that, as for any Hardy space H^p $(1 \leq p < \infty, p \neq 2)$, these isometries are represented by weighted composition operators.

The final section summarizes and completes some results established in [23] for strongly continuous semigroups of linear isometries of H^∞.

1. CONTINUOUS LINEAR FORMS ON SOME UNIFORM ALGEBRAS

Let m be a positive regular Borel measure on a compact Hausdorff space M with $m(M) \leq 1$, and let $v \in L^1_{\mathbb{R}}(M, m) = L^1_{\mathbb{R}}(M)$ be such that $|v| \leq 1$ almost everywhere on M and

$$\int v\, dm = 1. \tag{1}$$

That implies, first of all, that $v \geq 0$ a.e. on M and that $m(M) = 1$.

Let $N \subset M$ be the measurable set

$$N = \{x \in M : v(x) \neq 1\}. \tag{2}$$

We will show that

$$m(N) = 0. \tag{3}$$

Let $w(x) = 1 - v(x)$. Then

$$w(x) \geq 0 \tag{4}$$

and (1), (2) become

$$\int w \, dm = 0, \tag{5}$$

$$N = \{x \in M : w(x) > 0\}. \tag{6}$$

Suppose $m(N) > 0$. By the Lusin theorem (see, e.g., [18, Theorem 2.23]), for any $\epsilon \in (0, 1/2)$ there exists a real-valued, continuous function $\tilde{w}$ on M such that

$$\sup\{\tilde{w}(x) : x \in M\} \leq \sup\{|\tilde{w}(x)| : x \in M\} \leq \sup\{|w(x)| : x \in M\};$$

and

$$m(L(\epsilon)) < \epsilon m(N),$$

where $L(\epsilon)$ is the measurable set

$$L(\epsilon) = \{x \in M : \tilde{w}(x) \neq w(x)\}.$$

Hence,

$$\begin{aligned} m(N \backslash L(\epsilon)) &= m(N) - m(N \cap L(\epsilon)) \geq m(N) - m(L(\epsilon)) \\ &> (1 - \epsilon)m(N) > 0. \end{aligned}$$

The positive Borel measure m being regular, there exists a compact set $K \subset N \backslash L(\epsilon)$ such that

$$m(K) > (1 - 2\epsilon)m(N) > 0. \tag{7}$$

Since w is continuous on K, (4) and (6) imply that there is a positive constant k such that $w(x) \geq k$ for all $x \in K$. Thus, by (7),

$$\begin{aligned} \int_M w \, dm &= \int_{M \backslash K} w \, dm + \int_K w \, dm \geq \int_K w \, dm \\ &\geq k \int_K dm = km(K) > (1 - 2\epsilon)km(N) > 0, \end{aligned}$$

contradicting (5) and thereby proving that (3) holds.

In conclusion, the following lemma has been established.

LEMMA 1. *If $f \in L^1(M, m) = L^1(M)$ is such that $|f| \leq 1$ almost everywhere on M, $\|m\| \leq 1$ and*

$$\left| \int f\, dm \right| = 1,$$

then $\|m\| = m(M) = 1$, $\|f\| = 1$, and $f = e^{i\vartheta}$ a.e. for some $\vartheta \in \mathbb{R}$.

Let $\mathcal{A}$ be a uniform algebra on a compact Hausdorff space X, whose Shilov boundary $\partial\mathcal{A}$ coincides with X. Any continuous linear form λ on $\mathcal{A}$ is represented by a complex, regular Borel measure μ on X such that $\|\mu\| = \|\lambda\|$.

If

$$d\mu = h\, d|\mu|$$

is the polar representation of μ (see, e.g., [18]), where h is a complex-valued, measurable function with $|h| = 1$ a.e. $|\mu|$ on X, then for any $f \in \mathcal{A}$,

$$\langle f, \lambda \rangle = \int_X f\, d\mu = \int_X fh\, d|\mu|.$$

Suppose now that $\|\mu\| = \|\lambda\| \leq 1$, and let $u \in \mathcal{A}$ be such that

$$\|u\| = |\langle u, \lambda \rangle| = 1.$$

Then

$$1 = |\langle u, \lambda \rangle| = \left| \int_X uh\, d|\mu| \right| \leq \int_X |u|\, d|\mu| = \|u\| = 1.$$

As a consequence, $\|\mu\| = |\mu|(X) = 1$ and

$$u(x)h(x) = \langle u, \lambda \rangle \quad \text{a.e. } |\mu|.$$

By Lemma 1, hu must be constant on the support of μ.

2. LINEAR ISOMETRIES BETWEEN TWO UNIFORM ALGEBRAS

Let $\mathcal{A}$, $\mathcal{B}$ be uniform algebras on two compact Hausdorff spaces X, Y, and let $\Sigma(\mathcal{A})$, $\Sigma(\mathcal{B})$ and $\partial\mathcal{A} = X$, $\partial\mathcal{B} = Y$ be the spaces of maximal ideals and the Shilov boundaries of $\mathcal{A}$, $\mathcal{B}$. Let $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ be a linear isometry of $\mathcal{A}$ into $\mathcal{B}$.

For any $x \in X$, let

$$\Omega(x) = \{f \in \mathcal{A} : |f(x)| = \|f\| = 1\}$$

Since $\mathcal{A}$ contains the constants, $\Omega(x) \neq \emptyset$ for all $x \in X$.

Following an idea of W. Holsztyński [13], we now prove

LEMMA 2. *For any $x \in X$ the set*

$$\Upsilon(x) = \{y \in Y : |(Af)(y)| = 1\ \forall f \in \Omega(x)\}$$

is closed and not empty.

PROOF. Let n be a positive integer and let $u_1, \ldots, u_n$ be elements of $\mathcal{A}$ such that

$$|u_j(x)x| = \|u_j\| = 1, \qquad j = 1, \ldots, n. \tag{8}$$

The function

$$u = \sum_{j=1}^{n} \overline{u_j(x)}\, u_j \in \mathcal{A}$$

is such that

$$|u(t)| \le \sum_{j=1}^{n} |u_j(t)| \le \sum_{j=1}^{n} \|u_j\| = n \qquad \forall t \in X$$

and

$$u(x) = \sum_{j=1}^{n} |u_j(x)|^2 = n.$$

Therefore $\|u\| = n$.

Since A is an isometry, there is some $y \in Y$ for which $|(Au)(y)| = n$. As

$$(Au)(y) = \sum_{j=1}^{n} \overline{u_j(x)}(Au_j)(y),$$

we have

$$n = |(Au)(y)| \le \sum_{j=1}^{n} |(Au_j)(y)| \le \sum_{j=1}^{n} \|Au_j\| = n,$$

showing that $|(Au_j)(y)| = 1$ for $j = 1, \ldots, n$, i.e.

$$\{y \in Y : |(Au_j)(y)| = 1,\ j = 1, \ldots, n\} \neq \emptyset \tag{9}$$

for every choice of $u_1, \ldots, u_n$ in $\mathcal{A}$ satisfying (8). The conclusion follows from the fact that Y is compact and the set (9) is closed. □

An element $u \in \mathcal{A}$ such that $|u(x)| = 1$ for all $x \in X$ is called a *unimodular* (or *unitary*) *function*.

Let $\mathfrak{U} = \mathfrak{U}(\mathcal{A})$ be the set of all unitary functions in $\mathcal{A}$. Clearly,

$$\emptyset \neq \mathfrak{U} = \bigcap \{\Omega(x) : x \in M\}.$$

Let Q be the closed set of all $y \in Y$ such that $|(Au)(y)| = 1$ for all $u \in \mathfrak{U}(\mathcal{A})$. Since

$$\begin{aligned} y \in \Upsilon(x) \ &\Leftrightarrow\ |(Af)(y)| = 1\ \forall f \in \Omega(x) \\ &\Rightarrow\ |(Au)(y)| = 1\ \forall u \in \mathfrak{U} \ \Leftrightarrow\ y \in Q, \end{aligned}$$

it follows that

$$\Upsilon(x) \subset Q \qquad \forall x \in X, \tag{10}$$

and therefore $Q \neq \emptyset$.

Let $y \in Y$. Denoting by λ the continuous linear form on $\mathcal{A}$ defined by

$$\langle f, \lambda \rangle = (Af)(y) \quad \forall f \in \mathcal{A},$$

let μ be the complex regular Borel measure on X which represents λ. Then, for any $u \in \mathfrak{U}$ and all $y \in Q$,

$$(Au)(y) = u(t)h(t) \quad \text{a.e. in } \operatorname{Supp}(\mu).$$

If, given any two distinct points t' and t'' of X, there is $u \in \mathfrak{U}$ such that $u(t') \neq u(t'')$, then $\operatorname{Supp}(\mu)$ is reduced to one point, and the following lemma holds.

LEMMA 3. *If the set $\mathfrak{U}$ separates points in X, then there is a map $\psi : Q \to X$ such that*

$$(Au)(y) = (A1)(y)u(\psi(y)) \quad \forall u \in \mathfrak{U},\ y \in Q. \tag{11}$$

It has been shown by A. Bernard ([2], [3], [10, pp. 195–196]) that if the set $\mathfrak{U}$ generates $\mathcal{A}$, then the closed unit ball of $\mathcal{A}$ is the closed convex hull of $\mathfrak{U}$.

Thus, if $\mathfrak{U}$ generates $\mathcal{A}$, then for every $f \in \mathcal{A}$ and for every $\epsilon > 0$ there are a positive integer n, positive numbers $t_1, \ldots, t_n$ with $\sum_{\nu=1}^{n} t_\nu = 1$ and functions $u_1, \ldots, u_n \in \mathfrak{U}$ such that

$$\left\| f - \sum_{\nu=1}^{n} t_\nu u_\nu \right\| \leq \epsilon.$$

Choose $y \in Q$. Since $|(A1)(y)| = 1$, (11) yields

$$\begin{aligned} |(Af)(y) - (A1)(y)f(\psi(y))| &\leq \left| \left(f - \sum_{\nu=1}^{n} t_\nu u_\nu \right)(y) \right| + \left| \left(f - \sum_{\nu=1}^{n} t_\nu u_\nu \right)(\psi(y)) \right| \\ &= 2 \left\| f - \sum_{\nu=1}^{n} t_\nu u_\nu \right\| \leq 2\epsilon. \end{aligned}$$

Furthermore, since $\mathcal{A}$ separates points, if $\mathfrak{U}$ generates $\mathcal{A}$ then also $\mathfrak{U}$ separates points. All that yields the first part of the following theorem.

THEOREM 1. *If A is a linear isometry of $\mathcal{A}$ into $\mathcal{B}$, and if $\mathfrak{U}(\mathcal{A})$ generates $\mathcal{A}$, then there are a closed subset Q of Y and a continuous surjective map $\psi : Q \to X$ such that*

$$(Af)(y) = (A1)(y)f(\psi(y)) \quad \forall f \in \mathcal{A},\ y \in Q. \tag{12}$$

PROOF. To establish the continuity of ψ, let $\{y_j\}$ be a net in Q converging to $y \in Q$, and suppose that there are two subnets $\{y_r\}$ and $\{y_s\}$ such that the nets $\{\psi(y_r)\}$ and $\{\psi(y_s)\}$ converge to two distinct elements t' and t'' of X. Then, for any $f \in \mathcal{A}$, the nets

$$\{f(\psi(y_r))\} = \{(Af)(y_r)\,\overline{(A1)(y_r)}\}$$

and

$$\{f(\psi(y_s))\} = \{(Af)(y_s)\,\overline{(A1)(y_s)}\}$$

converge to

$$f(t') = (Af)(y)\,\overline{(A1)(y)} = f(t''),$$

contradicting the fact that, since $t' \neq t''$, there is some $f \in \mathcal{A}$ such that $f(t') \neq f(t'')$.

All that is left to prove is the surjectivity of ψ. Since ψ is continuous, $\psi(Q)$ is a compact subset of X. If $\psi(Q) \neq X$, the open set $X \setminus \psi(Q)$ is non-empty. Let V and W be two open sets in X such that $V \neq \emptyset$ and

$$\overline{V} \subset W \subset X \setminus \psi(Q).$$

Since the open set V contains some strong boundary point x of $\mathcal{A}$, there exists $h \in \mathcal{A}$ with $\|h\| \leq 1$ and

$$|h(x)| > 3/4 \quad \text{and} \quad |h(t)| < 1/4 \quad \forall t \in X \setminus V.$$

Let $s \in X$ be such that $|h(s)| = \|h\|$. Then

$$\|h\| = |h(s)| > 3/4.$$

Set

$$f = \frac{1}{\|h\|} h.$$

Since $1/\|h\| < 4/3$, we have

$$|f(t)| = \frac{|h(t)|}{\|h\|} < \frac{1}{\|h\|}\frac{1}{4} < \frac{1}{3} \quad \forall t \in X \setminus V.$$

Thus,

$$1 = \|Af\| \leq \sup\{|(A1)(y)|.|(f(\psi(y))| : y \in Q\} < 1/3,$$

a contradiction. $\square$

If A maps all unimodular functions on X to unimodular functions on Y, then $Q = Y$,

$$|(A1)(y)| = 1 \quad \forall y \in Y, \tag{13}$$

$$(Af)(y) = (A1)(y) f(\psi(y)) \quad \forall f \in \mathcal{A},\ y \in Y, \tag{14}$$

and the following proposition holds.

PROPOSITION 1. *If $\mathfrak{U}(\mathcal{A})$ generates $\mathcal{A}$ and is mapped by A into $\mathfrak{U}(\mathcal{B})$, then* (13) *is satisfied, and there is a continuous surjective map $\psi : Y \to X$ for which* (14) *holds.*

Conversely, if $\mathfrak{U}(\mathcal{A})$ and $\mathfrak{U}(\mathcal{B})$ satisfy the above hypotheses, then $|(Au)(y)| = 1$ for all $u \in \mathfrak{U}(\mathcal{A})$ and all $y \in Y$, i.e. $Q = Y$.

In particular, if A is an isometric homomorphism of the algebra $\mathcal{A}$ into the algebra $\mathcal{B}$, then $Q = Y$ if, and only if, there is a continuous surjective map $\psi : Y \to X$ such that

$$Af = f \circ \psi \quad \forall f \in \mathcal{A}.$$

Going back to the linear isometry A in Theorem 1, for all $f \in \mathcal{A}$ and any $y \in Q$,

$$|(Af)(y)| = |f(\psi(y))|.$$

Since $\psi(Q) = X$, this implies that if $Af \in \mathfrak{U}(\mathcal{B})$, then $f \in \mathfrak{U}(\mathcal{A})$, i.e.

$$A^{-1}(\mathfrak{U}(\mathcal{B}) \cap A(\mathcal{A})) \subset \mathfrak{U}(\mathcal{A}).$$

Hence, if A is surjective, then $\mathfrak{U}(\mathcal{B}) \subset A(\mathfrak{U}(\mathcal{A}))$ and also $\mathfrak{U}(\mathcal{A}) \subset A^{-1}(\mathfrak{U}(\mathcal{B}))$. Proposition 1 yields

COROLLARY 1. *If $\mathfrak{U}(\mathcal{A})$ generates $\mathcal{A}$ and if the isometry A is surjective, then $A(\mathfrak{U}(\mathcal{A})) = \mathfrak{U}(\mathcal{B})$, and A is expressed by* (14), *where ψ is now a homeomorphism of Y onto X.*

If $X = Y$, $\mathcal{A} = \mathcal{B}$ and if the isometry A is not surjective, then its spectrum $\sigma(A)$ is the closed unit disc $\overline{\Delta}$ and Δ is contained in the residual spectrum. As a consequence, $A(\mathcal{A})$ is contained in a proper closed subspace of $\mathcal{A}$. Thus, denoting by $\mathcal{A}^{-1}$ the set of all invertible elements of $\mathcal{A}$, we have proved

LEMMA 4. *If $X = Y$, $\mathcal{A} = \mathcal{B}$ and if $A(\mathcal{A}^{-1})$ contains a non-empty open set, then the isometry A is surjective.*

Let $\Sigma(\mathcal{A})$ and $\Sigma(\mathcal{B})$ be the sets of all characters of $\mathcal{A}$ and $\mathcal{B}$, i.e. all homomorphisms of the abelian Banach algebras $\mathcal{A}$ and $\mathcal{B}$ into $\mathbb{C}$. Let $P \subset \Sigma(\mathcal{B})$ be the set of all $\chi \in \Sigma(\mathcal{B})$ having a representing measure m_χ (i.e. a regular probability measure which represents χ) whose support is contained in Q. Obviously, $Q \subset P$ and, if $Q = Y$, then $P = \Sigma(\mathcal{B})$.

Let $\mathcal{A}$ and $\mathcal{B}$ satisfy the hypotheses of Theorem 1. For $\chi \in P$ let m_χ be a representing measure of χ whose support is contained in Q.

For any $f \in \mathcal{A}$,

$$\begin{aligned}\langle Af, \chi\rangle &= \int (Af)(y)\, dm_\chi(y) = \int (A1)(y)\, dm_\chi(y) \int f(\psi(y))\, dm_\chi(y)\\ &= \langle A1, \chi\rangle \int f(\psi(y))\, dm_\chi(y)\end{aligned}$$

because m_χ is multiplicative. Since furthermore $\psi(y) \in \partial\mathcal{A}$, we have

$$\begin{aligned}\int (f_1 f_2)(\psi(y))\, dm_\chi(y) &= \int f_1(\psi(y)) f_2(\psi(y))\, dm_\chi(y)\\ &= \int f_1(\psi(y))\, dm_\chi(y) \int f_2(\psi(y))\, dm_\chi(y)\end{aligned}$$

for all $f_1, f_2 \in \mathcal{A}$. Hence, there exists a character $\omega(\chi)$ of $\mathcal{A}$ such that

$$\int f(\psi(y))\, dm_\chi(y) = \langle f, \omega(\chi)\rangle.$$

Assuming that every $\chi \in P$ has a unique representing measure m_χ whose support is contained in Q, $\chi \mapsto \omega(\chi)$ defines a map $\omega : P \to \Sigma(\mathcal{A})$ such that

$$\langle Af, \chi\rangle = \langle A1, \chi\rangle \langle f, \omega(\chi)\rangle \quad \forall f \in \mathcal{A}, \chi \in P. \tag{15}$$

The same kind of argument as in the proof of Theorem 1 shows that the map ω is continuous, and, in conclusion, the following theorem holds.

THEOREM 2. *Let A be a linear isometry of the algebra $\mathcal{A}$ into $\mathcal{B}$. If every $\chi \in \Sigma(\mathcal{B})$ has a unique representing measure and if the set $\mathfrak{U}$ of all unimodular functions in $\mathcal{A}$ generates $\mathcal{A}$, then there is a subset P of $\Sigma(\mathcal{B})$ containing the closed subset Q of $\partial\mathcal{B} = Y$, and a continuous map $\omega : P \to \Sigma(\mathcal{A})$ such that $\omega_{|Q} = \psi$ (whence $\omega(P) \supset \partial\mathcal{A} = \omega(Q) = X$) and* (15) *holds.*

Furthermore, if $Q = Y$, then $P = \Sigma(\mathcal{B})$ and (15) *holds for all $\chi \in \Sigma(\mathcal{B})$.*

If A, $\mathcal{A}$ and $\mathcal{B}$ are as in Theorem 2 and if moreover

$$A(\mathcal{A}^{-1}) \subset \mathcal{B}^{-1}, \tag{16}$$

then by the Gleason–Kahane–Żelazko theorem (see, e.g., [22]) there exists a continuous map $\varphi : \Sigma(\mathcal{B}) \to \Sigma(\mathcal{A})$ such that

$$\langle Af, \chi\rangle = \langle A1, \chi\rangle\langle f, \varphi(\chi)\rangle \quad \forall f \in \mathcal{A},\ \chi \in \Sigma(\mathcal{B}).$$

Comparison of the last equation with (15) shows that, since $\langle A1, \chi\rangle \neq 0$ for all $\chi \in Q$,

$$\langle f, \varphi(\chi)\rangle = \langle f, \omega(\chi)\rangle \quad \forall f \in \mathcal{A},\ \chi \in Q;$$

since $\mathcal{A}$ separates points in Q, it follows that $\varphi = \omega$ on Q, i.e. φ is a continuous extension of $\omega_{|Q}$, and therefore $\varphi(\partial\mathcal{B}) \supset \partial\mathcal{A}$.

If furthermore

$$\Sigma(\mathcal{A}) = \partial\mathcal{A} = X, \tag{17}$$

and if $A1 \in \mathfrak{U}(\mathcal{B})$, then for any $u \in \mathfrak{U}(\mathcal{A})$ and any $\chi \in \partial\mathcal{B}$,

$$|\langle Au, \chi\rangle| = |\langle u, \varphi(\chi)\rangle| = 1.$$

Thus $Q = Y$ and the following proposition holds:

PROPOSITION 2. *If A, $\mathcal{A}$, $\mathcal{B}$ satisfy the hypotheses of Theorem* 2*, and if furthermore* (16) *and* (17) *hold and $A1$ is a unimodular function in $\mathcal{B}$, then $A(\mathfrak{U}(\mathcal{A})) \subset \mathfrak{U}(\mathcal{B})$, and* (15) *holds for all $\chi \in \Sigma(\mathcal{B})$.*

COROLLARY 2. *If A is an isometric homomorphism of $\mathcal{A}$ into $\mathcal{B}$ and if* (17) *holds, then there is a continuous surjective map $\varphi : \Sigma(\mathcal{B}) \to X$ such that*

$$\langle Af, \chi\rangle = \langle f, \varphi(\chi)\rangle \quad \forall f \in \mathcal{A},\ \chi \in \Sigma(\mathcal{B}).$$

We point out for future reference that if $X = Y$, $\mathcal{A} = \mathcal{B}$ and $Q = Y$, then for any positive integer n,

$$A^n 1 = A1 \cdot (A1 \circ \omega) \cdots (A1 \circ \omega^{\circ(n-1)}) \tag{18}$$

and

$$A^n f = A^n 1 \cdot f \circ \omega^{\circ n} \quad \forall f \in \mathcal{A}, \tag{19}$$

where, for any positive integer n, $A^n = A \circ \cdots \circ A$ (n times) and $\omega^{\circ n}$ is the n-th iterate of ω.

3. Gleason parts

Let now $\mathcal{C}$ be a uniform algebra on a compact Hausdorff space M. According to Bishop's theorem (see, e.g., [19, Theorem 16.6]), if χ_1, χ_2 are two characters of $\mathcal{C}$ which are contained in the same Gleason part of $\mathcal{C}$, and μ_1, μ_2 are representing measures of χ_1, χ_2, then μ_1, μ_2 are mutually absolutely continuous (and the Radon–Nikodym derivatives $d\mu_1/d\mu_2$, $d\mu_2/d\mu_1$ are both bounded). Therefore,

$$\mathrm{Supp}(\mu_1) = \mathrm{Supp}(\mu_2)$$

and, as a consequence, we have

LEMMA 5. *Under the hypotheses of Theorem 2, let Π be a Gleason part of $\mathcal{B}$. If $\Pi \cap P \neq \emptyset$, then $\Pi \subset P$.*

Let Π be a Gleason part of $\mathcal{C}$ and let $F : \Delta \to \Pi$ be a continuous one-to-one map such that the function

$$\hat{f} \circ F : z \mapsto \langle f, F(z)\rangle$$

is holomorphic on Δ for all $f \in \mathcal{C}$. Since for every continuous linear form λ on $\mathcal{C}$,

$$\|\lambda\| = \sup\{|\langle f, \lambda\rangle| : f \in \mathcal{C},\ \|f\| \leq 1\},$$

$\mathcal{C}$ is a determining manifold for the topological dual $\mathcal{C}'$ of $\mathcal{C}$. By Dunford's theorem (see, e.g., [9, Theorem II.3.10]), F is a holomorphic map of Δ into $\mathcal{C}'$.

Let now $F : \Delta \to \mathcal{C}'$ be a holomorphic map such that $F(\Delta) \subset \Sigma(\mathcal{C})$. For every $z \in \Delta$, we have $\|F(z)\| = 1$, and therefore, for $z_1, z_2 \in \Delta$,

$$\|F(z_1) - F(z_2)\| \leq \|F(z_1)\| + \|F(z_2)\| = 2.$$

If

$$\|F(z_1) - F(z_2)\| = 2$$

for some $z_1, z_2 \in \Delta$, then by the maximum principle

$$\|F(z_1) - F(z)\| = 2 \quad \forall z \in \Delta.$$

On the other hand,

$$\lim_{z \to z_1} \|F(z_1) - F(z)\| = 0$$

because F is continuous. This contradiction shows that

$$\|F(z_1) - F(z_2)\| < 2$$

for all $z_1, z_2 \in \Delta$, proving

PROPOSITION 3. *Let $F : \Delta \to \Sigma(\mathcal{C})$ be a continuous map such that $z \mapsto \langle f, F(z)\rangle$ is holomorphic in Δ for all $f \in \mathcal{C}$. Then F is a holomorphic map of Δ into $\mathcal{C}'$, and $F(\Delta)$ is contained in a Gleason part of $\mathcal{C}$.*

COROLLARY 3. *If F is as in Proposition* 3, *and if $F(z)$ is a one-point Gleason part of $\mathcal{C}$ for some $z \in \Delta$, then F is constant.*

Let $\mathcal{A}$, $\mathcal{B}$ and A be as in Theorem 2 and let Π be a Gleason part contained in P. If Π contains more than one point, thenWermer's embedding theorem (Theorem 17.1 of [19]) and Proposition 3 show that there is a holomorphic map $F : \Delta \to \mathcal{B}'$ such that $F(\Delta) = \Pi$ and

$$\langle Af, F(z)\rangle = \langle A1, F(z)\rangle\langle f, \omega(F(z))\rangle \quad \forall f \in \mathcal{A},\ z \in \Delta. \tag{20}$$

Since the functions $z \mapsto \langle Af, F(z)\rangle$ and $z \mapsto \langle A1, F(z)\rangle$ are holomorphic on Δ, so are $z \mapsto \langle f, \omega(F(z))\rangle$ for all $f \in \mathcal{A}$, and therefore also $z \mapsto \omega(F(z))$.

Hence:

THEOREM 3. *If $\mathcal{A}$, $\mathcal{B}$ and the linear isometry A satisfy the hypotheses of Theorem* 2, *and if Π is a Gleason part of $\mathcal{B}$ which contains more than one point and is contained in P, then there is a holomorphic map $F : \Delta \to \mathcal{B}'$ mapping Δ one-to-one onto Π such that $\omega \circ F$ is holomorphic on Δ, $\omega(F(\Delta))$ is contained in a Gleason part of $\mathcal{A}$, and* (20) *holds.*

4. EXAMPLES

I. By a theorem of R. Phelps ([17], see also [5]), the closed unit ball of the uniform algebra $C(M)$ of all complex-valued, continuous functions on any compact Hausdorff space M is the closed convex hull of the set of all the unitary functions in $C(M)$.

Let X and Y be compact Hausdorff spaces, and let $A \in \mathcal{L}(C(X), C(Y))$ be an isometry of $C(X)$ into $C(Y)$. Since all characters of $C(Y)$ are point evaluations, the sets P and Q coincide, and Theorem 2 yields W. Holsztyński's theorem [13]:

THEOREM 4. *There exists a closed subset $P \subset Y$ and a continuous surjective map $\omega : P \to X$ for which*

$$(Af)(y) = (A1)(y)f(\omega(y)) \tag{21}$$

for all $f \in C(X)$ and all $y \in P$.

The set $Q = P$ consists of all points $y \in Y$ such that $|(Au)(y)| = 1$ for all unitary functions $u \in C(X)$. Thus, if A maps all unitary functions to unitary functions, then $P = Q = Y$, and Theorem 4 yields Theorem 1 of [20] (see also [7]).

Corollary 2 yields

COROLLARY 4. *If A is an isometric homomorphism of $C(X)$ into $C(Y)$, then there is a continuous surjective map $\varphi : Y \to X$ such that*

$$(Af)(y) = f(\varphi(y)) \quad \forall f \in C(X),\, y \in Y.$$

Let now $X = Y$ and suppose that the linear self-isometry A of $C(X)$ maps $\mathfrak{U}(C(X))$ into itself and that the sequence $\{A^n\}$ of the iterates of A converges to the identity for the weak operator topology:

$$\lim_{n\to\infty}(A^n f)(y) = f(y) \quad \forall f \in C(M),\ y \in X. \tag{22}$$

In particular,

$$\lim_{n\to\infty} (A^n 1)(y) = 1 \quad \forall y \in X. \tag{23}$$

If $|(A^p 1)(y)| < 1$ for some $y \in X$ and some positive integer p, then, by (18), $|(A^n 1)(y)| < 1$ for all $n \geq p$, contradicting (23). Thus,

$$|(A^n 1)(y)| = 1 \quad \forall n \in \mathbb{N},\ y \in X, \tag{24}$$

and (19) yields

$$|(A^n f)(y)| = |f(\omega^{\circ n}(y))| \quad \forall f \in C(X),\ y \in X,\ n \in \mathbb{N},$$

and, by (22),

$$\lim_{n\to\infty} |f(\omega^{\circ n}(y))| = |f(y)|. \tag{25}$$

Since X is compact, for any $y \in X$, there are an increasing sequence of positive integers $\{n_0, n_1, \ldots\}$ and a point $y' \in X$ such that

$$\lim_{j\to\infty} \omega^{\circ n_j}(y) = y',$$

and therefore, by (25),

$$|f(y)| = \lim_{j\to\infty} |f(\omega^{\circ n_j}(y))| = |f(y')|$$

for all $f \in C(X)$. This implies that $y = y'$, i.e.

$$\lim_{n\to\infty} \omega^{\circ n}(y) = y \quad \forall y \in X. \tag{26}$$

Hence, for any $y \in X$,

$$\omega(y) = \lim_{n\to\infty} \omega^{\circ n}(\omega(y)) = \lim_{n\to\infty} \omega^{\circ(n+1)}(y) = y,$$

showing that the map ω is the identity, and therefore A is a surjective isometry represented by

$$Af = A1 \cdot f \quad \forall f \in C(X).$$

In particular,

$$A^n 1 = (A1)^n \quad \text{for } n = 0, 1, 2, \ldots. \tag{27}$$

Since $|(A1)(y)| = 1$ for all $y \in X$, we deduce that

$$\lim_{n\to\infty} (A^n 1)(y) = 1 \quad \forall y \in X$$

if, and only if, $A1 = 1$.

Thus:

PROPOSITION 4. *If the isometry $A \in \mathcal{L}(C(X))$ maps all unitary functions to unitary functions, and if the sequence $\{A^n\}$ converges to the identity for the weak operator topology, then A is the identity.*

II. By a theorem of R. Phelps ([17]), if the function algebra $\mathcal{A}$ is logmodular and if there is a Gleason part L of $\Sigma(\mathcal{A})$ which is total over $\mathcal{A}$, then the closed unit ball of $\mathcal{A}$ is the closed convex hull of its exposed points.

Examples of such algebras are the disc algebra $\mathcal{A}^0$, i.e., the uniform algebra of all complex-valued, continuous functions on the closure $\overline{\Delta}$ of Δ whose restrictions to Δ are holomorphic, and the uniform algebra H^∞ of all bounded holomorphic functions on Δ.

As is well known, $\mathcal{A}^0$ is a Dirichlet algebra whose set of maximal ideals and the Shilov boundary are respectively $\overline{\Delta}$ and the unit circle $\partial\Delta$.

As a consequence of a theorem by A. Bernard (see, e.g., [10, Corollary 2.4]), the closed unit ball of $\mathcal{A}^0$ is the closed convex hull of the set of all finite Blaschke products. By Proposition 1, if $A \in \mathcal{L}(\mathcal{A}^0)$ is an isometry for which

$$A(\mathfrak{U}(\mathcal{A}^0)) \subset \mathfrak{U}(\mathcal{A}^0), \tag{28}$$

then $A1 \in \mathfrak{U}(\mathcal{A}^0)$ and there is a surjective continuous map $\omega : \partial\Delta \to \partial\Delta$ such that

$$(Af)(z) = (A1)(z)\langle f, \omega(\delta_z)\rangle$$

for all $f \in \mathcal{A}^0$ and all $z \in \Delta$.

If $\iota \in \mathcal{A}^0$ is the "coordinate function", $\iota : \overline{\Delta} \ni z \mapsto z$, then

$$(A\iota)(z) = (A1)(z)\varpi(z),$$

where $\varpi(z) = \langle \iota, \omega(\delta_z)\rangle$. As a consequence, ϖ is holomorphic at all $z \in \Delta$ where $(A1)(z) \neq 0$. Hence it is holomorphic on Δ, that is, an inner function contained in $\mathcal{A}^0$.

If f is the restriction to $\overline{\Delta}$ of an analytic polynomial $\sum_{n=0}^{N} a_n \iota^n$ for some positive integer N and $a_n \in \mathbb{C}$, then

$$\begin{aligned}(Af)(z) &= \sum_{n=0}^{N} a_n (A\iota^n)(z) = (A1)(z) \sum_{n=0}^{N} a_n \iota^n(\varpi(z)) \\ &= (A1)(z) \sum_{n=0}^{N} a_n (\varpi(z))^n = (A1)(z) f(\varpi(z))\end{aligned} \tag{29}$$

for all $z \in \Delta$.

Since analytic polynomials are dense in $\mathcal{A}^0$, (29) holds for all $f \in \mathcal{A}^0$ and all $z \in \Delta$ and therefore for all $z \in \overline{\Delta}$. Furthermore, the fact that $Q = \partial\Delta$ entails that $|(A1)(z)| = 1$ for all $z \in \partial\Delta$.

Conversely, if A is a linear isometry of $\mathcal{A}^0$ expressed by (29) for all $f \in \mathcal{A}^0$ and all $z \in \overline{\Delta}$, where ϖ is an inner function contained in $\mathcal{A}^0$, then $|A1| = 1$ at all points of $\partial\Delta$, because if the set

$$V = \{\theta \in [0, 2\pi) : |(A1)(e^{i\theta})| < 1\}$$

is not empty, then choosing $f \in \mathcal{A}^0$, with $\|f\| = 1$, peaking only at a point of $\varpi(V)$ we have $\|Af\| < 1 = \|f\|$, contradicting the fact that A is an isometry.

In conclusion, the following theorem holds.

THEOREM 5. *For any isometry $A \in \mathcal{L}(\mathcal{A}^0)$ satisfying* (28), *$A1$ is an inner function contained in $\mathcal{A}^0$, and there is a non-constant inner function $\varpi \in \mathcal{A}^0$ such that*

$$Af = A1 \cdot f \circ \varpi \quad \forall f \in \mathcal{A}^0.$$

Conversely, if $A1 \in \mathcal{A}^0$ is an inner function and $\varpi \in \mathcal{A}^0$ is a non-constant inner function, then the operator A represented by the last equation is a linear isometry of $\mathcal{A}^0$ into itself.

If the isometry A is surjective, then A^{-1} is represented by

$$A^{-1}f = A^{-1}1 \cdot f \circ \varsigma \quad \forall f \in \mathcal{A}_0,$$

where $\varsigma \in \mathcal{A}^0$ is a non-constant inner function. For $f = 1$, the condition

$$A^{-1} \circ Af = A \circ A^{-1}f = f \tag{30}$$

is equivalent to

$$(A^{-1}1)(z)(A1)(\varsigma(z)) = (A1)(z)(A^{-1}1)(\varpi(z)) = 1 \quad \forall z \in \Delta.$$

The fact that $\|A1\| = \|A^{-1}1\| = 1$ and the maximum principle imply that $A1$ is constant: $A1 = c \in \partial\Delta$, and therefore $A^{-1}1 = \overline{c}$. Thus, by (30),

$$f(\varpi(\varsigma(z))) = f(\varsigma(\varpi(z))) = f(z) \quad \forall z \in \Delta,\ f \in \mathcal{A}^0,$$

i.e. ϖ is a holomorphic automorphism of Δ and $\varsigma = \varpi^{-1}$.

Thus, the following theorem holds:

THEOREM 6. *The operator $A \in \mathcal{L}(\mathcal{A}^0)$ is a bijective isometry of $\mathcal{A}^0$ into itself if, and only if, there exist a constant $c \in \partial\Delta$ and a Möbius transformation ϖ of Δ such that*

$$Af = c \cdot f \circ \varpi \quad \forall f \in \mathcal{A}^0. \tag{31}$$

We will now see how this result and the Wolff–Denjoy theorem [4] yield some information on the point spectrum of a surjective linear isometry A of $\mathcal{A}^0$.

Note first that if, and only if, $A1$ is constant ($A1 = c1$ for some $c \in \partial\Delta$), then 1 is an eigenvector of A (with eigenvalue c).

Let now A be a surjective isometry expressed by (31), where c is a unimodular constant and ϖ is a Möbius transformation with no fixed point in Δ.

By the Wolff–Denjoy theorem, for any $z \in \Delta$ the sequence $\{\varpi^{\circ n}(z) : n = 0, 1, 2, \ldots\}$ converges to a point $\zeta \in \partial\Delta$. Since, for any $f \in \mathcal{A}^0$ and any $z \in \Delta$,

$$\lim_{n\to\infty} f(\varpi^{\circ n}(z)) = f(\zeta),$$

by (31),

$$\lim_{n\to\infty} |(A^n f)(z)| = |f(\zeta)| \quad \forall f \in \mathcal{A}^0, z \in \Delta. \tag{32}$$

If κ is an eigenvalue of the isometry A, then $|\kappa| = 1$ and, by (31), if f is any one of its eigenvectors, then

$$|f(z)| = |\kappa^n f(z)| = |c^n f(\varpi^{\circ n}(z))| = |f(\varpi^{\circ n}(z))|$$

for all $z \in \Delta$ and $n = 1, 2, \ldots$. Thus, by (32),

$$|f(z)| = |f(\zeta)| \quad \forall z \in \Delta.$$

The maximum principle then implies that f is constant and $\kappa = c$, proving the following proposition.

PROPOSITION 5. *If $A \in \mathcal{L}(\mathcal{A}^0)$ is a surjective isometry and if the Möbius transformation ϖ has no fixed points in Δ, then the complex line spanned by the constant function* 1 *is the only eigenspace of A.*

Thus κ is the only eigenvalue of A, and its geometric multiplicity is one.

A direct computation yields

PROPOSITION 6. *If the linear isometry $A \in \mathcal{L}(\mathcal{A}^0)$ is surjective and if the Möbius transformation ϖ in* (31) *fixes one point in Δ, then there is $\theta \in [0, 2\pi)$ such that the point spectrum $p\sigma(A)$ of A is given by*

$$p\sigma(A) = \{ce^{in\theta} : n = 0, 1, \ldots\}.$$

Hence, if the Möbius transformation ϖ is not periodic, then the spectrum contains densely the point spectrum of A and coincides with $\partial\Delta$. If ϖ is periodic, then

$$p\sigma(A) = \{1, e^{2\pi i/N}, \ldots, e^{2\pi(N-1)i/N}\}$$

for some positive integer N.

III. Let H^∞ be the vector space of all bounded holomorphic functions on the open unit disc Δ of $\mathbb{C}$. Uniform norm and pointwise composition make H^∞ a uniform function algebra on the compact Hausdorff space $M = \partial H^\infty$.

For $\alpha \in \partial\Delta$ let Σ_α be the “fiber” over α ([11, p. 161], [10, p 190]):

$$\Sigma_\alpha = \{\chi \in \Sigma(H^\infty) : \langle \iota, \chi \rangle = \alpha\}, \qquad \partial H^\infty = \bigcup_{\alpha \in \partial\Delta} (\Sigma_\alpha \cap \partial H^\infty),$$

and let π be the continuous map of $\Sigma(H^\infty)$ onto $\overline{\Delta}$ mapping δ_z onto z for all $z \in \Delta$ and $\pi(\Sigma_\alpha) = \{\alpha\}$ for all $\alpha \in \partial\Delta$.

If $\mathcal{A}_\alpha$ is the restriction to Σ_α of the algebra $\widehat{H^\infty}$ of the Gelfand transforms of the elements of H^∞, then ([11, p. 192]) Σ_α and $\Sigma_\alpha \cap \partial H^\infty$ are respectively the maximal ideal space and the Shilov boundary of $\mathcal{A}_\alpha$.

For any $\theta \in \mathbb{R}$ the “rotation” $z \mapsto e^{i\theta} z$ maps bijectively Σ_α onto $\Sigma_{e^{i\theta}\alpha}$ and the Shilov boundary of $\mathcal{A}_\alpha$ onto the Shilov boundary of $\mathcal{A}_{e^{i\theta}\alpha}$.

By D. J. Newman’s characterization of the Shilov boundary of H^∞ (see, e.g., [11, Theorem, pp. 179–180]), $|\langle u, \chi \rangle| = 1$ for all inner functions u and all characters χ in ∂H^∞, i.e., u is unimodular.

Conversely, let $u \in H^\infty$ be unimodular. If u is not inner, there is a Borel set $C \subset \partial\Delta$ with positive Lebesgue measure such that

$$\lim_{r\uparrow 1} |u(r\zeta)| < 1 \quad \forall\zeta \in C.$$

By the Lusin theorem, there is a Borel set $E \subset C$ with positive Lebesgue measure such that u is continuously extendable to $\Delta \cup E$. As a consequence ([12, Theorem, p. 161]), u is constant on the fiber Σ_ζ, and therefore has modulus less than one on Σ_ζ, and in particular on $\Sigma_\zeta \cap \partial H^\infty$, for any $\zeta \in E$, contradicting the hypothesis that u is unimodular. This proves

LEMMA 6. *The set $\mathfrak{U}(H^\infty)$ consists of all inner functions in H^∞.*

Since H^∞ is a logmodular algebra [12], any character of H^∞ has a unique representing measure. Since moreover the closed unit ball of H^∞ is the closed convex hull of the set of all inner functions ([10, p. 196]), Theorem 2 yields the following result.

LEMMA 7. *For any isometry $A \in \mathcal{L}(H^\infty)$ of H^∞ into itself there exist a subset P of $\Sigma(H^\infty)$ and a continuous map $\omega : P \to \Sigma(H^\infty)$ such that*

$$\langle Af, \chi\rangle = \langle A1, \chi\rangle\langle f, \omega(\chi)\rangle \tag{33}$$

for all $f \in H^\infty$ and all $\chi \in P$. Moreover, $Q = P \cap \partial H^\infty$ is non-empty and closed in ∂H^∞, and $\omega(Q) = \partial H^\infty$.

Similar arguments to those developed for $\mathcal{A}^0$ yield

LEMMA 8. *For any isometry $A \in \mathcal{L}(H^\infty)$ such that*

$$A(\mathfrak{U}(H^\infty)) \subset \mathfrak{U}(H^\infty) \tag{34}$$

there is a continuous map $\omega : \Sigma(H^\infty) \to \Sigma(H^\infty)$ such that

$$\langle Af, \chi\rangle = \langle A1, \chi\rangle\langle f, \omega(\chi)\rangle \tag{35}$$

for all $f \in H^\infty$ and all $\chi \in \Sigma(H^\infty)$. Moreover, $A1$ is an inner function and ω maps ∂H^∞ onto itself.

By (35), for every $z \in \Delta$,

$$(Af)(z) = (A1)(z)\langle f, \omega(\delta_z)\rangle \tag{36}$$

for all $f \in H^\infty$ and all $z \in \Delta$. Choosing as f the coordinate function ι yields

$$\langle A\iota, \chi\rangle = \langle A1, \chi\rangle\langle \iota, \omega(\chi)\rangle$$

for all $\chi \in \Sigma(H^\infty)$. In particular if $\chi = \delta_z$, then

$$(A\iota)(z) = \langle A1, \delta_z\rangle\langle \iota, \omega(\delta_z)\rangle = (A1)(z)\varpi(z),$$

where $\varpi(z) = \langle \iota, \omega(\delta_z)\rangle$. As a consequence, ϖ is holomorphic at all $z \in \Delta$ where $(A1)(z) \neq 0$, and therefore it is holomorphic in Δ; so $\varpi \in H^\infty$. The fact that ω maps ∂H^∞ into itself shows that ϖ is an inner function.

If p is an analytic polynomial of degree N,

$$p(z) = \sum_{n=0}^{N} a_n z^n \qquad (a_n \in \mathbb{C}),$$

then

$$\begin{aligned}(Ap)(z) &= (A1)(z)\langle p, \omega(\delta_z)\rangle = (A1)(z)\sum_{n=0}^{N} a_n \langle \iota^n, \omega(\delta_z)\rangle \\ &= (A1)(z)\sum_{n=0}^{N} a_n \langle \iota, \omega(\delta_z)\rangle^n = (A1)(z)\sum_{n=0}^{N} a_n \varpi(z)^n \\ &= (A1)(z)p(\varpi(z))\end{aligned} \tag{37}$$

for all $z \in \Delta$.

If $f \in H^\infty$, then for $0 < r < 1$ the function $f_r : \Delta \ni z \mapsto f(rz)$ can be approximated pointwise by the sequence $\{c_N p_N : N = 0, 1, \ldots\}$, where p_N is the Taylor polynomial of degree N of f_r and

$$c_N = \frac{\|f_r\|}{\|p_N\|}.$$

Thus, by (37),

$$(Af_r)(z) = (A1)(z) f_r(\varpi(z)),$$

and because

$$\|Af_r - Af\| = \|A(f_r - f)\| = \|f_r - f\| \to 0$$

as $r \uparrow 1$, we have

$$\begin{aligned}(Af)(z) &= \lim_{r\uparrow 1}(Af_r)(z) = (A1)(z)\lim_{r\uparrow 1} f_r(\varpi(z)) \\ &= (A1)(z) f(\varpi(z)) \qquad \forall z \in \Delta.\end{aligned}$$

In conclusion, the following theorem holds.

THEOREM 7. *Any linear isometry A of H^∞ into itself satisfying* (34) *is a weighted composition operator represented by*

$$Af = A1 \cdot f \circ \varpi \qquad \forall f \in H^\infty, \tag{38}$$

where $A1$ is an inner function and ϖ is a non-constant inner function.

Conversely, if u and ϖ are inner functions and ϖ is not constant, then the weighted composition operator

$$H^\infty \ni f \mapsto u \cdot f \circ \varpi$$

is a linear isometry of H^∞.

A similar argument to that leading to Theorem 6 yields the following theorem ([16], [6] and [11, Corollary, p. 147]).

THEOREM 8. *The operator $A \in \mathcal{L}(H^\infty)$ is a surjective isometry if, and only if, there exist a constant $c \in \partial\Delta$ and a Möbius transformation ϖ of Δ such that* (31) *holds for all $f \in H^\infty$.*

The Gleason–Kahane–Żelazko theorem [22] then yields

COROLLARY 5. *An isometry $A \in \mathcal{L}(H^\infty)$ satisfying* (34) *is surjective if, and only if, it maps onto itself the set $H^{\infty-1}$ of all invertible elements of the Banach algebra H^∞.*

Let $A1 \in H^{\infty-1}$. Since any $f \in H^\infty$ is invertible in H^∞ if, and only if, $|f(z)| \geq a$ for some $a > 0$ and all $z \in \Delta$, Theorem 7 implies that if $f \in H^{\infty-1}$, then there is $b > 0$ such that

$$|(Af)(z)| = |(A1)(z)| \cdot |f(\varpi(z))| \geq b$$

for all $z \in \Delta$. That implies

PROPOSITION 7. *The isometry $A \in \mathcal{L}(H^\infty)$ satisfying* (34) *maps $H^{\infty-1}$ into itself if, and only if, $A1 \in H^{\infty-1}$.*

5. PERIODIC LINEAR ISOMETRIES OF H^∞

Let $A \in \mathcal{L}(H^\infty)$ be an isometry satisfying (34) for which there exists an integer $n > 1$ and some $z \in \Delta$ such that

$$(A^n f)(z) = f(z) \quad \forall f \in H^\infty. \tag{39}$$

By Theorem 7, A is represented by

$$A : f \mapsto A1 \cdot f \circ \varpi,$$

where $A1 \in \mathfrak{U}(H^\infty)$ and $\varpi : \Delta \to \Delta$ is a non-constant inner function.

Equations (18) and (19) imply that

$$(A^n 1)(z) = 1, \tag{40}$$

and $\varpi^{\circ n}(z) = z$. Hence, either $\varpi(z) = z$ or

$$\mathrm{Card}\{z, \varpi(z), \dots, \varpi^{\circ(n-1)}(z)\} > 1,$$

in which case (see [21]) ϖ is a holomorphic automorphism of Δ fixing a point $\alpha \in \Delta$. Thus, if there is an infinite set $E \subset \Delta$ with a cluster point in Δ such that (39) holds whenever $z \in E$, then ϖ is a holomorphic periodic automorphism of Δ fixing a point $\alpha \in \Delta$, whose period is a divisor of n.

Since

$$\varpi(\alpha) = \alpha, \tag{41}$$

we have

$$1 = (A^n 1)(\alpha) = ((A1)(\alpha))^n.$$

Thus, the maximum principle implies that the function $A1$ is constant: $A1 = \kappa$ for some $\kappa \in \partial\Delta$, which, by (39), is $\kappa = e^{2\pi i/n}$. Hence A is represented by

$$A(f) = e^{2\pi i/n} \cdot f \circ \varpi \quad \forall f \in H^\infty, \tag{42}$$

and the following theorem holds:

THEOREM 9. *If the linear isometry $A \in \mathcal{L}(H^\infty)$ satisfying* (34) *is such that* (39) *holds for some integer $n > 1$, any $f \in H^\infty$, and all $z \in E$, where $E \subset \Delta$ is an infinite set with a cluster point in Δ, then A is periodic with period n, and is represented by* (42)*, where ϖ is a periodic Möbius transformation of Δ.*

Conversely, for any Möbius transformation ϖ of Δ such that $\varpi^{\circ n} = \mathrm{id}$, (42) *defines an n-periodic surjective isometry of H^∞.*

We will now show that if the iterates of the linear isometry A of H^∞ satisfying (34) converge to the identity for the weak operator topology, then A itself is the identity.

The hypothesis implies that

$$\lim_{n\to\infty} \langle A^n f, \delta_z\rangle = \lim_{n\to\infty} (A^n f)(z) = f(z) \quad \forall f \in H^\infty,\ z \in \Delta. \tag{43}$$

If $|(A^m 1)(z)| \le a$ for some $z \in \Delta$, $a \in (0,1)$ and $m \ge 1$, then, by (18), $|(A^n 1)(z)| \le a < 1$ when $n \gg 1$, contradicting (43). Thus $|(A1)(z)| = 1$ for all $z \in \Delta$, and therefore, by the maximum principle, there is some constant $c \in \partial\Delta$ such that

$$(A1)(z) = c \quad \forall z \in \Delta.$$

Since, by (43), $c^n \to 1$ as $n \to \infty$, it follows that $c = 1$ and (38) yields

$$Af = f \circ \varpi. \tag{44}$$

Thus, again by (43),

$$\lim_{n\to\infty} \varpi^{\circ n}(z) = \lim_{n\to\infty} \iota(\varpi^{\circ n}(z)) = z$$

for all $z \in \Delta$. The Wolff–Denjoy theorem [4] then implies that

$$\varpi(z) = z \quad \forall z \in \Delta,$$

proving the following theorem.

THEOREM 10. *The identity operator is the only linear isometry A of H^∞ into itself which satisfies* (34)*, and whose iterates converge to the identity for the weak operator topology.*

6. CONTINUOUS SEMIGROUPS OF LINEAR ISOMETRIES OF H^∞

We will now investigate strongly continuous semigroups of linear isometries of H^∞.

Let $T : \mathbb{R}_+ \to \mathcal{L}(H^\infty)$ be a strongly continuous semigroup of linear isometries of H^∞ into itself. According to a result by H. P. Lotz [14], [15], T, as any strongly continuous semigroup of linear operators acting on H^∞, is uniformly continuous. Hence, it is the restriction to $\mathbb{R}_+$ of a uniformly continuous group on $\mathbb{R}$, with values in $\mathcal{L}(H^\infty)$, which will be denoted by the same symbol T.

Since for any $t < 0$ and any $f \in H^\infty$,

$$\|f\| = \|T(-t)T(t)f\| = \|T(t)f\|,$$

$T : \mathbb{R} \to H^\infty$ is a group of surjective isometries. According to Theorem 8, there are a function $c : \mathbb{R} \to \partial\Delta$ and a family $\{\rho_t : t \in \mathbb{R}\}$ of holomorphic automorphisms of Δ such that

$$T(t)f = c(t) f \circ \rho_t \quad \forall t \in \mathbb{R},\ f \in H^\infty, \tag{45}$$

As $c(t) = T(t)1$, c is a continuous homomorphism of $\mathbb{R}$ into $\partial\Delta$. Therefore there is $\delta \in \mathbb{R}$ such that

$$c(t) = e^{i\delta t} \quad \forall t \in \mathbb{R}. \tag{46}$$

Furthermore

$$\rho_{s+t} = \rho_s \circ \rho_t \quad \forall s, t \in \mathbb{R}$$

and the map $t \mapsto \rho_t(z)$ is continuous for every $z \in \Delta$; thus, ρ is a continuous flow of holomorphic automorphisms of Δ (which is called the *conformal flow* of T).

Hence the following theorem holds:

THEOREM 11. *Any strongly continuous semigroup of linear isometries of H^∞ is the restriction to $\mathbb{R}_+$ of a uniformly continuous group of surjective isometries.*

The continuous flow ρ is defined by a one-parameter subgroup $t \mapsto \exp t\Theta$ of $SU(1, 1)$ defined by a 2×2 matrix

$$\Theta = \begin{pmatrix} i\gamma & c \\ \bar{c} & -i\gamma \end{pmatrix}$$

with $\gamma \in \mathbb{R}$ and $c \in \mathbb{C}$. As was shown in [23], if $\gamma^2 - |c|^2$ is positive, negative or zero, then $\rho_t(z)$ is expressed respectively by:

$$\rho_t(z) = \frac{\big(\cos(rt) + i\frac{\gamma}{r}\sin(rt)\big)z + \frac{c}{r}\sin(rt)}{\frac{\bar{c}}{r}\sin(rt)z + \cos(rt) - i\frac{\gamma}{r}\sin(rt)}$$

with $r = \sqrt{\gamma^2 - |c|^2}$;

$$\rho_t(z) = \frac{\big(\cosh(st) + i\frac{\gamma}{s}\sinh(st)\big)z + \frac{c}{s}\sinh(st)}{\frac{\bar{c}}{s}\sinh(st)z + \cosh(st) - i\frac{\gamma}{s}\sinh(st)}$$

with $s = \sqrt{|c|^2 - \gamma^2}$;

$$\rho_t(z) = \frac{(1 + it\gamma)z + tc}{t\bar{c}z + 1 - it\gamma}.$$

In the first case, the flow ρ is elliptic, i.e. fixes one point in Δ and is periodic with period $2\pi/r$. In the second and third cases, the flow ρ is respectively hyperbolic and parabolic and has no fixed point in Δ.

In the elliptic case, the periodicity of ρ and (46) show that the group T is almost periodic.

It was shown in [23] that if the flow ϕ is not elliptic, then there is some $k > 0$ such that

$$\|T(t)\iota\| > 1/2 \quad \forall t > k.$$

In conclusion, the following theorem holds (see [23]).

THEOREM 12. *The group T is almost periodic if, and only if, its conformal flow ρ is elliptic.*

Furthermore, the group T is periodic if, and only if, δ and the period of ρ are linearly dependent over $\mathbb{Q}$.

Let now $T : \mathbb{R} \to \mathcal{L}(\mathcal{A}^0)$ be a strongly continuous group of (surjective) linear isometries of $\mathcal{A}^0$. Arguing as in the case of H^∞ one shows that T is represented by

$$T(t)f = e^{i\delta t} \cdot f \circ \rho_t$$

for all $t \in \mathbb{R}$ and all $f \in \mathcal{A}^0$, where $\delta \in \mathbb{R}$ and $\rho : t \mapsto \rho_t$ is a continuous flow of Möbius transformations of Δ.

As before we see that if the continuous flow ρ is elliptic, then the group T is almost periodic (periodic when δ and the period of ρ are linearly dependent over $\mathbb{Q}$).

On the other hand, if the flow ρ is hyperbolic or parabolic, then by Proposition 5 the only eigenspace of $T(t)$ is the complex line $e^{i\delta t}\mathbb{C}$. Theorem 2 of [1] then shows that the group T is not almost periodic, thus extending Theorem 12 to the strongly continuous groups of linear isometries acting on $\mathcal{A}^0$.

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