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Partial differential equations. — *On weak solutions for p-Laplacian equations with weights*, by PATRIZIA PUCCI and RAFFAELLA SERVADEI, communicated on 9 February 2007.

ABSTRACT. — We summarize the results obtained in the forthcoming papers [32, 33], in which we prove theorems on existence and non-existence of weak solutions of quasilinear singular elliptic equations with weights. We also establish regularity and qualitative properties of the solutions.

KEY WORDS: p-Laplacian operator; quasilinear equations; weak solutions.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 35J15; Secondary 35J70.

In this note we present a survey of the main results established in [32, 33]. Let us first consider p-Laplacian equations in the entire $\mathbb{R}^n$ of the type

$$\Delta_p u = g(x, u), \tag{1}$$

where $\Delta_p = \operatorname{div}(|Du|^{p-2}Du)$, $Du = (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$, $1 < p < n$, and $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

A function $u \in H^{1,p}_{\text{loc}}(\mathbb{R}^n)$, or $u \in D^{1,p}_{\text{loc}}(\mathbb{R}^n)$, is said to be a *weak solution* of (1) if

$$\int_{\mathbb{R}^n} |Du(x)|^{p-2}\langle Du(x), D\varphi(x)\rangle\, dx + \int_{\mathbb{R}^n} g(x, u(x))\varphi(x)\, dx = 0$$

for any $\varphi \in H^{1,p}(\mathbb{R}^n)$ compactly supported in $\mathbb{R}^n$. A *ground state* of (1) is a non-trivial non-negative weak solution of (1) which tends to zero as $|x| \to \infty$.

Finally, a *fast decay solution* of (1) is a non-trivial weak solution u of (1) such that

$$\lim_{|x|\to\infty} |x|^{(n-p)/(p-1)} u(x) \quad \text{exists and is finite.}$$

In [33] we give several qualitative and regularity properties of weak solutions u of (1), as well as of weak solutions of more general quasilinear elliptic equations. In particular, by using the Moser iteration scheme (see [21, 36]), the following three regularity results for weak solutions of (1) are proved.

THEOREM 1. *Let $u \in H^{1,p}_{\text{loc}}(\mathbb{R}^n)$ be a weak solution of* (1).

(i) *If $|g(x, u)| \le a(x)(1+|u|^{p-1})$ for a.a. $x \in \mathbb{R}^n$ and for all $u \in \mathbb{R}$, with $a \in L^{n/p}_{\text{loc}}(\mathbb{R}^n)$, then $u \in L^m_{\text{loc}}(\mathbb{R}^n)$ for any $m \in [1, \infty)$.*

(ii) *If* $|g(x,u)| \le C(|u|^{p-1} + |u|^{p^*-1})$ *for a.a.* $x \in \mathbb{R}^n$ *and for all* $u \in \mathbb{R}$, *with* $C > 0$, *then* $u \in L^\infty_{\rm loc}(\mathbb{R}^n)$.

(iii) *If* u *is also of class* $C(\mathbb{R}^n \setminus \{0\})$ *and the assumption of* (i) *holds with* $a \in L^{n/p(1-\varepsilon)}_{\rm loc}(\mathbb{R}^n)$ *for some* $\varepsilon \in (0,1]$,[1] *then for any bounded domain* Ω *of* $\mathbb{R}^n$ *containing* 0,

$$|u(x)| \le C^{n/p\varepsilon}(\|u\|_{L^p(\Omega)} + \|a\|^{1/(p-1)}_{L^{n/p(1-\varepsilon)}(\Omega)}|\Omega|) + \sup_{\partial\Omega}|u(x)| \quad \text{in } \Omega \setminus \{0\},$$

where $C = K[n/(n-p)]^{(n-p)/p}$ *and* K *is a suitable positive constant depending on* ε. *In particular* $u \in L^\infty_{\rm loc}(\mathbb{R}^n)$.

(iv) *If* $1 < p \le 2$ *and* $g(\cdot, u(\cdot)) \in L^{p'}_{\rm loc}(\mathbb{R}^n \setminus \{0\})$,[2] *then* $u \in H^{2,p}_{\rm loc}(\mathbb{R}^n \setminus \{0\})$; *while if* $1 < p \le 2$ *and* $g(\cdot, u(\cdot)) \in L^{p'}_{\rm loc}(\mathbb{R}^n)$, *then* $u \in H^{2,p}_{\rm loc}(\mathbb{R}^n)$.

In [31, Lemma 6.2.1] Pucci and Serrin prove Theorem 1(iii) for general divergence elliptic inequalities in bounded domains, with some control at the boundary. The inequality given in Theorem 1(iii) shows that any weak solution $u \in H^{1,p}_{\rm loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n \setminus \{0\})$ of (1) is bounded in any bounded domain Ω of $\mathbb{R}^n$ containing 0, that is, a sort of maximum principle holds for (1).

Clearly, Theorem 1(iv) also holds when $\mathbb{R}^n \setminus \{0\}$ or $\mathbb{R}^n$ are replaced by any domain $\Omega \subseteq \mathbb{R}^n$. Here the result is stated in $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^n$ for the later main applications (see Theorems 2 and 4).

A result similar to Theorem 1(iv) is first established in [35], where Simon proves in particular that every solution $u \in H^{1,p}(\mathbb{R}^n)$ of (1) is of class $H^{2,p}(\mathbb{R}^n)$, provided that $1 < p \le 2$ and $g(x,u) = d(x)|u|^{p-2}u + \phi(x)$ with $d \in L^\infty(\mathbb{R}^n)$, $\operatorname{ess\,inf}_{\mathbb{R}^n} d(x) > 0$, and $\phi \in L^{p'}(\mathbb{R}^n)$. For theorems of this type in bounded domains Ω of $\mathbb{R}^n$ see [12, 13]. In particular, in [12] de Thélin proves that if $1 < p \le 2$, g is independent of u and $g \in L^{p'}(\Omega)$, then any solution $u \in H^{1,p}(\Omega)$ of (1) is of class $H^{2,p}_{\rm loc}(\Omega)$. Then he applies this regularity result to a special case of (1) (see [13, Theorem 1]).

Our proof of Theorem 1(iv) is based on an inequality proved by Simon in [35] and on an argument taken from [12].

Next, we give some regularity results and qualitative properties for radial weak solutions of (1) when

$$(G) \qquad g = g(r,u),\ r = |x|,\ \text{is continuous in } \mathbb{R}^+ \times \mathbb{R}.$$

These results are particular consequences of the main theorems obtained in [33] for more general quasilinear elliptic equations.

PROPOSITION 1. *Let* $u \in D^{1,p}_{\rm rad}(\mathbb{R}^n)$ *be a radial weak solution of* (1). *Assume* (G) *and that* $g(\cdot, u(\cdot)) \in L^1_{\rm loc}(\mathbb{R}^n)$. *Then*

(i) $|Du(x)| = O(|x|^{-(n-1)/(p-1)})$ *as* $|x| \to 0$;

[1] The space $L^{n/p(1-\varepsilon)}_{\rm loc}(\mathbb{R}^n)$, $\varepsilon \in (0,1]$, reduces to the usual space $L^\infty_{\rm loc}(\mathbb{R}^n)$ when $\varepsilon = 1$.

[2] We denote by p' the Hölder conjugate of p, that is, $1/p + 1/p' = 1$.

(ii) *$u \in C^1(\mathbb{R}^n \setminus \{0\})$ and $|Du|^{p-2}Du \in C^1(\mathbb{R}^n \setminus \{0\})$;*
(iii) *u solves* (1) *pointwise in $\mathbb{R}^n \setminus \{0\}$;*
(iv) *$u \in C^{1,\theta}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ for some $\theta \in (0,1)$;*
(v) *moreover, if $u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ and $g(\cdot,0) = 0$ in $\mathbb{R}^+$, $g > 0$ in $(R,\infty) \times (0,\delta)$ and $0 < u < \delta$ in (R,∞) for some $R, \delta > 0$, then $u' < 0$ in (R,∞) and $u'(r) \to 0$ as $r \to \infty$, where $' = d/dr$.*

As a particular case of (1) we consider the following quasilinear singular elliptic equation:

$$\begin{gathered}\Delta_p u - \lambda |u|^{p-2}u + \mu |x|^{-\alpha}|u|^{q-2}u + h(|x|)f(u) = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\},\\ \lambda, \mu \in \mathbb{R}, \quad 1 < p < n,\end{gathered} \tag{2}$$

where either $0 \le \alpha < p \le q < p^*_\alpha = p(n-\alpha)/(n-p)$ or $\alpha = q = p\ (= p^*_\alpha)$, and $h : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Special cases of (1) were recently widely studied in the literature. For the existence and non-existence, as well as qualitative properties, of non-trivial non-negative solutions for elliptic equations with singular coefficients in bounded domains see [2, 6, 8, 17, 22, 34] for $p = 2$ and [15, 18, 23, 40] for general $p > 1$, and in unbounded domains cf. [25, 27, 34, 38] for $p = 2$ and [1, 7, 10, 19, 24, 30, 37] for general $p > 1$.

Homogeneous Dirichlet problems associated to equations of type (2) are studied by Ekeland and Ghoussoub in [15] and by Ghoussoub and Yuan in [23] in smooth bounded domains of $\mathbb{R}^n$ containing zero, when $\lambda = 0$, $h \equiv 1$, $f(u) = c|u|^{s-2}u$ with $c > 0$ and $p \le s < p^* = pn/(n-p)$. They give existence and multiplicity results for non-trivial non-negative solutions by using variational methods and the Hardy–Sobolev inequality (see, e.g., [2, 8, 18, 28]), when either $0 \le \alpha < p \le q < p^*_\alpha$ or $\alpha = q = p\ (= p^*_\alpha)$.

In [32] we extend the existence results of [15, 23] to the entire $\mathbb{R}^n$, and to the case in which $\lambda > 0$, h is a general non-trivial weight such that $h(|x|) = o(|x|^{-\beta})$ as $|x| \to 0$, with $\beta \in [0,p)$, bounded at infinity, while f is possibly different from a pure power. In particular, we prove the existence of a radial ground state u of (2) by the celebrated Mountain Pass Theorem of Ambrosetti and Rabinowitz [3] and the Hardy–Sobolev inequality. More precisely, on f we assume the canonical conditions required in [3], that is

(F1) f is continuous in $\mathbb{R}^+_0$;
(F2) there exist $a \ge 0$, $b > 0$ and $p < s$ such that $|f(u)| \le au^{p-1} + bu^{s-1}$ in $\mathbb{R}^+_0$;
(F3) $\lim_{u \to 0^+} u^{-p}F(u) = 0$, where $F(u) = \int_0^u f(v)\,dv$ for all $u \in \mathbb{R}^+_0$;
(F4) $0 < sF(u) \le uf(u)$ for all $u \in \mathbb{R}^+$;

while on the weight function h we require the following assumption:

(H1) $h = h(|x|) \in \mathcal{W}_\beta$ for some $\beta \in [0,p)$, where $\mathcal{W}_\beta$ is the function space

$$\mathcal{W}_\beta = \{w \in L^\infty(\Omega_R) \text{ for any } R > 0 : w \not\equiv 0,\ w \ge 0 \text{ a.e. in } \mathbb{R}^n,\ \lim_{|x|\to 0} |x|^\beta w(x) = 0\},$$

with $\Omega_R = \mathbb{R}^n \setminus B_R$ and $B_R = \{x \in \mathbb{R}^n : |x| \le R\}$, $R > 0$.

THEOREM 2. *Assume* (F1)–(F4) *and* (H1). *Consider* (2) *with*

$$(3)\qquad \begin{aligned} &0 \le \beta < p < s < p^*_\beta = p(n-\beta)/(n-p),\ \lambda > 0,\ 0 \le \mu p C^q_{\mathrm{HS}} < q \min\{1,\lambda\},\\ &\textit{and either}\quad 0 \le \alpha < p \le q < p^*_\alpha \quad \textit{or} \quad \alpha = q = p\ (= p^*_\alpha), \end{aligned}$$

where $C_{\mathrm{HS}} = C_{\mathrm{HS}}(n, p, \alpha, q)$ *is the constant of the embedding* $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L^q_\alpha(\mathbb{R}^n)$.[3] *Then* (2), (3) *admits a radial ground state* $u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$. *Moreover,*

(i) $u \in C^{1,\theta}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ *for some* $\theta \in (0,1)$*;*
(ii) $|Du|^{p-2}Du \in C^1(\mathbb{R}^n \setminus \{0\})$*;*
(iii) u *is positive, solves* (2), (3) *pointwise in* $\mathbb{R}^n \setminus \{0\}$, $\langle x, Du(x)\rangle < 0$ *for all* x *with* $|x|$ *sufficiently large and* $|Du(x)| \to 0$ *as* $|x| \to \infty$*;*
(iv) u *is a fast decay solution of* (2), (3)*;*
(v) *if* $0 \le \alpha < p \le q < p^*_\alpha$, *then* $u \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$ *for any* $m \in [1,\infty)$*;*
(vi) *if* $0 \le \max\{\alpha,\beta\} < p$, *then* $u \in L^\infty(\mathbb{R}^n)$*;*
(vii) *if* $1 < p \le 2$, *then* $u \in H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$*; if furthermore* $0 \le \max\{\alpha, \beta\} \le p-1$, *then* $u \in H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

The regularity properties of the solution u constructed in Theorem 2 are a consequence of Theorem 1 and Proposition 1.

Since in the degenerate case $p > 2$ the uniform ellipticity of Δ_p is lost at zeros of Du, the best we can expect with respect to the regularity of solutions, even in the standard non-weighted case of (2), is to have solutions of class $C^{1,\theta}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ (see [14]). Of course, for (2) much less could be expected and regularity was an open problem. A partial result is given in the following proposition for radial ground states of (2), provided they are assumed a priori *bounded*.

PROPOSITION 2. *Assume* (F1)–(F4) *and* (H1). *Consider* (2) *with* $\lambda > 0$, $\mu \ge 0$, $q > 1$ *and*

$$0 \le \max\{\alpha,\beta\} < p.$$

Let $u \in C^1(\mathbb{R}^n \setminus \{0\})$, *with* $|Du|^{p-2}Du \in C^1(\mathbb{R}^n \setminus \{0\})$, *be a bounded radial ground state which solves* (2) *also pointwise in* $\mathbb{R}^n \setminus \{0\}$. *Then* u *is positive in* $\mathbb{R}^n \setminus \{0\}$. *Moreover*

(i) *if* $\alpha, \beta \in [0,1)$, *then* $u \in C^1(\mathbb{R}^n)$, *with* $u(0) > 0$ *and* $Du(0) = \mathbf{0}$*;*
(ii) *if* $\alpha = \beta = 1$, *then* $u \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^n)$*;*
(iii) *if* $1 < \max\{\alpha, \beta\} < p$, *then* $u \in C^{0,\theta}_{\mathrm{loc}}(\mathbb{R}^n)$, *with* $\theta = (p - \max\{\alpha, \beta\})/(p-1)$.

Therefore u *is continuous at* $x = 0$ *in all the cases* (i)–(iii).

[3] For $1 \le q < \infty$ and $\alpha \in \mathbb{R}$ consider the weighted Lebesgue space

$$L^q_\alpha(\mathbb{R}^n) = L^q(\mathbb{R}^n, |x|^{-\alpha}\,dx) = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |u(x)|^q |x|^{-\alpha}\,dx < \infty \right\}$$

endowed with the norm $\|u\|_{q,\alpha} = (\int_{\mathbb{R}^n} |u(x)|^q |x|^{-\alpha}\,dx)^{1/q}$. Embeddings of $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ into $L^q_\alpha(\mathbb{R}^n)$ are proved in Section 2 of [32].

In particular this proposition applies to the bounded radial ground state constructed in Theorem 2 when $0 \le \max\{\alpha, \beta\} < p$.

We point out that Proposition 2 does not cover the case $\alpha = q = p$ in (2), which remains open.

In [32] we also give some non-existence results for (2) by a Pohozaev–Pucci–Serrin type identity when condition (F1) holds, $h : \mathbb{R}^+ \to \mathbb{R}$ is continuous and either

$$\alpha = q = p \ (= p_\alpha^*) \quad \text{or}$$
$$\alpha \in [0, p) \quad \text{if } q \in [p, p_\alpha^*],\ p_\alpha^* = p(n-\alpha)/(n-p) > p.$$

More precisely, we establish the following result (see Lemma 4.6 of [32]):

PROPOSITION 3. *Let $u \in H^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfy (2) a.e. in $\mathbb{R}^n$ and assume $F \circ u \in L^1_h(\mathbb{R}^n)$.[4] Then*

$$\lambda pq \|u\|_p^p + q \int_{\mathbb{R}^n} [(n-p)u(x)f(u(x)) - npF(u(x))]h(|x|)\,dx - pq \int_{\mathbb{R}^n} F(u(x))|x|h'(|x|)\,dx = \mu(n-p)(p_\alpha^* - q)\|u\|_{q,\alpha}^q$$

for any $\lambda, \mu \in \mathbb{R}$.

Analogously, if $u \in D^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfies

$$\Delta_p u + h(|x|)f(u) = 0 \quad \text{a.e. in } \mathbb{R}^n$$

and $F \circ u \in L^1_h(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} [(n-p)u(x)f(u(x)) - npF(u(x))]h(|x|)\,dx = p \int_{\mathbb{R}^n} F(u(x))|x|h'(|x|)\,dx.$$

The non-existence results for (2) proved in [32, Section 4] are consequences of the Pohozaev–Pucci–Serrin inequality given in Proposition 3.

Now, we consider (2) when $\lambda = \mu = 0$ and the weight function h is a power, that is, we treat the equation

$$\Delta_p u + |x|^{-\beta} f(u) = 0 \quad \text{in } \Omega, \tag{4}$$

where $\beta < p$, $f : \mathbb{R}_0^+ \to \mathbb{R}$ is continuous and $\Omega = \mathbb{R}^n$ if $\beta \le 0$, while $\Omega = \mathbb{R}^n \setminus \{0\}$ if $\beta \in (0, p)$.

If f is a *pure power*, that is, (4) reduces to

$$\Delta_p u + \gamma |x|^{-\beta} |u|^{s-2} u = 0 \quad \text{in } \Omega, \qquad \beta < p,\ s > 1, \tag{5}$$

with $\gamma \in \mathbb{R}$, then the following result holds (see Corollary 4.13 and Theorem 5.1 of [32]):

[4] As before, for $1 \le q < \infty$ and $h \in \mathcal{W}_\beta$, $\beta \in \mathbb{R}$, consider the weighted Lebesgue space

$$L_h^q(\mathbb{R}^n) = L^q(\mathbb{R}^n, h(|x|)\,dx) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |u(x)|^q h(|x|)\,dx < \infty \right\}$$

endowed with the norm $\|u\|_{q,h} = (\int_{\mathbb{R}^n} |u(x)|^q h(|x|)\,dx)^{1/q}$.

THEOREM 3. *If either $\gamma \le 0$, or $\gamma > 0$ and $s \neq p^*_\beta$, then* (5) *admits in $D^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ only the trivial solution $u \equiv 0$.*

When $\gamma > 0$ and $s = p^*_\beta$, the function

$$u(x) = c(1 + |x|^{(p-\beta)/(p-1)})^{-(n-p)/(p-\beta)},$$

where the constant c is given by

$$c = \left[\frac{n-\beta}{\gamma} \left(\frac{n-p}{p-1} \right)^{p-1} \right]^{(n-p)/p(p-\beta)},$$

is a positive radial fast decay ground state of (5) *of class*

$$D^{1,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\}),$$

which solves (5) *pointwise in $\mathbb{R}^n \setminus \{0\}$.*

Finally, $u \in H^{1,p}(\mathbb{R}^n)$ if and only if $n > p^2$.

If $\beta = 0$ and $s = p^*$, then (5) reduces to the classical critical equation

$$\Delta_p u + \gamma |u|^{p^*-2} u = 0 \quad \text{in } \mathbb{R}^n. \tag{6}$$

The existence of a non-trivial solution for (6) was considered by many authors which have also given an explicit form of such solution (for the case $p = 2$ see, for instance, [26, 36, 39] and references therein). When $\gamma > 0$ and $\beta \in [0, p)$ problem (5) was studied in several papers for general p (see, for instance, [15, 23]) and for $p = 2$ (see, for example, [6, 25]).

When $\gamma = 1$ and $\beta \in [0, p)$, the explicit solution u in Theorem 3 was first given in Theorem 3.1 of [23] by a different argument and approach.

The regularity at $x = 0$ of the solution u constructed in Theorem 3 can be expressed in terms of the parameters p and β and is summarized in the following table:

	$1 < p \le 2$	$p > 2$
$\beta < 2 - p$	$C^2(\mathbb{R}^n)$	
$\beta = 2 - p$	$C^2(\mathbb{R}^n)$	$C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$
$2 - p < \beta < 1$	$C^2(\mathbb{R}^n)$	$C^{1,(1-\beta)/(p-1)}_{\mathrm{loc}}(\mathbb{R}^n)$
$1 \le \beta < p$	$C^{0,(p-\beta)/(p-1)}_{\mathrm{loc}}(\mathbb{R}^n)$	

If f is negative in all $\mathbb{R}^+_0$, then equation (4) admits only the trivial solution $u \equiv 0$, as a consequence of Lemma 4.2 of [32].

Now we give a result relating to the existence of positive radial ground states of (4) by means of the constrained minimization method (see [4, 11]), when f is not modelled by a pure power, but actually f is negative near the origin and positive at infinity. This is usually called the *normal case* (see [29]).

After the papers [4, 5] relating to elliptic problems for the Laplace operator, equations with no weights, that is, when $\beta = 0$ in (4), involving the p-Laplacian operator in $\mathbb{R}^n$ were treated widely in the literature when f is negative near the origin and positive at infinity; see e.g. [9, 16, 20] for the non-weighted case and [7] for general weighted equations.

Let us introduce the following conditions on the non-linear term f:

(F5) there exist $a > 0$ and $q > 1$ such that $\lim_{u \to 0^+} u^{1-q} f(u) = -a$;
(F6) there exists $\overline{u} > 0$ such that $f(\overline{u}) \geq 0$ and $F(\overline{u}) > 0$
(F7) $\lim_{u \to \infty} u^{1-p^*_\beta} f(u) = 0$, $p^*_\beta = p(n-\beta)/(n-p)$, $1 < p < n$.

THEOREM 4. *Assume* (F1) *and* (F5)–(F7). *Then equation* (4) *with* $\beta < p$ *admits a radial ground state* $u \in D^{1,p}_{\text{rad}}(\mathbb{R}^n) \cap L^q_\beta(\mathbb{R}^n)$ *bounded above by* $\overline{u}$. *Moreover,*

(i) $u \in C^{1,\theta}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ *for some* $\theta \in (0, 1)$*;*
(ii) $|Du|^{p-2} Du \in C^1(\mathbb{R}^n \setminus \{0\})$ *and* u *solves* (4) *pointwise in* $\mathbb{R}^n \setminus \{0\}$*;*
(iii) $|Du(x)| \to 0$ *as* $|x| \to \infty$ *and* $|Du(x)| = O(|x|^{-(n-1)/(p-1)})$ *as* $|x| \to 0$*;*
(iv) u *is continuous at* $x = 0$, $\langle x, Du(x)\rangle \leq 0$ *in* $\mathbb{R}^n \setminus \{0\}$, *and* $\|u\|_\infty = u(0) \in (u_0, \overline{u}]$, *where* $u_0 = \inf\{v > 0 : F(v) > 0\}$*;*
(v) *if* $1 < p \leq 2$, *then* $u \in H^{2,p}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$*; if furthermore* $\beta < n/p'$, *then* $u \in H^{2,p}_{\text{loc}}(\mathbb{R}^n)$.

If $1 < q < p$, *then* u *is compactly supported in* $\mathbb{R}^n$, *and of course is a fast decay solution of* (4) *of class* $H^{1,p}(\mathbb{R}^n)$. *Furthermore,* u *has the regularity in* $\mathbb{R}^n$ *as described in the following table:*

	$1 < p \leq 2$	$p > 2$
$\beta < 1$	$C^2(\mathbb{R}^n)$	$C^1(\mathbb{R}^n)$
$1 \leq \beta < p$	$C^{0,(p-\beta)/(p-1)}_{\text{loc}}(\mathbb{R}^n)$	

If $q \geq p$, *then* u *is positive in* $\mathbb{R}^n$, $u \in C^2(\mathbb{R}^n \setminus \{0\})$ *and* u *has the regularity in* $\mathbb{R}^n$ *as described in the following table:*

	$1 < p \leq 2$	$p > 2$
$\beta < 2 - p$	$C^2(\mathbb{R}^n)$	
$\beta = 2 - p$	$C^2(\mathbb{R}^n)$	$C^{1,1}_{\text{loc}}(\mathbb{R}^n)$
$2 - p < \beta < 1$	$C^2(\mathbb{R}^n)$	$C^{1,(1-\beta)/(p-1)}_{\text{loc}}(\mathbb{R}^n)$
$1 \leq \beta < p$	$C^{0,(p-\beta)/(p-1)}_{\text{loc}}(\mathbb{R}^n)$	

Moreover, u *is a fast decay solution of* (4) *and in particular* $r^{(n-p)/(p-1)}u$ *is decreasing in* $[R, \infty)$ *for* R *sufficiently large and approaches a limit* $\ell \geq 0$ *as* $r \to \infty$. *If* $\ell > 0$ *then* $u \in H^{1,p}(\mathbb{R}^n)$ *if and only if* $n > p^2$*; while if* $\ell = 0$ *and* $n > p^2$ *then* $u \in H^{1,p}(\mathbb{R}^n)$.

Condition (F6) is necessary for the existence of non-trivial weak solutions u for (4) of class $D^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, with $F \circ u \in L^1_\beta(\mathbb{R}^n)$, $\beta < p$. Indeed, by Lemma 4.2 of [32], the following identity holds for u:

$$(n-p)\|Du\|_p^p = p(n-\beta)\|F \circ u\|_{1,\beta}.$$

Hence, if (F6) does not hold, that is, $F(u) \leq 0$ for any $u \in \mathbb{R}$, then (4) has only the trivial solution, since $1 < p < n$.

Theorem 4 extends the results given in [20] for the non-weighted version of (4) when local Lipschitz continuity of f is assumed. Theorem 4 extends to the weighted case also the existence results obtained by Berestycki and Lions in [4] when $p = 2$, and by Citti in [9] for general $p > 1$ (see also [16]).

The regularity results given in Theorem 4 extend to the general non-linear weighted equation (4) in the normal case the regularity established for the critical problem (5) with $s = p_\beta^*$, when $\gamma > 0$ and the explicit ground state is known (see Theorem 3 and the related table). We also improve the regularity properties first obtained by Citti (see Remarks 1.2 and 1.3 of [9]) in the non-weighted case $\beta = 0$.

An interesting model for f is when f is of *polynomial type*, e.g.

$$f(u) = -a|u|^{q-2}u - b|u|^{l-2}u + c|u|^{s-2}u, \quad a \geq 0,\ b \geq 0,\ c > 0,\ a + b > 0. \tag{7}$$

In this case (F1), (F5)–(F7) are satisfied provided $1 < q \leq l < s < p_\beta^*$, and so Theorem 4 applies. When f is as in (7), in order to apply Theorem 4 we need that its growth exponent at zero is $q < p_\beta^*$, but there are functions satisfying (F1), (F5)–(F7) whose growth at zero is critical or supercritical. For example,

$$f(u) = \begin{cases} -qu^{q-1} & \text{if } u \in [0, \tilde{u}],\ \tilde{u} > 0, \\ q\tilde{u}^{q-2}(u - 2\tilde{u}) & \text{if } u \in (\tilde{u}, 2\tilde{u}), \\ s(u - 2\tilde{u})^{s-1} & \text{if } u \in [2\tilde{u}, \infty), \end{cases}$$

satisfies (F1), (F5)–(F7) for all $q > 1$ and $s \in (1, p_\beta^*)$. Thus, in particular, Theorem 4 applies for such f also when $q \geq p_\beta^*$.

Clearly, if we extend f as an odd function, then $-u$, where u is given by Theorem 4, is a non-trivial non-positive weak solution of (4) which tends to zero as $|x| \to \infty$.

As a consequence of Proposition 3, the solution $u \in D^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ of (4) such that $F \circ u \in L^1_\beta(\mathbb{R}^n)$ satisfies the identity

$$\int_{\mathbb{R}^n} [u(x)f(u(x)) - p_\beta^* F(u(x))]|x|^{-\beta}\, dx = 0. \tag{8}$$

Theorem 4 and identity (8) finally yield

THEOREM 5. *Consider* (4) *with f given by* (7)*, where $q, l, s > 1$ and $\beta < p$.*

If $1 < q \leq l < s < p_\beta^*$ then (4), (7) *admits a radial continuous ground state u of class $D^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \cap L^q_\beta(\mathbb{R}^n)$, with $\|u\|_\infty = u(0) \in (u_0, \overline{u}]$, where u_0 is defined as*

$$u_0 = \inf\{v > 0 : F(v) > 0\}$$

and $\overline{u}$ is any number satisfying

$$\overline{u} > \begin{cases} C^{1/(s-q)} \geq u_0 & \textit{if } 0 < C \leq 1, \\ C^{1/(s-l)} \geq u_0 & \textit{if } C \geq 1, \end{cases} \quad \textit{and} \quad C = s\,\frac{al + bq}{cql} > 0. \tag{9}$$

Moreover, u has the regularity as stated in Theorem 4, and if $1 < q < p$ the solution u is compactly supported in $\mathbb{R}^n$, while if $q \geq p$ the solution u is positive in $\mathbb{R}^n$.

On the other hand, (4), (7) *admits in* $D^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ *only the trivial solution* $u \equiv 0$ *whenever* $(q - p^*_\beta)(l - p^*_\beta) \geq 0$ *and either*

$$\begin{aligned} &s = p^*_\beta \quad \text{and} \quad (q - p^*_\beta) + (l - p^*_\beta) \neq 0, \quad \text{or} \\ &s \neq p^*_\beta \quad \text{and} \quad (s - p^*_\beta)[(q - p^*_\beta) + (l - p^*_\beta)] \leq 0. \end{aligned}$$

In particular, if $1 < q \leq l < s$, *then* (4), (7) *admits a bounded radial continuous ground state when* $p^*_\beta > s$ *and only the trivial solution* $u \equiv 0$ *when* $p^*_\beta \in [l, s]$. *The case* $p^*_\beta \in (p, l)$ *is left open.*

Furthermore, if $l = q$ *then* (4), (7) *admits a bounded radial continuous ground state* u *when* $1 < q = l < s < p^*_\beta$, *with* $\|u\|_\infty = u(0) \in (C^{1/(s-q)}, \overline{u}]$ *and* $C = s(a + b)/cq$; *while only the trivial solution* $u \equiv 0$ *when* $p^*_\beta \in [q, s]$. *It remains an open problem whether there are solutions of* (4), (7) *when* $p < p^*_\beta < q < s$. *On the other hand, the case* $l = q = p$ *is completely treated, that is,* (4), (7) *admits a bounded radial ground state when* $p < s < p^*_\beta$ *and only the trivial solution* $u \equiv 0$ *when* $s \geq p^*_\beta$.

Theorem 5 extends to the weighted p-Laplacian case the existence and non-existence results given by Berestycki and Lions in [4, Example 2] for the non-weighted Laplacian case, i.e. $p = 2$ and $\beta = 0$.

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REFERENCES

[1] B. ABDELLAOUI - V. FELLI - I. PERAL, *Existence and non-existence results for quasilinear elliptic equations involving the p-Laplacian*. Boll. Un. Mat. Ital. Sez. B 9 (2006), 445–484.

[2] ADIMURTHI - N. CHAUDHURI - M. RAMASWAMY, *An improved Hardy–Sobolev inequality and its application*. Proc. Amer. Math. Soc. 130 (2002), 489–505.

[3] A. AMBROSETTI - P. RABINOWITZ, *Dual variational methods in critical point theory and applications*. J. Funct. Anal. 14 (1973), 349–381.

[4] H. BERESTYCKI - J.-L. LIONS, *Nonlinear scalar field equations, I. Existence of a ground state*. Arch. Ration. Mech. Anal. 82 (1983), 313–345.

[5] H. BERESTYCKI - J.-L. LIONS, *Nonlinear scalar field equations, II. Existence of infinitely many solutions*. Arch. Ration. Mech. Anal. 82 (1983), 347–375.

[6] P. CALDIROLI - A. MALCHIODI, *Singular elliptic problems with critical growth*. Comm. Partial Differential Equations 27 (2002), 847–876.

[7] E. CALZOLARI - R. FILIPPUCCI - P. PUCCI, *Existence of radial solutions for the p-Laplacian elliptic equations with weights*. Discrete Contin. Dyn. Syst. 15 (2006), 447–479.

[8] N. CHAUDHURI - M. RAMASWAMY, *Existence of positive solutions of some semilinear elliptic equations with singular coefficients*. Proc. Roy. Soc. Edinburgh Sect. A Math. 131 (2001), 1275–1295.

[9] G. CITTI, *Positive solutions for a quasilinear elliptic equation in* $\mathbb{R}^n$. Rend. Circ. Mat. Palermo 35 (1986), 364–375.

[10] PH. CLÉMENT - R. MANÁSEVICH - E. MITIDIERI, *Some existence and non-existence results for a homogeneous quasilinear problem*. Asymptotic Anal. 17 (1998), 13–29.

[11] S. COLEMAN - V. GLASER - A. MARTIN, *Action minima among solutions to a class of euclidean scalar field equations*. Comm. Math. Phys. 58 (1978), 211–221.

[12] F. DE THÉLIN, *Local regularity properties for the solutions of a nonlinear partial differential equation*. Nonlinear Anal. 6 (1982), 839–844.

[13] F. DE THÉLIN, *Résultats d'existence et de non-existence pour la solution positive et bornée d'une e.d.p. elliptique non linéaire*. Ann. Fac. Sci. Toulouse Math. 8 (1986/87), 375–389.

[14] E. DIBENEDETTO, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*. Nonlinear Anal. 7 (1983), 827–850.

[15] I. EKELAND - N. GHOUSSOUB, *Selected new aspects of the calculus of variations in the large*. Bull. Amer. Math. Soc. 39 (2002), 207–265.

[16] A. FERRERO - F. GAZZOLA, *On subcriticality assumptions for the existence of ground states of quasilinear elliptic equations*. Adv. Differential Equations 8 (2003), 1081–1106.

[17] A. FERRERO - F. GAZZOLA, *Existence of solutions for singular critical growth semilinear elliptic equations*. J. Differential Equations 177 (2001), 494–522.

[18] J. P. GARCÍA AZORERO - I. PERAL, *Hardy inequalities and some critical elliptic and parabolic problems*. J. Differential Equations 144 (1998), 441–476.

[19] J. P. GARCÍA HUIDOBRO - R. MANÁSEVICH - C. YARUR, *On the structure of positive radial solutions to an equation containing a p-Laplacian with weight*. J. Differential Equations 223 (2006), 51–95.

[20] F. GAZZOLA - J. SERRIN - M. TANG, *Existence of ground states and free boundary problems for quasilinear elliptic operators*. Adv. Differential Equations 5 (2000), 1–30.

[21] D. GILBARG - N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*. 2nd ed., Springer, Berlin, 1983.

[22] N. GHOUSSOUB - F. ROBERT, *Concentration estimates for Emden–Fowler equations with boundary singularities and critical growth*. Int. Math. Res. Pap. 2006, 21867, 1–85.

[23] N. GHOUSSOUB - C. YUAN, *Multiple solutions for quasilinear PDEs involving the critical Sobolev and Hardy exponents*. Trans. Amer. Math. Soc. 352 (2000), 5703–5743.

[24] J. V. GONCALVES - C. A. P. SANTOS, *Positive solutions for a class of quasilinear singular equations*. Electron. J. Differential Equations 2004, no. 56, 15 pp.

[25] E. JANNELLI - S. SOLIMINI, *Critical behaviour of some elliptic equations with singular potentials*. Rapporto n. 41, Università degli Studi di Bari, 1996.

[26] O. KAVIAN, *Introduction à la théorie des points critiques*. Springer, Paris, 1983.

[27] J. LI, *Equation with critical Sobolev–Hardy exponents*. Int. J. Math. Math. Sci. 20 (2005), 3213–3223.

[28] E. MITIDIERI, *A simple approach to Hardy inequalities*. Mat. Zametki 67 (2000), 563–572 (in Russian); English transl.: Math. Notes 67 (2000), 479–486.

[29] W.-M. NI - J. SERRIN, *Non-existence theorems for quasilinear partial differential equations*. Rend. Circ. Mat. Palermo (2) Suppl. 8 (1985), 171–185.

[30] P. PUCCI - M. GARCÍA-HUIDOBRO - R. MANÁSEVICH - J. SERRIN, *Qualitative properties of ground states for singular elliptic equations with weights*. Ann. Mat. Pura Appl. (4) 185 (2006), 205–243.

[31] P. PUCCI - J. SERRIN, *The Strong Maximum Principle*. Progr. Nonlinear Differential Equations Appl., Birkhäuser, Basel, to appear.

[32] P. PUCCI - R. SERVADEI, *Existence, non-existence and regularity of radial ground states for p-Laplacian equations with singular weights*. Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.

[33] P. PUCCI - R. SERVADEI, *Regularity of weak solutions of quasilinear elliptic equations*. In preparation.

[34] D. RUIZ - M. WILLEM, *Elliptic problems with critical exponents and Hardy potentials*. J. Differential Equations 190 (2003), 524–538.

[35] J. SIMON, *Régularité de la solution d'une équation non linéaire dans* $\mathbb{R}^n$. In: Journées d'Analyse Non Linéaire, P. Bénilan and J. Robert (eds.), Lecture Notes in Math. 665, Springer, Berlin, 1978, 205–227.

[36] M. STRUWE, *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. 3rd ed., Ergeb. Math. Grenzgeb. 34, Springer, Berlin, 2000.

[37] C. A. SWANSON - L. S. YU, *Critical p-Laplacian problems in* $\mathbb{R}^n$. Ann. Mat. Pura Appl. (4) 169 (1995), 233–250.

[38] S. TERRACINI, *On positive entire solutions to a class of equations with a singular coefficient and critical exponent*. Adv. Differential Equations 1 (1996), 241–264.

[39] M. WILLEM, *Minimax Theorems*. Progr. Nonlinear Differential Equations Appl. 24, Birkhäuser, Boston, 1996.

[40] B. XUAN, *The solvability of quasilinear Brézis–Nirenberg-type problems with singular weights*. Nonlinear Anal. 62 (2005), 703–725.

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