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Mathematical analysis. — *Ostrowski type inequalities over Euclidean domains*, by GEORGE A. ANASTASSIOU and JEROME A. GOLDSTEIN, communicated on 11 May 2007.

ABSTRACT. — The classical Ostrowski inequality for functions on intervals is extended to functions on general domains in Euclidean space. For radial functions on balls the inequality is sharp.

KEY WORDS: Ostrowski inequality; sharp inequality; multivariate inequality.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 26D10, 26D15.

1. INTRODUCTION

The classical Ostrowski inequality (of 1938) [6] is

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \le \left(\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)\|f'\|_\infty, \quad \text{for } f \in C^1([a,b]),\ x \in [a,b],$$

and it is sharp. It was extended from intervals to rectangles in $\mathbb{R}^N$, $N \ge 1$ (see [2, p. 507]). For other recent results related to Ostrowski's inequality, see [3], [4] and [7], [8], [9].

The extension to general domains in $\mathbb{R}^N$ has remained an open problem. Our purpose here is to solve this problem. We deduce Ostrowski type inequalities on general bounded domains in $\mathbb{R}^N$, and the inequalities are shown to be sharp on balls.

2. MAIN RESULTS

Let $N > 1$, $B(0,R) := \{x \in \mathbb{R}^N : |x| < R\}$ be the ball in $\mathbb{R}^N$ centered at the origin and of radius $R > 0$. Let $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ be the unit sphere in $\mathbb{R}^N$.

Let $d\omega$ be the element of surface measure on S^{N-1} and let $\omega_N = \int_{S^{N-1}} d\omega = 2\pi^{N/2}/\Gamma(N/2)$. For $x \in \mathbb{R}^N - \{0\}$ we can write $x = r\omega$, where $r = |x| > 0$ and $\omega = x/r \in S^{N-1}$. Note that $\int_{B(0,R)} dy = \omega_N R^N/N$ is the Lebesgue measure of the ball.

For $f \in C(\overline{B(0,R)})$ let

$$⨍_{B(0,R)} f(y)\,dy := \frac{1}{\mathrm{Vol}(B(0,R))}\int_{B(0,R)} f(y)\,dy$$

and

$$\fint_{S^{N-1}} f(r\omega)\,d\omega = \frac{1}{\omega_N}\int_{S^{N-1}} f(r\omega)\,d\omega$$

be the averages of f over the ball and the sphere, respectively. Here f can be real or complex valued.

Let

$$\tilde{f}(r) := \fint_{S^{N-1}} f(r\omega)\,d\omega$$

be the average of $f(x)$ as x ranges over $\{y \in \mathbb{R}^N : |y| = r\}$. Then

$$\mathcal{N}(f) := \sup_{x\in\overline{B(0,R)}} |f(x) - \tilde{f}(r)| = \|f - \tilde{f}\|_\infty \tag{2.1}$$

measures how far f is from being a radial function. More precisely, $\mathcal{N}$ is a seminorm on $C\big(\overline{B(0,R)}\big)$, and $\mathcal{N}(f) = 0$ if and only if f is a radial function, i.e. $f(x) = g(r)$ for some function $g \in C([0,R])$.

We view how close f is to being radial by computing $\mathcal{N}(f)$: the closer f is to being radial, the smaller $\mathcal{N}(f)$ is, and conversely.

Let Ω be a domain in $\mathbb{R}^N$ and let

$$\text{Lip}(\Omega) = \{f \in C(\overline{\Omega}) : |f(x) - f(y)| \le K|x-y| \quad \text{for some } K > 0 \text{ and all } x, y \in \Omega\}. \tag{2.2}$$

The Lipschitz constant of $f \in \text{Lip}(\Omega)$ is

$$\|f\|_{\text{Lip}} = \inf\{K : K \text{ as in (2.2)}\}.$$

Then $X := \text{Lip}(\Omega)$ is a Banach space under the norm

$$f \mapsto \|f\|_\infty + \|f\|_{\text{Lip}} =: \|f\|_X.$$

Equivalently, X is the Sobolev space $W^{1,\infty}(\Omega)$ (cf. [5]).

Our first main result is the following:

THEOREM 2.1. *Let $f \in \text{Lip}(B(0,R)) = W^{1,\infty}(B(0,R))$. Then for $x = r\omega$ as above,*

$$\left| f(x) - \fint_{B(0,R)} f(y)\,dy \right| \le \mathcal{N}(f) + \frac{N}{R^N}\|\nabla f\|_\infty \left[\frac{2|x|^{N+1}}{N(N+1)} + R^N\left(\frac{R}{N+1} - \frac{|x|}{N}\right)\right]. \tag{2.3}$$

The constants in (2.3) *are best possible, and equality can be attained for nontrivial radial functions at any $r \in [0, R]$.*

PROOF. Let $f \in \mathrm{Lip}(B(0,R))$. Then

$$
\begin{aligned}
(2.4)\qquad & \left| f(x) - \fint_{B(0,R)} f(y)\,dy \right| \\
&\le |f(x) - \tilde{f}(r)| + \left| \fint_{S^{N-1}} f(r\omega')\,d\omega' - \frac{N}{\omega_N R^N} \int_{S^{N-1}} \int_0^R f(s\omega')s^{N-1}\,ds\,d\omega' \right| \\
&\le \mathcal{N}(f) + \frac{N}{R^N} \fint_{S^{N-1}} \left[\int_0^R |f(r\omega') - f(s\omega')| s^{N-1}\,ds \right] d\omega' \\
&\le \mathcal{N}(f) + \frac{N}{R^N} \fint_{S^{N-1}} \int_0^R \left\| \frac{\partial f}{\partial r}(\omega') \right\|_{L^\infty((0,R))} |s-r| s^{N-1}\,ds\,d\omega' \\
&\le \mathcal{N}(f) + \left\| \frac{\partial f}{\partial r} \right\|_{L^\infty(B(0,R))} \frac{N}{R^N} \left(\int_0^R |s-r| s^{N-1}\,ds \right) \\
&= \mathcal{N}(f) + \left\| \frac{\partial f}{\partial r} \right\|_{L^\infty(B(0,R))} \frac{N}{R^N} \left[\frac{2r^{N+1}}{N(N+1)} + R^N \left(\frac{R}{N+1} - \frac{r}{N} \right) \right].
\end{aligned}
$$

Then (2.3) follows since

$$\left\| \frac{\partial f}{\partial r} \right\|_\infty \le \|\nabla f\|_\infty .$$

In particular, a stronger form of (2.3) actually holds in all cases, with $\|\nabla f\|_\infty$ replaced by $\|\partial f/\partial r\|_\infty$. Let $r \in [0,R]$ and $g^*(z) = |z-r|$. We can view g^* as a radial function on $\overline{B(0,R)}$. Then

$$
g^{*\prime}(z) = \begin{cases} \mathrm{sign}(z-r),\ z \ne r, & 0 < r < R, \\ 1, & r = 0, \\ -1 & r = R. \end{cases}
$$

Thus $\|g^{*\prime}\|_\infty = 1$. Therefore

$$
\begin{aligned}
\text{L.H.S.(2.3)} &= \left| g^*(z) - \frac{N}{R^N} \int_0^R g^*(s) s^{N-1}\,ds \right| = \left| |z-r| - \frac{N}{R^N} \int_0^R |s-r| s^{N-1}\,ds \right| \\
&= \left| |z-r| - \frac{N}{R^N} \left[\frac{2r^{N+1}}{N(N+1)} + R^N \left(\frac{R}{N+1} - \frac{r}{N} \right) \right] \right|.
\end{aligned}
$$

Also

$$\text{R.H.S.(2.3)} = \frac{N}{R^N} \left[\frac{2r^{N+1}}{N(N+1)} + R^N \left(\frac{R}{N+1} - \frac{r}{N} \right) \right].$$

Hence equality holds in (2.3) at $z = r$.

Note that the function $g^*(z) = |z-r|$ is in $C^1([0,R])$ only for $r = 0$ and $r = R$; for $0 < r < R$, $g^* \in \mathrm{Lip}([0,R]) - C^1([0,R])$. Of course for $0 < r < R$, g^* can be approximated by C^1 functions, namely $g_n(z) = |z-r|^{1+1/n}$. □

REMARK 1. A key step in the proof is the fact that we can evaluate exactly

$$Q(r) = \int_0^R |s-r| p(s)\,ds$$

for $0 \leq r \leq R$, where p is a nonnegative continuous function satisfying $\int_0^R p(s)\,ds = 1$. In the Ostrowski case ($N = 1$), $p(s) = 1/R$, while in our N-dimensional case, $p(s) = Ns^{N-1}/R^N$.

This works for many other cases including: linear combinations $p(s) = \sum_{j=1}^m a_j s^{q_j}$, where $a_j > 0$, $q_j \geq 0$ (not necessarily an integer) and

$$\sum_{j=1}^m a_j \frac{R^{q_j+1}}{q_j+1} = 1;$$

$p(s) = \sum_{j=1}^{m_1} a_j e^{\lambda_j s}$, where $a_j > 0$, $\lambda_j \in \mathbb{R} - \{0\}$,

$$\sum_{j=1}^{m_1} a_j \left(\frac{e^{\lambda_j R} - 1}{\lambda_j}\right) = 1;$$

and sums $p(s)$ of the form

$$a_j \sin(b_j s + c_j) + d_j \cos(e_j s + f_j),$$

where the coefficients are such that $p(s) \geq 0$ and $\int_0^R p(s)\,ds = 1$.

The space $\operatorname{Lip}(\Omega) \cap C_0(\Omega)$ consists of all Lipschitz continuous functions on $\overline{\Omega}$ vanishing on the boundary $\partial\Omega$ of Ω. Note that

$$\operatorname{Lip}(\Omega) \cap C_0(\Omega) = \{f \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega}) : f = 0 \text{ on } \partial\Omega\}$$

(cf. [5]).

Next comes our more general result where we consider functions over general domains.

THEOREM 2.2. *Let $f \in \operatorname{Lip}(\Omega) \cap C_0(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. Extend f by zero to F on $B(0, R)$, the smallest ball centered at the origin and containing Ω. Then for all $x \in \Omega$,*

$$\left| f(x) - \fint_\Omega f(y)\,dy \right| \leq \mathcal{N}(F) + \left(1 - \frac{\operatorname{Vol}(\Omega)}{\operatorname{Vol}(B(0,R))}\right) \left| \fint_\Omega f(y)dy \right| + \frac{N}{R^N} \|\nabla f\|_{L^\infty(\Omega)} \left[\frac{2|x|^{N+1}}{N(N+1)} + R^N \left(\frac{R}{N+1} - \frac{|x|}{N}\right)\right]. \tag{2.5}$$

PROOF. Let $R := \inf\{R_0 > 0 \colon \Omega \subset B(0, R_0)\}$. Then

$$F(x) := \begin{cases} f(x), & x \in \overline{\Omega}, \\ 0, & x \in \overline{B(0,R)} - \Omega, \end{cases}$$

satisfies

$$F \in \operatorname{Lip}(B(0,R)) \cap C_0(B(0,R)).$$

Then for $x \in \Omega$,

$$\begin{aligned} \left| f(x) - \fint_\Omega f(y)\,dy \right| &\leq \left| F(x) - \fint_{B(0,R)} F(y)\,dy \right| + \left| \fint_{B(0,R)} F(y)\,dy - \fint_\Omega f(y)\,dy \right| \\ &= \mathcal{J}_1 + \mathcal{J}_2, \end{aligned}$$

where

$$\mathcal{J}_1 := \left| F(x) - \fint_{B(0,R)} F(y)\,dy \right|,$$

$$\mathcal{J}_2 := \left| \left(\frac{1}{\mathrm{Vol}(B(0,R))} - \frac{1}{\mathrm{Vol}(\Omega)} \right) \int_{\Omega} f(y)\,dy \right|.$$

By Theorem 2.1,

$$\mathcal{J}_1 \le \mathcal{N}(F) + \frac{N}{R^N} \|\nabla f\|_\infty \left[\frac{2|x|^{N+1}}{N(N+1)} + R^N \left(\frac{R}{N+1} - \frac{|x|}{N} \right) \right].$$

and

$$\mathcal{J}_2 = \left[1 - \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(B(0,R))} \right] \left| \fint_{\Omega} f(y)\,dy \right|.$$

This completes the proof of Theorem 2.2. □

REMARK 2. Note that $\mathcal{N}(F)$ appears in (2.5). In this context, $\mathcal{N}(f)$ does not make sense. Also, $\mathcal{N}(F)$ need not be small (of course, it is small if f is approximately spherically symmetric).

Here is a simple example to illustrate that $\mathcal{N}(F)$ can be large. Let $x_0 \in \Omega$ and choose $\varepsilon > 0$ small enough so that $B(x_0, \varepsilon) \subset \Omega$. Let $f \in C^\infty(\Omega)$ have support in $B(x_0, \varepsilon)$ and satisfy $f(y) = g(\rho)$ where $\rho = |y - x_0|$ for $0 \le \rho \le \varepsilon$. Assume further that g is nonincreasing and $g(0) = f(x_0) = M > 0$. Fix M. Then

$$0 < f(x_0) - \fint_{\Omega} f(y)\,dy \to M = \|f\|_\infty \quad \text{as } \varepsilon \to 0+.$$

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