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Number theory. — *A transcendence criterion for infinite products*, by PIETRO CORVAJA and JAROSLAV HANČL, communicated on 11 May 2007.

ABSTRACT. — We prove a transcendence criterion for certain infinite products of algebraic numbers. Namely, for an increasing sequence of positive integers a_n and an algebraic number $\alpha > 1$, we consider the convergent infinite product $\prod_n ([\alpha^{a_n}]/\alpha^{a_n})$, where $[\cdot]$ stands for the integer part. We prove (Theorem 1) that its value is transcendental under certain hypotheses; Theorem 3 will show that such hypotheses are in a sense unavoidable.

KEY WORDS: Transcendental numbers; diophantine approximation.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 11J81.

1. STATEMENTS

Transcendence criteria are usually based on diophantine approximation: for instance the celebrated theorem of Roth provides a transcendence criterion stating that if a real number is "too well" approximated by a sequence of rationals, then it is transcendental. As a consequence, one obtains several transcendence results for sums of convergent series of rational numbers, like the series $\sum_n 2^{-3^n}$. Generalizations of Roth's theorem like the one by Ridout naturally lead to sharper transcendence criteria. In [1] Schmidt's subspace theorem was applied to prove the transcendence of certain lacunary series, going beyond what could be done by other known methods.

The purpose of this paper is the application of a new diophantine approximation result proved in [2] (also using the subspace theorem) to the transcendence of infinite *products* of algebraic numbers. Our main result is the following:

THEOREM 1. *Let $\alpha > 1$ be a real algebraic number such that no power of α is a Pisot number; let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers with*

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 2.$$

Then the real number

$$\prod_{n=1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}} \tag{1}$$

is transcendental.

Here the symbol $[\cdot]$ stands for the integer part. Of course, if α is an integer, then the above product equals 1. Theorem 3 below will show that the condition that no power of

α is a Pisot number cannot be replaced by the weaker condition that no power of α is an integer.

The above mentioned diophantine approximation tool is the following Theorem CZ, which can be easily deduced from the main theorem in [2]. Its statement requires a definition, introduced in [2]:

DEFINITION. *We say that a real algebraic number $\alpha > 1$ is a* pseudo-Pisot number *if all its (complex) conjugates have absolute values < 1 and it has an integral trace:* $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha) \in \mathbb{Z}$.

With the above definition we have

THEOREM CZ. *Let α, δ be real algebraic numbers with $\alpha > 1$. If for some positive real number ϵ the inequality*

$$0 < \left|\delta - \frac{p}{\alpha^N}\right| < \frac{1}{\alpha^{N(1+\epsilon)}} \tag{2}$$

has infinitely many solutions $(p, N) \in \mathbb{N}^2$, then for all but finitely many such integers N, the algebraic number $\alpha^N\delta$ is a pseudo-Pisot number. Also the numerator p is the trace of $\delta\alpha^N$:

$$p = \mathrm{Tr}_{\mathbb{Q}(\delta\alpha^N)/\mathbb{Q}}(\delta\alpha^N).$$

We note that the absolute value involved is the ordinary one (i.e. normalized with respect to $\mathbb{Q}$, not to $\mathbb{Q}(\alpha)$), so the result is sharper than the classical Roth–Ridout theorem whenever α is irrational.

Under some mild restrictive conditions on the sequence a_n appearing in Theorem 1, one can relax the condition on α, and require just that it is not an integer (clearly this last requirement cannot be avoided). We shall consider a sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ such that:

(i) $\liminf_{n\to\infty} a_{n+1}/a_n > 2$;
(ii) *for every prime p, there exist infinitely many indices n such that p does not divide a_n*;
(iii) *for every triple $(A, B, C) \in \mathbb{Z}^3$ with $A, B > 0$, the line of equation $Ax - By = C$ contains only finitely many pairs (a_n, a_{n+1}).*

Observe that all conditions (i)–(iii) are satisfied for instance when the sequence $(a_{n+1}/a_n)_{n\geq 1}$ of rational numbers converges to an irrational number > 2 and the fractions a_{n+1}/a_n are reduced.

We shall prove the following

THEOREM 2. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence satisfying conditions* (i)–(iii) *above. Then the function $f : [1, \infty) \to (0, 1]$ defined as*

$$f(x) = \prod_{n=1}^{\infty} \frac{[x^{a_n}]}{x^{a_n}}$$

takes transcendental values at every non-integral algebraic point.

On the other hand, it is clear that $f(x) = 1$ whenever x is an integer. We observe that transcendence statements like Theorem 2 are quite common for values of transcendental analytic functions, but our function f is not even continuous.

As remarked, Theorem CZ in the particular case of a rational number α follows from Ridout's generalization of Roth's theorem, i.e. from the one-dimensional subspace theorem. As a consequence, one could prove by using just Ridout's theorem that the function f in Theorem 2 takes transcendental values at every *rational* non-integral point $x > 1$.

As promised, we show by a concrete example that the conclusion of Theorem 1 does not hold if one just assumes that no power of α is an integer.

THEOREM 3 (Example). *Let $t \geq 3$ be an integer, and α be the largest solution to the quadratic equation*

$$x^2 - tx + 1 = 0.$$

Then α is an irrational Pisot number; in particular no power of α is an integer. Let $a_1, a_2, \ldots$ be the sequence defined by $a_1 = 1$ and $a_n = 2 \cdot 3^{n-2}$ for $n \geq 2$. Then

$$\lim_{n\to\infty} (a_{n+1}/a_n) = 3$$

and

$$\prod_{n=1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}} = \frac{t-1}{t}.$$

2. PROOFS

We begin with the following elementary lemma of purely algebraic nature:

LEMMA 4. *Let δ, α be real algebraic numbers with $\alpha > 1$. Suppose that for infinitely many positive integers $n \in \mathbb{N}$, the algebraic number $\delta\alpha^n$ is pseudo-Pisot. Then there exists an integer $D > 0$ such that all the conjugates of α^D, distinct from α^D, have absolute value ≤ 1. Also, every power α^n has this property if and only if $n \equiv 0 \pmod{D}$. There exists an integer $h \in \{0, \ldots, D-1\}$ and an integer $n_0 > 0$ such that for every $n > n_0$, if $\delta\alpha^n$ is pseudo-Pisot, then $n \equiv h \pmod{D}$.*

PROOF. Let $K \subset \mathbb{C}$ be the Galois closure over $\mathbb{Q}$ of the field $\mathbb{Q}(\alpha, \delta)$. Let $G := \{\sigma_1, \ldots, \sigma_r\}$ (where $r = [K : \mathbb{Q}]$) be the corresponding Galois group, with σ_1 the identity automorphism of K. Let $\mathcal{N} \subset \mathbb{N}$ be the set of integers $n > 0$ such that $\delta\alpha^n$ is pseudo-Pisot. Note that if a positive integer n belongs to $\mathcal{N}$ then for all $i = 1, \ldots, r$, either

$$\sigma_i(\delta\alpha^n) = \delta\alpha^n$$

or

$$|\sigma_i(\delta\alpha^n)| < 1.$$

We begin to prove that, under our assumption that the set $\mathcal{N}$ is infinite, no conjugate $\sigma_i(\alpha)$ has (complex) absolute value $> |\alpha|$. If this were the case, then for all large n, we would have $|\sigma_i(\delta\alpha^n)| > |\delta\alpha^n|$, contrary to the assumption that $\delta\alpha^n$ is pseudo-Pisot for all $n \in \mathcal{N}$. Hence for all $i = 1, \dots, r$, $|\sigma_i(\alpha)| \le |\alpha|$.

Let us now consider the set $\mathcal{A} \subset \{1, \dots, r\}$ of indices i for which we have the equality of absolute values $|\sigma_i(\alpha)| = |\alpha|$. Since $|\alpha| > 1$, for all $i \in \mathcal{A}$ and all large n, $|\sigma_i(\delta\alpha^n)| > 1$. Then, for all but finitely many $n \in \mathcal{N}$, $\sigma_i(\delta\alpha^n) = \delta\alpha^n$, since $\delta\alpha^n$ is pseudo-Pisot. Rewriting the last equality as

$$\frac{\sigma_i(\delta)}{\delta} = \left(\frac{\alpha}{\sigma_i(\alpha)}\right)^n$$

we see that $\alpha/\sigma_i(\alpha)$ is a root of unity; letting D_i be the order of $\alpha/\sigma_i(\alpha)$, the exponents n for which the above equality holds form an arithmetic progression $n \equiv h_i \pmod{D_i}$, for a suitable integer $h_i \in \{0, \dots, D_i - 1\}$. As already mentioned, all the exponents $n \in \mathcal{N}$, except possibly finitely many, satisfy such congruence.

Let now $\mathcal{B} \subset \{1, \dots, r\}$ be the set of indices i such that $|\sigma_i(\alpha)| < |\alpha|$; then necessarily $|\sigma_i(\alpha)| \le 1$, as otherwise we would have $1 < |\sigma_i(\delta\alpha^n)| < |\delta\alpha^n|$ for large n, contrary to the assumption that $\delta\alpha^n$ is pseudo-Pisot for infinitely many n.

We thus obtain the partition $\{1, \dots, r\} = \mathcal{A} \cup \mathcal{B}$, with the property that $|\sigma_i(\alpha)| \le 1$ whenever $i \in \mathcal{B}$, and all large integers $n \in \mathcal{N}$ satisfy the system of congruences

$$n \equiv h_i \pmod{D_i}$$

for $i \in \mathcal{A}$. Such a system either has no solution, which is excluded by the fact that the set $\mathcal{N}$ is infinite, or is equivalent to a single congruence $n \equiv h \pmod{D}$, where the modulus D is the least common multiple of the D_i. Note that D is also the order of the (finite) subgroup of the multiplicative group generated by the numbers $\alpha/\sigma_i(\alpha)$ for $i \in \mathcal{A}$. So, if $n \equiv 0 \pmod{D}$, the conjugates $\sigma_i(\alpha^n)$ either coincide with α (when $i \in \mathcal{A}$) or have absolute value ≤ 1 (when $i \in \mathcal{B}$), as required. On the other hand, if $n \not\equiv 0 \pmod{D}$, then α has at least one conjugate $\sigma_i(\alpha)$ which is different from α but has absolute value equal to $|\alpha|$, so > 1. □

The following statement sharpens the previous lemma, under the additional hypothesis that the trace of $\delta\alpha^n$ be a good approximation to the pseudo-Pisot number $\delta\alpha^n$; it depends on the subspace theorem.

LEMMA 5. *Let δ, α be as in Lemma 4. Suppose that for all integers n in an infinite set $\mathcal{N} \subset \mathbb{N}$ the algebraic number $\delta\alpha^n$ is pseudo-Pisot and its trace*

$$p(n) := \mathrm{Tr}_{\mathbb{Q}(\delta\alpha^n)/\mathbb{Q}}(\delta\alpha^n)$$

satisfies

$$|p(n) - \delta\alpha^n| < l^n \tag{3}$$

for some fixed real number $0 < l < 1$. Let D, h be the integers defined in Lemma 4. Then α^D is a Pisot number, in particular α is an algebraic integer. Also, α^n is a pseudo-Pisot number if and only if $n \equiv 0 \pmod{D}$, in which case it is a Pisot number. Finally, every large integer $n \in \mathbb{N}$ satisfies (3) if and only if $n \equiv h \pmod{D}$.

PROOF. In view of Lemma 4, the set $\mathcal{N}$ is contained in the union of a finite set and the arithmetic progression $\{n \in \mathbb{N} : n \equiv h \pmod D\}$. Neglecting the finitely many integers $n \in \mathcal{N}$ not satisfying such a congruence, we write $n = h + mD$. Using the notation of the proof of Lemma 4, let us remark that the automorphisms σ_i, with $i \in \mathcal{A}$, are precisely those fixing α^D; they form a subgroup H of G, and they fix also $\delta\alpha^h$. After renumbering the σ_i, we can suppose that $\sigma_1, \dots, \sigma_t$ (with $t = [\mathbb{Q}(\delta\alpha^h, \alpha^D) : \mathbb{Q}])$ form a complete set of representatives for G/H. Then the trace $p(n)$ can be written as

$$p(n) = p(h + mD) = \mathrm{Tr}_{\mathbb{Q}(\delta\alpha^n)/\mathbb{Q}}(\delta\alpha^n) = \sum_{j=1}^{t} \sigma_j(\delta\alpha^h) \cdot (\sigma_j(\alpha^D))^m.$$

The linear recurrence sequence on the right hand side above is non-degenerate by definition of the integer D. Then Lemma 1 from [2] (applied with $n = t$, $u = \alpha^{mD}$, $\lambda_j = \sigma_j(\delta\alpha^h)$ for $j = 1, \dots, t$ and with w equal to the complex absolute value) implies that if the inequality (3) has infinitely many solutions then all roots $\sigma_j(\alpha^D)$ for $j \neq 1$ have absolute value strictly less than 1, as desired. Also, the same Lemma 1 from [2] in the ultrametric case implies that α is an algebraic integer. Hence α^D is a Pisot number, while α^n is not, for n not dividing D, because it admits some conjugate with the same absolute value. Finally, it is clear that for all large n with $n \equiv h \pmod D$, inequality (3) is satisfied if we take for l any number with $\max_{1<j\leq t} |\sigma_j(\alpha^D)|^{1/D} < l < 1$. □

PROOF OF THEOREM 1. Let α be as in Theorem 1 and δ be the value of the infinite product in (1); suppose by contradiction that δ is an algebraic number. We assume that (1) holds, so there exist positive real numbers ϵ, N_0 such that

$$\frac{a_{n+1}}{a_n} > 2 + \epsilon$$

for all $n \geq N_0$. In particular the sequence is strictly increasing for $n > N_0$.

For $m \geq N_0$ put

$$p = p(m) = \prod_{n=1}^{m} [\alpha^{a_n}]$$

and let $N = N(m) = \sum_{n=1}^{m} a_m$. Then

$$\left|\delta - \frac{p}{\alpha^N}\right| = \left|\frac{p}{\alpha^N}\right| \cdot \left|1 - \prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}}\right|; \tag{4}$$

using the inequality $|1 - t| \leq |\log t|$ for $0 < t < 1$ we deduce from the above that

$$\left|1 - \prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}}\right| \leq \left|\log\left(\prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}}\right)\right|.$$

On the other hand,

$$\log\left(\prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}}\right) = \sum_{n=m+1}^{\infty} \log\left(1 - \frac{\{\alpha^{a_n}\}}{\alpha^{a_n}}\right),$$

where the symbol $\{\cdot\}$ stands for the fractional part. Using the inequality $|\log(1-t)| \leq |2t|$ for $0 < t < 1/2$, and the fact that the fractional part $\{\cdot\}$ is always < 1, we find that the right hand side above is bounded by

$$\sum_{n=m+1}^{\infty} \left| \log\left(1 - \frac{\{\alpha^{a_n}\}}{\alpha^{a_n}}\right) \right| < \sum_{n=m+1}^{\infty} \frac{2}{\alpha^{a_n}} = \frac{2}{\alpha^{a_{m+1}}} \cdot \sum_{n=m+1}^{\infty} \frac{1}{\alpha^{a_n - a_{m+1}}} < \frac{2}{\alpha^{a_{m+1}}} \cdot \frac{1}{\alpha - 1}.$$

So finally we obtain from (4) and the above inequalities, recalling that $p/\alpha^N \leq 1$,

$$\left| \delta - \frac{p}{\alpha^N} \right| < \frac{2}{\alpha^{a_{m+1}}} \cdot \frac{1}{\alpha - 1}. \tag{5}$$

We shall now compare the integer $N = \sum_{n=1}^{m} a_n$ with a_{m+1}. For this purpose we recall that for $m \geq N_0$, $a_{m+1} > (2+\epsilon)a_m$ while $a_{m-i} \leq (2+\epsilon)^{-i} a_m$ for $i = 1, \ldots, m - N_0$. Then for large m we have $a_{m+1} \geq (1+\epsilon)N$. Hence in particular we obtain a rational approximation p/α^N to the algebraic number δ satisfying the second inequality in (2). Our hypothesis that no power of α is an integer guarantees that the difference $\delta - p/\alpha^N$ is non-zero, so both inequalities in (2) are satisfied. Now Theorem CZ shows that $\delta\alpha^N$ is a pseudo-Pisot number and that the numerator $p = p(N)$ is the trace of $\delta\alpha^N$, so inequality (3) of Lemma 5 is also satisfied. If this happens for infinitely many exponents N then by Lemma 5 some power of α is a Pisot number, which we have excluded. □

PROOF OF THEOREM 2. Let $\alpha \in [1, \infty)$ be an algebraic point, not a rational integer, and let $\delta = f(\alpha)$. We have to prove that δ is transcendental. Suppose by contradiction it is algebraic.

Arguing as in the proof of Theorem 1, we deduce that the sequence of rational numbers

$$\prod_{n=1}^{m} \frac{[\alpha^{a_n}]}{\alpha^{a_n}} = \frac{p(m)}{\alpha^N},$$

where $N = N(m) = \sum_{n=1}^{m} a_n$, satisfies inequality (2) of Theorem CZ. Hence, by Theorem CZ, for all large m, $\delta\alpha^N$ is a pseudo-Pisot number. Now by Lemmas 4 and 5 this implies that all such integers N are congruent modulo D, where D is the minimal positive integer n such that α^n is a Pisot number; in particular, there exists a number n_0 such that all the integers a_n for $n > n_0$ are divisible by D. Due to the arithmetic condition (ii), this is possible only if $D = 1$, hence α is a Pisot number. Since by hypothesis α is not an integer, it must be irrational.

Now, α being a Pisot number, the integral part of every power α^n has the explicit expression

$$[\alpha^n] = \sum_{j=1}^{t} \alpha_j^n = \mathrm{Tr}(\alpha^n),$$

or

$$[\alpha^n] = \left(\sum_{j=1}^{t} \alpha_j^n\right) - 1 = \mathrm{Tr}(\alpha^n) - 1,$$

where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_t$ are the conjugates of α, while the integral part of $\delta\alpha^n$ is expressed as

$$[\delta\alpha^n] = \eta_1\alpha_1^n + \cdots + \eta_t\alpha_t^n \tag{6}$$

for suitable non-zero conjugate algebraic numbers $\eta_1, \ldots, \eta_t$. Writing the above identity for $n = N(m)$ and $N = N(m+1) = N(m) + a_{m+1}$ we obtain

$$\begin{aligned}(\eta_1\alpha_1^{N(m)} + \cdots + \eta_t\alpha_t^{N(m)}) \cdot (\alpha_1^{a_{m+1}} + \cdots + \alpha_t^{a_{m+1}})\\ = (\eta_1\alpha_1^{N(m)+a_{m+1}} + \cdots + \eta_t\alpha_t^{N(m)+a_{m+1}}),\end{aligned} \tag{7}$$

or, in the case $[\alpha^{a_n}] = \mathrm{Tr}(\alpha^n) - 1$,

$$\begin{aligned}(\eta_1\alpha_1^{N(m)} + \cdots + \eta_t\alpha_t^{N(m)}) \cdot (\alpha_1^{a_{m+1}} + \cdots + \alpha_t^{a_{m+1}} - 1)\\ = (\eta_1\alpha_1^{N(m)+a_{m+1}} + \cdots + \eta_t\alpha_t^{N(m)+a_{m+1}}).\end{aligned}$$

Since the two cases are very similar, we only treat the first one, so we suppose that (7) holds; we then obtain the S-unit equation

$$\sum_{i\neq j} \eta_i\alpha_i^{N(m)}\alpha_j^{a_{m+1}} = 0. \tag{8}$$

Note that the sum on the left, which ranges over the pairs $(i, j) \in \{1, \ldots, t\}^2$ with $i \neq j$, is non-empty since we have proved that α is irrational (which is equivalent to $t > 1$). By the S-unit equation theorem (see for instance [3, Ch. V, Theorem 2A]), there exist two distinct elements in the above sum for which the ratio is equal to a given number infinitely often. This in particular implies that there exist two pairs $(i, j) \neq (h, k)$ such that the equation

$$\frac{\alpha_i^{N(m)}\alpha_j^{a_{m+1}}}{\alpha_h^{N(m)}\alpha_k^{a_{m+1}}} = c$$

has infinitely many solutions. Since the ratios α_i/α_h and α_j/α_k are not both roots of unity, the above relation implies the existence of a non-trivial linear dependence relation of the form

$$c_1N(m) - c_2a_{m+1} = c_3, \tag{9}$$

where c_1, c_2, c_3 are real numbers with $(c_1, c_2) \neq (0, 0)$, satisfied by infinitely many pairs $(N(m), a_{m+1})$. Clearly, one can suppose without loss of generality that c_1, c_2, c_3 are integers and c_1, c_2 are positive. Let $\mathcal{N}$ be the (infinite) set of integers m such that the pair $(N(m), a_{m+1})$ satisfies (9). Since the pair $(N(m+1), a_{m+2}) = (N(m)+a_{m+1}, a_{m+2})$ also satisfies the S-unit equation (8), there is a non-trivial linear dependence relation of type (9) satisfied by infinitely many integers $m \in \mathcal{N}$. Namely there exist integers d_1, d_2, d_3, with $d_1, d_2 > 0$, such that for infinitely many $m \in \mathcal{N}$,

$$d_1N(m+1) - d_2a_{m+2} = d_3. \tag{10}$$

From (9) and (10), using the equality $N(m+1) = N(m) + a_{m+1}$ and eliminating $N(m)$, we obtain the dependence relation

$$(d_1c_1 + d_1c_2)a_{m+1} - c_1d_2a_{m+2} = d_3c_1 - d_1c_3,$$

which would have infinitely many solutions. This is excluded by our hypothesis (iii), so the theorem is proved. □

The proof of Theorem 3 rests on the following result:

LEMMA 6. *Assume the hypotheses of Theorem* 3 *hold. For all* $m \geq 1$ *put* $N(m) = \sum_{n=1}^{m} a_n$. *Also, denote by* α' *the algebraic conjugate of* α, *i.e.* $\alpha' = t - \alpha$. *Then for all integers* $m \geq 1$,

$$\prod_{n=1}^{m} [\alpha^{a_n}] = \frac{t-1}{t} \cdot (\alpha^{N(m)} + \alpha'^{\,N(m)}).$$

PROOF. The proof by induction on m. For $m = 1$, we have $a_1 = 1 = N(1)$, $[\alpha] = t - 1$ and $\alpha + \alpha' = t$, so the relation holds. Suppose it holds for m, so

$$\prod_{n=1}^{m} [\alpha^{a_n}] = \frac{t-1}{t} \cdot (\alpha^{N(m)} + \alpha'^{\,N(m)}).$$

Then we have to prove that

$$(\alpha^{N(m)} + \alpha'^{N(m)})[\alpha^{a_{m+1}}] = \alpha^{N(m+1)} + \alpha'^{N(m+1)}.$$

For this purpose we use the property of the sequence a_n that for each $m \geq 1$, $a_{m+1} = 2N(m) = 2\sum_{n=1}^{m} a_n$, so $N(m+1) = 3N(m)$. Then the above identity becomes

$$(\alpha^{N(m)} + \alpha'^{N(m)}) \cdot [\alpha^{2N(m)}] = \alpha^{3N(m)} + \alpha'^{3N(m)},$$

which follows easily from the fact that for all $n \geq 1$, $[\alpha^n] = \alpha^n - 1 + \alpha'^{\,n}$. □

PROOF OF THEOREM 3. In view of the above lemma, for each $m \geq 1$,

$$\prod_{n=1}^{m} \frac{[\alpha^{a_n}]}{\alpha^{a_n}} = \frac{t-1}{t} \cdot \frac{\alpha^{N(m)} + \alpha'^{\,N(m)}}{\alpha^{N(m)}},$$

which clearly tends to the limit $(t-1)/t$ as $m \to \infty$. □

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P. Corvaja
Dipartimento di Matematica e Informatica
via delle Scienze 206
33100 UDINE, Italy
corvaja@dimi.uniud.it

J. Hančl
Department of Mathematics
and Institute for Research and Applications
of Fuzzy Modeling
University of Ostrava
30. dubna 22
701 03 OSTRAVA 1, Czech Republic
hancl@osu.cz