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Mathematical physics. — *Geometry and balance of hyperstresses*, by PAOLO MARIA MARIANO, communicated on 11 May 2007.

ABSTRACT. — Surface hyperstress is introduced along a layer with vanishing thickness, across which two second-grade elastic Cauchy bodies are glued. Surface balances are obtained by making use of the relative power of actions and of invariance requirements on the Lagrangian. The action of bulk hyperstress on vacancies, linear inclusions and cracks is also evaluated.

KEY WORDS: Second jet bundle field theories; non-simple bodies; hyperstresses.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 74A30, 74A50, 74B20.

1. INTRODUCTION

The possible dependence of constitutive relations on second or higher-order gradients of deformation and/or velocity in continuum mechanics has been widely discussed in the scientific literature, since the pioneer works by Korteweg (1901) [17] and Jaramillo (1929) [14]. Reasons for considering such a constitutive dependence are of various nature. For example, gradients of the density are used to represent capillarity effects in fluids after Korteweg (see, e.g., [17], [3], [2], [7], [40], [15], [8], [22]). The scheme admits a natural generalization to fluids with constitutive structures depending on higher-order gradients of the velocity field [29]. Solids undergoing phase transitions are characterized by diffuse interfaces; the effects of the presence of such interfaces can be considered scattered over the body and accounted for by gradients of deformation of various orders (see [27], [35], [36]). Experiments also show the existence of non-local effects in plastic phenomena [13], [30]; these effects have suggested the need of second-gradient theories of plasticity (see, e.g., [12], [5] and references therein). Second-gradient (non-local) contributions of the deformation can also be attributed to latent microstructures [2]. They determine a 'regularizing' effect on the motion. In general, when gradients of order N of the deformation and/or velocity are involved in the constitutive structures, bodies are called of *grade* N. The analysis of such bodies has theoretical interest in purely mechanical (see, e.g., [37], [39], [28], [20]), thermodynamical (see [3], [36]), and geometrical [18] aspects.

Measures of interaction, called hyperstresses, are power-conjugated with each gradient of order greater than or equal to 2. Their existence is a necessary consequence of the validity of the second law of thermodynamics [11], [3], [2]. Their balance has been widely discussed (see the basic contributions [38], [10], [6]).

Here attention is focused on materials of grade 2 (also called second-grade materials). A commentary to the standard treatment of them is contained in Section 2. The second jet bundle language used below is the natural setting in which one may emphasize clearly the balance equations discussed in what follows. The accomplishments in the subsequent sections are listed below.

- A *surface hyperstress* is introduced: it is defined along a thin layer gluing two second-grade bodies. Such a stress enters surface balances which are derived here in a covariant way.
- The influence of bulk hyperstresses on point defects and linear (rigid) inclusions is evaluated by making use of (i) invariance requirements on the Lagrangian density and (ii) balances of powers defined appropriately.
- Finally, the evolution of cracks in second-grade materials is analyzed. The presence of hyperstresses alters the structure of the force driving the crack tip, as pointed out by experiments (see companion results in [41]).

Notations. $A \cdot B$ denotes the scalar product between tensors of the same rank (covariant components act on contravariant components in any product defined here). If A and B are second-rank tensors, AB indicates the product which contracts only one index and generates a second-order tensor; for example $(AB)_{ij} = A_{ik}B^k_j$ (the sum over repeated indices is understood). For second-rank tensors the superscript $*$ indicates the *formal adjoint*. If u is a vector and A a second-rank tensor, the product Au contracts the index of u and furnishes a vector; for example $(Au)^i = A^i_j u^j$. If $\mathfrak{A}$ and $\mathfrak{G}$ are third-order tensors, the product $\mathfrak{A} : \mathfrak{G}$ contracts two indices and furnishes a second-order tensor; for example $(\mathfrak{A} : \mathfrak{G})_{il} = \mathfrak{A}_{ijk}\mathfrak{G}^{jk}_l$. If $\mathfrak{A}$ is a third-order tensor and A a second-order tensor, the product $\mathfrak{A}A$ contracts the indices of A and furnishes a vector (or a covector); for example $(\mathfrak{A}A)_i = \mathfrak{A}_{ijk}A^{jk}$; moreover, the product $\mathfrak{A}\llcorner A$ contracts only one index and generates a third-order tensor, for example $(\mathfrak{A}\llcorner A)_{ijl} = \mathfrak{A}_{ijk}A^k_l$ (in certain circumstances it will also be written $A \lrcorner \mathfrak{A}$ with the same meaning, say $(A \lrcorner \mathfrak{A})_{ijk} = A^l_i \mathfrak{A}_{ljk}$). If u is still a vector and $\mathfrak{A}$ a third-rank tensor, the product $\mathfrak{A}u$ furnishes a second-rank tensor, for example $(\mathfrak{A}u)_{ij} = \mathfrak{A}_{ijk}u^k$. For third-rank tensors the superscript $*$ indicates the major formal adjoint, i.e., given vectors (or co-vectors) a, b, c, one finds $((\mathfrak{A}a)b)c = ((\mathfrak{A}^*c)b)a$. The superscript t indicates the formal minor adjoint, namely $((\mathfrak{A}a)b)c = ((\mathfrak{A}^t b)a)c$.

For any pair of vector spaces V and W (with duals V^* and W^*), $\mathrm{Hom}(V, W)$ is the space of linear maps from V to W. For any smooth manifold M, $T_m M$ is the tangent space of M at $m \in M$, while $T^*_m M$ the corresponding cotangent space. ∂_r means partial derivative with respect to the generic entry r.

For a surface Σ in $\mathbb{R}^3$, oriented at almost every point by the normal m and for any differentiable field $\mathbb{R}^3 \ni x \mapsto a(x)$, the notation $\nabla_\Sigma a = \nabla a(I - m \otimes m)$ indicates its surface gradient at $x \in \Sigma$. The trace of $\nabla_\Sigma a$ defines the surface divergence of a at x, $\mathrm{Div}_\Sigma\, a = \mathrm{tr}\,\nabla_\Sigma a$. In particular, the negative of the surface gradient of m, $-\nabla_\Sigma m$, is denoted by L and is the curvature tensor of Σ. Its trace is the overall curvature $\mathcal{K}$.

If $x \mapsto a(x)$ is a piecewise differentiable field over $\mathbb{R}^3$ (or some regular region of it), taking values in a linear space and suffering a bounded discontinuity over Σ, its jump $[a]$ across Σ is defined by $[a] := a^+ - a^-$ at each $x \in \Sigma$. Here a^+ and a^- denote respectively the inner and outer traces of a at Σ given by the limits $a^\pm := \lim_{\varepsilon\to 0} a(x \pm \varepsilon m)$ for $\varepsilon \geq 0$. The average of a across Σ at each x is given by $2\langle a\rangle = a^+ + a^-$. For any pair of fields a_1 and a_2 over $\mathbb{R}^3$ with the same properties of a, the relation $[a_1 a_2] = [a_1]\langle a_2\rangle + \langle a_1\rangle[a_2]$ holds if the product $a_1 a_2$ is defined in distributive way.

2. COMMENTARY TO SECOND-GRADE ELASTICITY: VARIATIONAL FORMULATION ON THE SECOND JET BUNDLE

A body occupies a regular region $\mathcal{B}_0$ of $\mathbb{R}^3$ in a place taken as *reference*. New places are achieved by *transplacement maps*

$$\mathcal{B}_0 \ni x \mapsto y := y(x) \in \mathbb{R}^3. \tag{2.1}$$

Each y is a standard deformation. Common assumptions are that y is one-to-one, continuous and piecewise continuously differentiable (at least twice for our purposes). The current place $\mathcal{B} := y(\mathcal{B}_0)$ of the body is endowed with the same regularity properties of $\mathcal{B}_0$. Moreover, y is orientation preserving: the value of its gradient at each $x \in \mathcal{B}_0$, i.e. $F := \nabla y(x) \in \mathrm{Hom}(T_x\mathcal{B}_0, T_y\mathcal{B})$, has positive determinant (see [1] for further details).

For simplicity one may assume that $\mathcal{B}_0$ belongs to an isomorphic copy $\mathbb{R}^{\ddagger 3}$ of $\mathbb{R}^3$. Motions are maps $(x, t) \mapsto y := y(x, t) \in \mathbb{R}^3$ twice differentiable in time, with $x \in \mathcal{B}_0$, $t \in [0, \bar{t}]$ and $x = y(x, 0)$. The time derivative $\dot{y} := \frac{\partial}{\partial t} y(x, t)$ is the velocity.

A fiber bundle $\mathcal{Y}$ with $\pi : \mathcal{Y} \to \mathcal{B}_0 \times [0, \bar{t}]$ the natural projection and $\pi^{-1}(x, t) = \mathbb{R}^3$ the *prototype fiber* is the natural geometrical setting for describing the shape of a body and its motion when the generic material element is considered only as a mass point. In this case one may call such bodies *Cauchy bodies* to stress the difference with *complex bodies*, those for which the prototype material element needs to be considered as a system.

Mappings

$$\eta : \mathcal{B}_0 \times [0, \bar{t}] \to \mathcal{Y}, \qquad \eta(x, t) = (x, t, y(x, t)), \tag{2.2}$$

can then be selected and admit first and second prolongations

$$j^1(\eta)(x, t) = (x, t, y, \dot{y}, F), \tag{2.3}$$

$$j^2(\eta)(x, t) = (x, t, y, \dot{y}, F, \ddot{y}, \nabla F), \tag{2.4}$$

respectively, with $j^1(\eta)$ and $j^2(\eta)$ belonging to the first and second jet bundles $J^1\mathcal{Y}$ and $J^2\mathcal{Y}$. Consequently, one recovers the sequence

$$\mathcal{B}_0 \times [0, \bar{t}] \xleftarrow{\pi} \mathcal{Y} \xleftarrow{\pi^1} J^1\mathcal{Y} \xleftarrow{\pi^2} J^2\mathcal{Y}, \tag{2.5}$$

where π^1 and π^2 are the relevant natural projections.

The conservative mechanical behavior of second-grade Cauchy elastic bodies is described by a Lagrangian

$$L : J^2\mathcal{Y} \to \bigwedge^{3+1}(\mathcal{B}_0 \times [0, \bar{t}]), \tag{2.6}$$

such that

$$L(j^2(\eta)(x, t)) = \mathcal{L}\, dx \wedge dt, \tag{2.7}$$

with $\mathcal{L} := \mathcal{L}(x, y, \dot{y}, F, \nabla F)$ a C^2 density. In principle, one should take care in defining L because $\mathcal{B}_0 \times [0, \bar{t}]$ is a manifold with boundary $\mathcal{B}_0 \times \{0\} \cup \mathcal{B}_0 \times \{\bar{t}\}$. However, possible problems disappear because one is just interested in the variations of the action functional

$$\mathcal{E} := \int_{\mathcal{B}_0 \times [0, \bar{t}]} \mathcal{L}\, dx \wedge dt. \tag{2.8}$$

Note that, with the choice (2.8), the analysis is restricted to the description of *autonomous* behavior. In particular, $\mathcal{L}$ is selected of the form

$$\mathcal{L} := \frac{1}{2}\rho_0|\dot{y}|^2 - \rho_0\tilde{e}(x, F, \nabla F) - \rho_0\tilde{\mathfrak{w}}(y), \tag{2.9}$$

where ρ_0 is the referential mass density conserved along the motion, $\tilde{e}$ the elastic energy and $\tilde{\mathfrak{w}}$ the potential of external body forces, all taken per unit mass. As convenient notation, e and w will indicate the values $e := \rho_0\tilde{e}(x, F, \nabla F)$ and $\tilde{\mathfrak{w}} := \rho_0\tilde{\mathfrak{w}}(y)$.

The Euler–Lagrange equations derive from the requirement that the first variation of $\mathcal{E}$ vanishes:

$$\delta\mathcal{E} = 0. \tag{2.10}$$

Under conditions of sufficient smoothness and an appropriate definition of the variations, such equations read

$$\overline{\partial_{\dot{y}}\mathcal{L}}^{\cdot} = \partial_y\mathcal{L} - \mathrm{Div}(\partial_F\mathcal{L} - \mathrm{Div}\,\partial_{\nabla F}\mathcal{L}). \tag{2.11}$$

The derivative $\partial_{\nabla F}\mathcal{L}$ is the so-called *hyperstress* in the bulk.

Invariance requirements on the Lagrangian density under changes in observers and relabeling of the material elements placed in $\mathcal{B}_0$ play the essential role of first principles. In general, observers are representations of the geometrical environments necessary to describe the morphology of a body and its motion (see [24]). In this case such geometric environments are the ambient space $\mathbb{R}^3$, an isomorphic copy $\mathbb{R}^{\ddagger 3}$ containing the reference place $\mathcal{B}_0$, and the time scale $[0, \bar{t}]$. Here attention is focused only on synchronous changes in observers that leave invariant the reference place. They are described by smooth curves $\mathbb{R}^+ \ni s \mapsto \mathbf{f}_s \in \mathrm{Diff}(\mathbb{R}^3, \mathbb{R}^3)$ on the group of diffeomorphisms of $\mathbb{R}^3$, starting from the identity. The infinitesimal generator of such an action is indicated by v at each y. It is the derivative with respect to s of $\mathbf{f} := \mathbf{f}_s(y)$ at $s = 0$, that is, $\mathbf{f}'_s(y)|_{s=0}$.

When appropriate, s will be tacitly identified with the time t. Of course, in the above setting, classical isometric changes in observers, governed by the semidirect product $\mathbb{R}^3 \ltimes SO(3)$, are included.

By considering only changes of the ambient space $\mathbb{R}^3$, all observers register an *identical picture* of the reference place $\mathcal{B}_0$. However, the assignment of labels x to points in $\mathcal{B}_0$ is only instrumental and has no physical relevance. Then the requirement of invariance of the Lagrangian with respect to relabeling appears natural in these conditions. Moreover, when defects are present, relabeling implies a sort of ‘permutation’ of defects (for example point defects such as vacancies or inclusions). Formally, the relabeling is defined by smooth curves $\mathbb{R}^+ \ni s_1 \mapsto \mathbf{f}^1_{s_1} \in \mathrm{SDiff}(\mathcal{B}_0, \mathbb{R}^{\ddagger 3})$, with $\mathbf{f}^1_0$ the identity. $\mathrm{SDiff}(\mathcal{B}_0, \mathbb{R}^{\ddagger 3})$ is the special group of diffeomorphisms over $\mathcal{B}_0$ so that at each s_1 one gets $x \mapsto \mathbf{f}^1_{s_1}(x)$, with $\mathrm{Div}\,\mathbf{f}^{1\prime}_{s_1}(x) = 0$, where the prime denotes differentiation with respect to the parameter s_1. The notation $w := \mathbf{f}^{1\prime}_0(x)$ is useful and also the gradient with respect to $\mathbf{f}^1 := \mathbf{f}^1_{s_1}(x)$ is indicated by $\nabla_{\mathbf{f}^1}$. When appropriate, s_1 will be tacitly identified with the time t as well as with s.

LEMMA 1. *The combined action of* $\mathbf{f}^1_{s_1}$ *and* $\mathbf{f}_s$ *at any* s_1 *and* s *implies the following transformations:*

$$\dot{y} \mapsto (\operatorname{grad}\mathbf{f})\dot{y}, \tag{2.12}$$

$$F \mapsto \bar{F} = (\operatorname{grad}\mathbf{f})F(\nabla\mathbf{f}^1)^{-1}, \tag{2.13}$$

$$\nabla F \mapsto \overline{\nabla F} = ((\operatorname{grad}\mathbf{f})\lrcorner\nabla F \llcorner(\nabla\mathbf{f}^1)^{-1})^t \llcorner(\nabla\mathbf{f}^1)^{-1} + F\lrcorner\nabla_{\mathbf{f}^1}(\nabla\mathbf{f}^1)^{-1}. \tag{2.14}$$

PROOF. At each s and $y \in \mathcal{B}$, $\operatorname{grad}\mathbf{f} \in \operatorname{Hom}(T_y\mathcal{B}, T_{\mathbf{f}}\mathbf{f}_s(\mathcal{B}))$; then, since $\dot{y} \in T_y\mathcal{B}$, the relation (2.12) follows. To prove (2.13), first recall that $F \in \operatorname{Hom}(T_x\mathcal{B}_0, T_y\mathcal{B}) \cong T_y\mathcal{B} \otimes T^*_x\mathcal{B}_0$. Then, the left action of $\operatorname{grad}\mathbf{f}$ maps linearly $T_y\mathcal{B}$ in $T_{\mathbf{f}}\mathbf{f}_s(\mathcal{B})$ while the right action of $(\nabla\mathbf{f}^1)^{-1}$ transforms linearly $T^*_x\mathcal{B}_0$ in $T^*_{\mathbf{f}^1}\mathbf{f}^1_{s_1}(\mathcal{B}_0)$. The relation (2.14) can be obtained by calculating the second gradient of $\mathbf{f} = \mathbf{f}_s(y(\mathbf{f}^{1^{-1}}(x)))$ and taking into account (2.13). Notice that the twofold right application of $(\nabla\mathbf{f}^1)^{-1}$ is justified by the fact that $\nabla F \in T_y\mathcal{B} \otimes T^*_x\mathcal{B}_0 \otimes T^*_x\mathcal{B}_0$, so a twofold transformation of $T^*_x\mathcal{B}_0$ in $T^*_{\mathbf{f}^1}\mathbf{f}^1_{s_1}(\mathcal{B}_0)$ by means of $(\nabla\mathbf{f}^1)^{-1}$ is necessary. The second term on the right-hand side of (2.14) follows from the fact that the derivative ∇ is transported along the trajectories induced by $\mathbf{f}^1_{s_1}$. □

Note that the proof above holds even if one considers a generalized relabeling represented by curves on the entire group of diffeomorphisms $\operatorname{Diff}(\mathcal{B}_0, \mathbb{R}^{\ddagger 3})$ over $\mathcal{B}_0$, rather than its special subgroup.

DEFINITION 1. $\mathcal{L}$ *is said to be* invariant with respect to changes in observers and relabeling *if*

$$\mathcal{L}(x, y, \dot{y}, F, \nabla F) = \mathcal{L}(\mathbf{f}^1, \mathbf{f}, (\operatorname{grad}\mathbf{f})\dot{y}, \bar{F}, \overline{\nabla F}) \tag{2.15}$$

for any smooth curve $s \mapsto \mathbf{f}_s \in \operatorname{Aut}(\mathbb{R}^3)$ *and* $s_1 \mapsto \mathbf{f}^1_{s_1} \in \operatorname{SDiff}(\mathcal{B}_0, \mathbb{R}^{\ddagger 3})$.

From now on it is assumed that the map $x \mapsto \partial_{\nabla F}\mathcal{L} \in T^*_y\mathcal{B} \otimes T_x\mathcal{B}_0 \otimes T_x\mathcal{B}_0$, $x \in \mathcal{B}_0$, is of class $C^1(\mathcal{B}_0)$.

DEFINITION 2. $\mathcal{Q}$ *and* $\mathfrak{F}$ *are respectively scalar and vector densities defined by*

$$\mathcal{Q} := \partial_{\dot{y}}\mathcal{L} \cdot (v - Fw), \tag{2.16}$$

$$\mathfrak{F} := \mathcal{L}w + \partial_F\mathcal{L}^*(v - Fw) - (\operatorname{Div}\partial_{\nabla F}\mathcal{L})^*(v - Fw) + \partial_{\nabla F}\mathcal{L}^t\nabla(v - Fw). \tag{2.17}$$

THEOREM 1. *If* $\mathcal{L}$ *is invariant with respect to changes in observers and relabeling, then, when* $\mathfrak{F}$ *is of class* C^1 *in space and* $\mathcal{Q}$ *is of the same class in time,*

$$\dot{\mathcal{Q}} + \operatorname{Div}\mathfrak{F} = 0. \tag{2.18}$$

The theorem above is the Noether theorem for second-grade elasticity and is not a new result. A proof is presented in [18] within a general framework of second-order multisymplectic field theories [26] (see also [33] for higher order Euler operators and [34]). A simple proof is reported below.

PROOF. The requirement of invariance of $\mathcal{L}$ under the action of $\mathbf{f}^1_{s_1}$ and $\mathbf{f}_s$ implies the identities

$$\frac{d}{ds_1}\mathcal{L}\bigg|_{s_1=0,\, s=0} = 0, \qquad \frac{d}{ds}\mathcal{L}\bigg|_{s_1=0,\, s=0} = 0, \tag{2.19}$$

that correspond respectively to

$$\partial_x\mathcal{L}\cdot w - \partial_F\mathcal{L}\cdot F\nabla w - \partial_{\nabla F}\mathcal{L}\cdot(\nabla F^t \llcorner \nabla w) \\ - \partial_{\nabla F}\mathcal{L}\cdot(\nabla F \llcorner \nabla w) - \partial_{\nabla F}\mathcal{L}\cdot(F \lrcorner \nabla\otimes\nabla w) = 0, \tag{2.20}$$

$$\partial_y\mathcal{L}\cdot v + \partial_{\dot y}\mathcal{L}\cdot(\operatorname{grad} v)\dot y + \partial_F\mathcal{L}\cdot(\operatorname{grad} v)F + \partial_{\nabla F}\mathcal{L}\cdot(\operatorname{grad} v \lrcorner \nabla F) = 0. \tag{2.21}$$

By evaluating the time derivative of $\mathcal{Q}$ and the spatial divergence of $\mathfrak{F}$, the use of (2.20) and (2.22) implies, after some algebra, that

$$\dot{\mathcal{Q}} + \operatorname{Div}\mathfrak{F} = \frac{d}{ds_1}\mathcal{L}\bigg|_{s_1=0,\, s=0} + \frac{d}{ds_2}\mathcal{L}\bigg|_{s_1=0,\, s=0}. \qquad \square \tag{2.22}$$

Note that in the proof above use is made of the Euler–Lagrange equation, which requires a regularity greater than the one needed in a direct proof that does not make use of balance equations (see [18], also [9] for classical field theories on first jet bundles).

COROLLARY 1. *If $v \neq 0$ is left arbitrary and $w = 0$, it follows that*

$$\rho_0\ddot y = b + \operatorname{Div}(P - \operatorname{Div}\mathfrak{S}) \quad \textit{in } \mathcal{B}_0 \tag{2.23}$$

In (2.23), $P = -\partial_F L \in \operatorname{Hom}(T_x^*\mathcal{B}_0, \mathcal{T}_y^*\mathcal{B})$ is the first Piola–Kirchhoff stress, $b = \partial_y\mathcal{L} \in \mathcal{T}_y^*\mathcal{B}$ the vector of non-inertial body forces and $\mathfrak{S} = -\partial_{\nabla F}\mathcal{L} \in T_x\mathcal{B}_0\otimes T_x\mathcal{B}_0\otimes\mathcal{T}_y^*\mathcal{B}$ the bulk hyperstress (reasons for the existence of hyperstresses apart from this conservative setting have been discussed in [6], [32]). Of course, (2.23) is just the Euler–Lagrange equation associated with (2.8) but the way it is re-obtained ensures its covariance.

PROOF. When $v \neq 0$ and $w = 0$, in fact, one gets

$$\mathcal{Q} = \partial_{\dot y}\mathcal{L}\cdot v, \tag{2.24}$$

$$\mathfrak{F} = \partial_F\mathcal{L}^* v - (\operatorname{Div}\partial_{\nabla F}\mathcal{L})^* v + \partial_{\nabla F}\mathcal{L}^t\nabla v. \tag{2.25}$$

Then, from Theorem 1 and (2.22), equation (2.23) follows. $\square$

COROLLARY 2. *Let $\mathbf{f}_s$ be an isometry. Then $v = q\times(y - y_0)$ with $q\times \in \mathfrak{so}(3)$ and y_0 an arbitrary point. Let also q be constant. The action of the special choice of $\mathbf{f}_s$ just selected, leaving q arbitrary, implies*

$$\operatorname{skw}(PF^* + \mathfrak{S} : (\nabla F)^*) = 0. \tag{2.26}$$

The proof follows by direct calculation.

COROLLARY 3. *If $w \neq 0$ is left arbitrary and $v = 0$, it follows that*

$$\overline{F^* \partial_{\dot{x}} \mathcal{L}}^{\cdot} + \operatorname{Div}\left(\mathbb{P}_2 - \frac{1}{2} \rho_0 |\dot{x}|^2 I \right) + F^* b - \partial_X e = 0 \tag{2.27}$$

in $\mathcal{B}_0$, with

$$\mathbb{P}_2 = eI - F^* P + F^* \operatorname{Div} \mathfrak{S} - (\nabla F)^t : \mathfrak{S}. \tag{2.28}$$

The second-rank tensor $\mathbb{P}_2 \in T_x^* \mathcal{B}_0 \otimes T_x \mathcal{B}_0$ is the appropriate form for second-grade materials of the Eshelby stress (see e.g. [16], [19]). In the absence of defects evolving irreversibly within the body, equation (2.27) is nothing but the projection of (2.23) in the reference place by means of the inverse map y^{-1}, when the deformation is sufficiently smooth. On the other hand, when a bulk defect is present and evolves irreversibly in the body, equation (2.27), augmented by a driving force, governs the evolution of the defect itself.

PROOF. When $v = 0$ and $w \neq 0$, it follows that

$$\mathcal{Q} = -\partial_{\dot{y}} \mathcal{L} \cdot F w, \tag{2.29}$$

$$\mathfrak{F} = \mathcal{L} w - \partial_F \mathcal{L}^* (F w) + (\operatorname{Div} \partial_{\nabla F} \mathcal{L})^* (F w) - \partial_{\nabla F} \mathcal{L}^t \nabla (F w). \tag{2.30}$$

Then, by using Theorem 1 and (2.20), equation (2.27) follows. □

For a *homogeneous* second-grade elastic material, for any fixed control part $\mathfrak{b}$, equation (2.27) is implied by the integral balance

$$\frac{d}{dt} \int_{\mathcal{B}_0} f(x) \rho_0 F^* \dot{y} \, dx + \int_{\mathcal{B}_0} f(x) F^* b \, dx + \int_{\partial \mathcal{B}_0} f(x) \mathbb{P}_2 n \, d\mathcal{H}^2 = \frac{1}{2} \int_{\partial \mathcal{B}_0} f(x) \rho_0 |\dot{y}|^2 n \, d\mathcal{H}^2, \tag{2.31}$$

holding for any smooth (scalar) function $x \mapsto f(x)$ with compact support in $\mathcal{B}_0$. Here n is the outward unit normal to the boundary $\partial \mathcal{B}_0$ and $d\mathcal{H}^2$ the two-dimensional measure over $\partial \mathcal{B}_0$. Moreover, if inertia and body forces are absent, one gets

$$\int_{\partial \mathcal{B}_0} f(x) \mathbb{P}_2 n \, d\mathcal{H}^2 = 0. \tag{2.32}$$

3. GLUING TWO SECOND-GRADE BODIES: THE SURFACE HYPERSTRESS

Although the dependence of the energy on the second gradient of deformation ∇F allows one to account for the presence of minute interfaces scattered throughout the body in a regularized manner, there might be circumstances in which additional macroscopic sharp discontinuity surfaces occur. In this case, there is interaction between the minute interfaces and the macroscopic one.

As a paradigmatic example, consider two second-grade bodies glued to each other along a smooth surface Σ by means of a layer of glue with vanishing thickness. A surface energy density ϕ accounts for the properties of the glue distributed along Σ in order to attach the two bodies, and assumed to be made of a second-grade material.

Σ is the surface $\{x \in \mathrm{cl}\,\mathcal{B}_0 : f(x) = 0\}$, with f a scalar function assumed smooth for simplicity. The normal m to Σ is defined by

$$m = \frac{\nabla f(x)}{|\nabla f(x)|} \tag{3.1}$$

and orients Σ locally. Alternatively one may consider a body in a reference place $\mathcal{B}_0$; then one may cut it in two distinct pieces and then glue them. In this case Σ is the image in $\mathcal{B}_0$ of the glued cut in $\mathcal{B}$, obtained by means of the inverse motion y^{-1}.

It is assumed that both the maps $\mathbf{f}^1_{s_1}$ and $\mathbf{f}_s$, defining respectively relabeling and changes in observers, are continuous across Σ together with their derivatives $\mathbf{f}^{1\prime}_{s_1}$ and $\mathbf{f}'_s$ with respect to the relevant parameters (s_1 for $\mathbf{f}^1_{s_1}$ and s for $\mathbf{f}_s$). Moreover, it is assumed that the map

$$\mathcal{B}_0 \ni x \mapsto F = F(x) \in T_y\mathcal{B} \otimes T^*_x\mathcal{B}_0 \tag{3.2}$$

suffers bounded jumps at Σ, together with its gradient, and, as usual, the *surface deformation gradient* $\mathbb{F}$ is defined by $\mathbb{F} := \langle F\rangle(I - m \otimes m)$. It is the value of a C^1 map

$$\Sigma \ni x \mapsto \mathbb{F} = \mathbb{F}(x) \in \mathrm{Hom}(T_x\Sigma, T_y\mathcal{B}). \tag{3.3}$$

The surface gradient of $\mathbb{F}$ defined by $\nabla_\Sigma\mathbb{F} := \nabla_\Sigma\mathbb{F}(x) \in T^*_x\Sigma \otimes T^*_x\Sigma \otimes T_y\mathcal{B}$ is the *second surface gradient of deformation*. In particular, it is immediate to recognize that

$$\nabla_\Sigma\mathbb{F} = (\langle\nabla F\rangle \llcorner (I - m \otimes m))^t \llcorner (I - m \otimes m) + (\langle F\rangle \mathsf{L}) \otimes m + (\langle F\rangle m) \otimes \mathsf{L}. \tag{3.4}$$

The average of F across Σ defines the surface deformation gradient along Σ; its jump characterizes Σ, which is called *coherent* when $[F](I - m \otimes m) = 0$.

It is assumed that bulk stresses and velocity suffer bounded jumps across Σ while bulk forces are continuous throughout $\mathcal{B}_0$. The mass density is constant.

As mentioned above, Σ is assumed to be endowed with its own *surface energy* ϕ given by a sufficiently smooth map ϕ defined by

$$(m, \mathbb{F}, \nabla_\Sigma\mathbb{F}) \mapsto \phi = \phi(m, \mathbb{F}, \nabla_\Sigma\mathbb{F}). \tag{3.5}$$

The dependence on the second surface gradient of deformation $\nabla_\Sigma\mathbb{F}$ allows one to account for minute interfaces distributed over Σ or even for distributed wrinkles in a scattered sense. The presence of the normal m in the list of entries of ϕ accounts for possible *anisotropy* of Σ. In the case of isotropic surfaces, ϕ does not depend on m.

As shown in [4] and [24], in the presence of discontinuity surfaces the arbitrariness of the possible relabeling of $\mathcal{B}_0$ has to be restricted. I indicate by $s_1 \mapsto \hat{\mathbf{f}}^1_{s_1} \in \mathrm{SDiff}(\mathcal{B}_0)$ the 'restricted' relabeling defined by the properties listed below.

(1) The field $\mathcal{B}_0 \ni x \mapsto w := \hat{\mathbf{f}}^{1\prime}_0(x)$ is of class $C^1(\mathcal{B}_0)$.
(2) Each $\hat{\mathbf{f}}^1_{s_1}$ preserves the elements of area of Σ: if dA is the element of area of Σ in $\mathcal{B}_0$, then $dA = \hat{\mathbf{f}}^{1*}_{s_1} dA$, where the asterisk indicates push-forward.

(3) $(\nabla w)m = 0$ at $x \in \Sigma$.
(4) $\nabla_\Sigma \mathsf{v}_m = 0$, with $\mathsf{v}_m = w \cdot m$.

DEFINITION 3. *$\mathfrak{X}$ is a vector density over Σ defined by*

$$\begin{aligned}\mathfrak{X} := {} & -\phi(I - m \otimes m)w + (\partial_{\mathbb{F}}\phi)^*(v - \langle F\rangle w) - (\mathrm{Div}_\Sigma\, \partial_{\nabla_\Sigma \mathbb{F}}\phi)^*(v - \langle F\rangle w) \\ & + (\partial_{\nabla_\Sigma \mathbb{F}}\phi)^t \nabla_\Sigma (v - \langle F\rangle w) - (\partial_m \phi \otimes m)w. \end{aligned} \tag{3.6}$$

At each x, the vector $\mathfrak{X}$ is the surface counterpart of $\mathfrak{F}$. There is no surface counterpart of $\mathcal{Q}$ because Σ has no surface inertia of its own.

The projection of the vector $\partial_F \mathcal{L}^*(v - Fw)$ along the normal n is the *power* of the tension Pn in the difference between the virtual velocity v and the push-forward in $\mathcal{B}$ of the virtual rate w of material relabeling. Such a difference of velocities is a relative velocity; for this reason I call the power developed in a relative velocity or in its gradient *relative power*. This expression is also used below for the power of surface stresses. In fact, an analogous meaning can be attributed to the surface vector field $x \mapsto (\partial_{\mathbb{F}}\phi)^*(v - \langle F\rangle w)$. Its projection along any normal n to a generic smooth curve over Σ, with n in the tangent plane to Σ at x, is the relative power of the surface traction $(\partial_{\mathbb{F}}\phi)\mathsf{n}$ developed in the difference between the virtual velocity of change in observer v and the push-forward $\langle F\rangle w$ of the virtual velocity of surface relabeling. Analogous interpretations hold for the other terms. Notice that the surface hyperstress $\partial_{\nabla_\Sigma \mathbb{F}}\phi$ appears twice in the definition of $\mathfrak{X}$, because the hyperstress $\partial_{\nabla F} e$ appears twice in the definition of $\mathfrak{F}$. The divergence of $\partial_{\nabla F} e$ is a ‘standard’ stress and is a sort of second order perturbation to $\partial_F e$. The divergence of $\partial_{\nabla_\Sigma \mathbb{F}}\phi$ has the same interpretation with respect to $\partial_{\mathbb{F}}\phi$. On the other hand, the bulk and surface hyperstresses $\partial_{\nabla F} e$ and $\partial_{\nabla_\Sigma \mathbb{F}}\phi$ take into account inhomogeneity effects within the material. At any virtual surface (Cauchy cut) in $\mathcal{B}_0$ with normal n, the stress $(\partial_{\nabla F} e)n$ develops power in the gradients of the rates v and Fw. Moreover, at any curve (surface Cauchy cut) in Σ with normal n, the surface stress $(\partial_{\nabla_\Sigma \mathbb{F}}\phi)\mathsf{n}$ develops power in the surface gradient of v and $\langle F\rangle w$. The gradients of v, Fw and $\langle F\rangle w$ underline inhomogeneities in the virtual rates of changes in observers and relabeling. The variation in space of v and w allows one to account for the hyperstress effects. In fact, in the absence of body forces, homogeneous deformations are universal solutions in the bulk even for second-grade elasticity and no bulk hyperstress is associated with them. More precisely, $\mathfrak{F} \cdot n$ and $\mathfrak{X} \cdot \mathsf{n}$ represent the sums of the referential internal energy flow and the ‘relative power’ of stresses in the bulk and on the surface Σ respectively.

A *part* $\mathfrak{b}_\Sigma$, a generic subset of $\mathcal{B}_0$ with the same geometrical properties of $\mathcal{B}_0$ itself, is said to *cross* Σ when $\partial(\mathfrak{b}_\Sigma \cap \Sigma)$ is a closed curve. In particular here $\mathfrak{b}_\Sigma$ is selected in such a way that $\partial(\mathfrak{b}_\Sigma \cap \Sigma)$ is piecewise smooth and the normal n to it is defined everywhere except at a finite number of points and, at each point of $\partial(\mathfrak{b}_\Sigma \cap \Sigma)$, the vector n belongs to the plane tangent to Σ at the same point. By postulating the integral balance present in the theorem below, I presume that a generalized form of the integral version of (2.18) holds on a generic part $\mathfrak{b}_\Sigma$.

THEOREM 2. *Let $\mathcal{L}$ be invariant under changes in observers and relabeling. Suppose that the integral balance*

$$\frac{d}{dt}\int_{\mathfrak{b}_\Sigma} \mathcal{Q}\, dx + \int_{\partial \mathfrak{b}_\Sigma} \mathfrak{F} \cdot n\, d\mathfrak{H}^2 + \int_{\partial(\mathfrak{b}_\Sigma \cap \Sigma)} \mathfrak{X} \cdot \mathsf{n}\, d\mathcal{H}^1 = 0 \tag{3.7}$$

is also true for any part $\mathfrak{b}_\Sigma$ *of* $\mathcal{B}_0$ *crossing* Σ *and for any choice of the virtual rates* v *and* w*. Then across* Σ *the following interface balances hold:*

$$[P - \mathrm{Div}\, \mathfrak{S}]m + \mathrm{Div}_\Sigma(\mathbb{T} - \mathrm{Div}_\Sigma\, \mathfrak{T}) = 0, \tag{3.8}$$

where

$$\mathbb{T} = -\partial_{\mathbb{F}}\phi \in \mathrm{Hom}(T_x^*\Sigma, T_y^*\mathcal{B}) \cong T_x\Sigma \otimes T_y^*\mathcal{B} \tag{3.9}$$

is the surface Piola–Kirchhoff stress and

$$\mathfrak{T} = -\partial_{\nabla_\Sigma \mathbb{F}}\phi \in T_x\Sigma \otimes T_x\Sigma \otimes T_y^*\mathcal{B} \tag{3.10}$$

the surface hyperstress; moreover, if the surface configurational shear

$$\mathfrak{c}_2 := -\partial_m\phi - \mathbb{T}\langle F\rangle m + (\mathrm{Div}_\Sigma\, \mathfrak{T})(\langle F\rangle m) + \mathfrak{T}(\nabla_\Sigma\langle F\rangle)^t m - \mathfrak{T}\langle F\rangle \mathsf{L} \tag{3.11}$$

is such that the map $x \mapsto \mathfrak{c}_2(x)$ *is of class* $C^1(\mathcal{B}_0)$*, then*

$$m \cdot [\mathbb{P}_2]m + \mathbb{C}_{2\,\tan} \cdot \mathsf{L} + \mathrm{Div}_\Sigma\, \mathfrak{c}_2 = 0, \tag{3.12}$$

with

$$\mathbb{C}_{2\,\tan} := \phi(I - m \otimes m) - \mathbb{F}^*\mathbb{T} + \mathbb{F}^*(\mathrm{Div}_\Sigma\, \mathfrak{T}) - \nabla_\Sigma \mathbb{F}^t : \mathfrak{T}. \tag{3.13}$$

$\mathbb{C}_{2\,\tan}$ is a generalized version of the surface Eshelby stress, introduced here for second-grade materials; it accounts for the surface hyperstress. In (3.7), $d\mathcal{H}^1$ is the one-dimensional measure over the line $\partial(\mathfrak{b}_\Sigma \cap \Sigma)$.

The relation (3.7) states that the rate of the relative power of the momentum is balanced by the flux of the relative power of the stresses (relative in the sense specified above) and the material flux of the elastic energy, under assumption of invariance of the Lagrangian density (see Definition 1). As mentioned above, the integral balance (3.7) can be considered as a *principle of relative virtual power for second-grade elastic Cauchy bodies*. Note that, in postulating (3.7), I account for the conditions (2.19) specifying the requirement of invariance of the Lagrangian. One could omit explicit listing of the constitutive entries of the elastic potential and write the balance of virtual power by inserting directly standard stresses and hyperstresses. In this way, a global expression of the relative 'virtual' power of all actions (inertial and non-inertial) on a part $\mathfrak{b}_\Sigma$, augmented by the material flux of energy due to the permutation of homogeneities, would appear more complicated than the one used here.

When the surface energy is not taken into account, equation (3.12) is explicitly the projection along m of the Weierstrass–Erdmann corner condition for second-grade Cauchy bodies.

PROOF. The velocity $\dot{y}$ suffers bounded jumps across Σ, but, since Σ has no peculiar motion relative to the rest of the body, if a generic part $\mathfrak{b}_\Sigma$ is selected across Σ as a control volume fixed in time, one gets simply

$$\frac{d}{dt}\int_{\mathfrak{b}_\Sigma} \mathcal{Q}\, dx = \int_{\mathfrak{b}_\Sigma} \dot{\mathcal{Q}}\, dx. \tag{3.14}$$

Moreover, from the Gauss theorem it follows that

$$\int_{\partial \mathfrak{b}_\Sigma} \mathfrak{F} \cdot n \, d\mathcal{H}^2 = \int_{\mathfrak{b}_\Sigma} \operatorname{Div} \mathfrak{F} \, dx - \int_{\mathfrak{b}_\Sigma \cap \Sigma} [\mathfrak{F}] \cdot m \, d\mathcal{H}^2 \tag{3.15}$$

and

$$\int_{\partial(\mathfrak{b}_\Sigma \cap \Sigma)} \mathfrak{X} \cdot \mathsf{n} \, d\mathcal{H}^1 = \int_{\mathfrak{b}_\Sigma \cap \Sigma} \operatorname{Div}_\Sigma \mathfrak{X} \, d\mathcal{H}^2. \tag{3.16}$$

By substituting (3.14) and (3.16) in (3.7) and taking into account Theorem 1, one finds that the arbitrariness of $\mathfrak{b}_\Sigma$ implies the *weak local interfacial relative power balance*

$$[\mathfrak{F}] \cdot m + \operatorname{Div}_\Sigma \mathfrak{X} = 0 \tag{3.17}$$

for any choice of v and w. If $w = 0$ and v is arbitrary and *constant*, one deduces from (2.17) and (3.6) that

$$\mathfrak{F} = -P^* v + (\operatorname{Div} \mathfrak{S})^* v, \qquad \mathfrak{X} = -\mathbb{T}^* v + (\operatorname{Div}_\Sigma \mathfrak{T})^* v. \tag{3.18}$$

Substitution in (3.17) implies (3.8) thanks to the arbitrariness of v. Moreover, if $v = 0$ and w is arbitrary and *constant,* one obtains

$$\mathfrak{F} = -\mathbb{P}_2^* w - \rho_0 \tilde{\mathfrak{w}}(y) w, \tag{3.19}$$

$$\mathfrak{X} = -\mathbb{C}_{2\,\tan}^* w - \mathfrak{c}_2 \mathsf{v}_m, \tag{3.20}$$

where the second-order tensor $\mathbb{C}_{2\,\tan}$ and the vector $\mathfrak{c}_2$ are given by (3.13) and (3.11) respectively. Of course, the jump of w across Σ disappears because bulk forces are continuous throughout $\mathcal{B}_0$. When (3.19) and (3.20) are substituted in (3.7), one must take into account first that

$$\operatorname{Div}_\Sigma (\mathbb{C}_{2\,\tan}^* w + \mathfrak{c}_2 \mathsf{v}_m) = w \cdot (\operatorname{Div}_\Sigma \mathbb{C}_{2\,\tan} + (\operatorname{Div}_\Sigma \mathfrak{c}_2) m) \tag{3.21}$$

and

$$m \cdot \operatorname{Div}_\Sigma \mathbb{C}_{2\,\tan} = \mathbb{C}_{2\,\tan} \cdot \mathsf{L}. \tag{3.22}$$

The first relation is immediate because w is constant. To obtain the second relation, one should take into account that, by definition, $\mathbb{C}_{2\,\tan}$ is a superficial second-rank tensor, i.e. $\mathbb{C}_{2\,\tan} m = 0$ at each x. Formula (3.22) holds for any second-rank superficial tensor field over Σ (see Lemma 2 in [4]). Both formulas (3.21) and (3.22) are useful because, to derive (3.12), it is necessary to (i) insert (3.19) and (3.20) in (3.7), (ii) make use of the arbitrariness of w and its continuity across Σ, (iii) evaluate the component along m of the resulting equation, where the term $\rho_0 \tilde{\mathfrak{w}}(y) w$ disappears because it is continuous across Σ. In developing algebra connected with the use of (3.19) and (3.20), one finds the product $\mathfrak{T}^t \nabla_\Sigma (\langle F \rangle w)$. It contributes both to the explicit expression of $\mathbb{C}_{2\,\tan}$ and $\mathfrak{c}_2$. The reason is that since by definition

$$\langle F \rangle = \mathbb{F} + \langle F \rangle m \otimes m, \tag{3.23}$$

it follows that

$$\nabla_\Sigma \langle F \rangle = \nabla_\Sigma \mathbb{F} + ((\nabla_\Sigma \langle F \rangle)^t m) \otimes m - (\langle F \rangle \mathsf{L}) \otimes m - (\langle F \rangle m) \otimes \mathsf{L}. \tag{3.24}$$

The first term of the right-hand side contributes to $\mathbb{C}_{2\,\tan}$ while the other terms appear in the expression of $\mathfrak{c}_2$. In particular, notice that since

$$\nabla_\Sigma (\langle F \rangle w) = (\nabla_\Sigma \langle F \rangle)^t w \tag{3.25}$$

because w is selected constant amid all possibilities, one also gets

$$((\langle F \rangle m) \otimes \mathsf{L})^t w = (\langle F \rangle m) \otimes \nabla_\Sigma \mathsf{v}_m = 0, \tag{3.26}$$

thanks to the requirement (4) in the definition of relabeling with a restriction over Σ. This last statement concludes the proof. □

Theorems analogous to the previous one hold for both simple and complex bodies (see [4] and [24]), that is, for first jet bundle classical field theories. In the relevant proofs one may leave v and w arbitrary and exploit the requirement of invariance of the surface energy under changes in observers and relabeling. Here, one is forced to select v and w constant to eliminate undesired terms like $[\mathfrak{S}]^t \nabla v$, which cannot be eliminated by invariance requirements on ϕ and $\mathcal{L}$.

Following Definition 1, ϕ is called *invariant under changes in observers and relabeling* when $\phi(m, \mathbb{F}, \nabla_\Sigma \mathbb{F}) = \phi((\nabla_\Sigma \mathbf{f}^1)^{-1} m, (\mathrm{grad}_\Sigma \mathbf{f}) \mathbb{F} (\nabla \mathbf{f}^1)^{-1}, \overline{\nabla_\Sigma \mathbb{F}})$, where $\overline{\nabla_\Sigma \mathbb{F}} = ((\mathrm{grad}_\Sigma \mathbf{f}) \lrcorner \nabla_\Sigma \mathbb{F} \llcorner (\nabla \mathbf{f}^1)^{-1})^t \llcorner (\nabla \mathbf{f}^1)^{-1}$. In particular, ϕ is called *objective* when $\mathbf{f}^1_{s_1}$ is the identity and $\mathbf{f}_s$ is an isometry with infinitesimal generator $v = q \times (y - y_0)$, $q \times \in \mathfrak{so}(3)$ (see Corollary 2). If ϕ is objective, its derivative with respect to s, evaluated at $s = 0$, must vanish. A straightforward calculation then implies that

$$\mathrm{skw}(\mathbb{T}\mathbb{F}^* + \mathfrak{T} : \nabla_\Sigma \mathbb{F}^*) = 0. \tag{3.27}$$

4. Point defects

Consider a point defect (an impurity or a vacancy) located at $\bar{x}$ in $\mathcal{B}_0$ when $t = 0$ and assume that it moves in $\mathcal{B}$ relative to $\mathcal{B}$ itself. The relative motion of the defect in $\mathcal{B}$ can be pictured in $\mathcal{B}_0$ by means of the inverse motion y^{-1}. The result is a non-material motion $t \mapsto \bar{x}(t)$ in $\mathcal{B}_0$, characterized by the velocity $\bar{w} := \frac{d}{dt}\bar{x}(t)$.

Special assumptions about the admissible classes of changes in observers and relabeling are here necessary. I presume that

(i) $\lim_{x \to \bar{x}} w(x) = \bar{w}$, at each t,
(ii) $v(y)$ is continuous.

A force $\mathfrak{f}$ (a *driving force*) is power-conjugated with the fictitious (in the sense of being non-material) kinematics $t \mapsto \bar{x}(t)$ in $\mathcal{B}_0$. The evolution of the point defect implies the breaking of material bonds in $\mathcal{B}$ so that $\mathfrak{f}$ should be *purely dissipative* in the sense that

$$\mathfrak{f} \cdot \bar{w} \geq 0 \tag{4.1}$$

for any choice of $\bar{w}$. Equality holds when $\bar{w} = 0$. As a consequence, $\mathfrak{f}$ admits a representation of the type

$$\mathfrak{f} = g(|\bar{w}|)\bar{w}, \tag{4.2}$$

with $g(\cdot)$ a positive definite isotropic scalar-valued function such that $g(0) = 0$. The relation (4.2) is, in fact, a solution to (4.1). Moreover g depends on the modulus $|\bar{w}|$ of $\bar{w}$ rather than $\bar{w}$ itself if one imposes objectivity (in the sense of $SO(3)$ invariance) on g. The driving force $\mathfrak{f}$ is associated with the sole dissipation mechanism occurring. Of course, since the motion $t \mapsto \bar{x}(t)$ is non-material (it is the fictitious representation in $\mathcal{B}_0$ of the real motion of the point defect in $\mathcal{B}$), the driving force $\mathfrak{f}$ is also fictitious and must be expressed in terms of the true stress and hyperstress acting in $\mathcal{B}$ to 'break' the material bounds around the point defect, by allowing it to move.

Below, $\mathfrak{b}_r$ denotes a sphere of radius r centered at $\bar{x}$ during its evolution, so that the boundary $\partial\mathfrak{b}_r$ is endowed with a uniform velocity $\bar{w}$. Consider an arbitrary part $\mathfrak{b}$ of $\mathcal{B}_0$, including $\bar{x}(t)$ in its interior so that there exist $r > 0$ and a ball $\mathfrak{b}_r \subset \mathfrak{b}$ such that $\partial\mathfrak{b}_r \cap \partial\mathfrak{b} = \emptyset$. For any field a which takes values in a linear space and is possibly singular at $\bar{x}$, the integral of a over $\mathfrak{b}$ is understood here in the limit sense

$$\int_{\mathfrak{b}} a(x)\,dx = \lim_{r\to 0}\int_{\mathfrak{b}\setminus\mathfrak{b}_r} a(x)\,dx. \tag{4.3}$$

From now on it is assumed that $\mathcal{Q}$ and $\operatorname{Div}\mathfrak{F}$ are integrable over $\mathfrak{b}$ in the sense above. $\mathcal{Q}$ is then called *regular* and $\mathfrak{F}$ *div-regular*.

For any scalar field a depending on space and time, for $\mathfrak{b}$ around $\bar{x}$ fixed in time, and $\mathfrak{b}_r \subset \mathfrak{b}$ varying in time to 'follow' virtually the motion of the point defect in its representation in $\mathcal{B}_0$, in a time interval in which $\partial\mathfrak{b}_r(t) \cap \partial\mathfrak{b} = \emptyset$, one then gets

$$\frac{d}{dt}\int_{\mathfrak{b}} a(x,t)\,dx = \int_{\mathfrak{b}} \dot{a}(x,t)\,dx - \lim_{r\to 0}\int_{\partial\mathfrak{b}_r(t)} a(x,t)(\bar{w}\cdot n)\,d\mathcal{H}^2. \tag{4.4}$$

Below it is assumed that the velocity field $x \mapsto \dot{y}$, the standard stress and the hyperstress may be singular in principle at $\bar{x}$.

THEOREM 3. *Let $\mathfrak{b}$ be an arbitrary part including in its interior the interstitial point defect located at $\bar{x}$ in $\mathcal{B}_0$. Consider a sphere $\mathfrak{b}_r$ centered at $\bar{x}$ and strictly included in $\mathfrak{b}$. Suppose also that*

$$\frac{d}{dt}\int_{\mathfrak{b}} \mathcal{Q}\,dx + \int_{\partial\mathfrak{b}} \mathfrak{F}\cdot n\,d\mathcal{H}^2 + \mathfrak{f}\cdot w = 0 \tag{4.5}$$

for any possible choice of $\mathfrak{b}$, including $\bar{x}$ as specified above. If $\mathcal{L}$ is invariant under changes in observers and relabeling, covariant pointwise balances for a point defect follow as in the list below:

$$\lim_{r\to 0}\int_{\partial\mathfrak{b}_r} (P - \operatorname{Div}\mathfrak{S})n\,d\mathcal{H}^2 = -\lim_{r\to 0}\int_{\partial\mathfrak{b}_r} (\rho_0\dot{y}\otimes\bar{w})n\,d\mathcal{H}^2, \tag{4.6}$$

$$\mathfrak{f} = \lim_{r\to 0}\int_{\partial\mathfrak{b}_r} (\mathbb{P}_2 - k_{\mathrm{rel}}I)n\,d\mathcal{H}^2, \tag{4.7}$$

with

$$k_{\rm rel} = \frac{1}{2}\rho_0|\dot{y} - F\bar{w}|^2.$$

Covariance is understood here in the sense of invariance with respect to the group of automorphisms of the ambient space and relabeling. Note that (4.5) is always a relative power balance. In the last term the infinitesimal generator v of changes in observer is absent because $\mathfrak{f}$ is configurational and is not influenced by the actions of the automorphisms of the ambient space, where actual places of the body are described.

PROOF. Let $\mathfrak{b}$ and $\mathfrak{b}_r$ be selected around $\bar{x}$ as described above (that is, $\mathfrak{b}_r \subset \mathfrak{b}$ and $\partial\mathfrak{b}_r(t) \cap \partial\mathfrak{b} = \emptyset$ in a given time interval). By the divergence theorem,

$$\int_{\partial\mathfrak{b}} \mathfrak{F}\cdot n\, d\mathcal{H}^2 = \int_{\mathfrak{b}} \mathrm{Div}\,\mathfrak{F}\, dx + \lim_{r\to 0}\int_{\partial\mathfrak{b}_r} \mathfrak{F}\cdot n\, d\mathcal{H}^2. \tag{4.8}$$

Then, by using (4.4) and (4.8), the relation (4.5) reduces to

$$\int_{\mathfrak{b}} (\dot{\mathcal{Q}} + \mathrm{Div}\,\mathfrak{F})\, dx - \lim_{r\to 0}\int_{\partial\mathfrak{b}_r} \mathcal{Q}(\bar{w}\cdot n)\, d\mathcal{H}^2 + \lim_{r\to 0}\int_{\partial\mathfrak{b}_r} \mathfrak{F}\cdot n\, d\mathcal{H}^2 + \mathfrak{f}\cdot w = 0, \tag{4.9}$$

so that Theorem 1 implies

$$\lim_{r\to 0}\int_{\partial\mathfrak{b}_r} (\mathfrak{F}\cdot n - \mathcal{Q}(\bar{w}\cdot n))\, d\mathcal{H}^2 + \mathfrak{f}\cdot w = 0. \tag{4.10}$$

If $v \neq 0$ and $w = 0$, one finds

$$\mathcal{Q} = \rho_0 \dot{y}\cdot v, \qquad \mathfrak{F} = -P^* v + (\mathrm{Div}\,\mathfrak{S})^* v \tag{4.11}$$

so that (4.10) reduces to

$$v\cdot \lim_{r\to 0}\int_{\partial\mathfrak{b}_r} (\rho_0 \dot{y}\otimes\bar{w} + P)n\, d\mathcal{H}^2 = 0 \tag{4.12}$$

as r goes to zero, and (4.6) follows thanks to the arbitrariness of v. The appearance of $\bar{w}$ is due to the assumption made above that $\lim_{x\to\bar{x}} w(x) = \bar{w}$.

When $w \neq 0$ and $v = 0$, one gets

$$\mathcal{Q} = -\rho_0 \dot{y}\cdot F w, \tag{4.13}$$

$$\begin{aligned} \mathfrak{F} &= \mathcal{L}w + P^*(Fw) - (\mathrm{Div}\,\mathfrak{S})^*(Fw) + \mathfrak{S}^t \nabla(Fw) \\ &= \frac{1}{2}\rho_0|\dot{y}|^2 w - \mathbb{P}_2^* w - \mathfrak{w} w. \end{aligned} \tag{4.14}$$

In this case (4.10) reduces to

$$\begin{aligned} \mathfrak{f}\cdot w &+ \lim_{r\to 0}\int_{\partial\mathfrak{b}_r} \rho_0 \dot{y}\cdot Fw(\bar{w}\cdot n)\, d\mathcal{H}^2 \\ &+ \lim_{r\to 0}\int_{\partial\mathfrak{b}_r} \left(\frac{1}{2}\rho_0|\dot{y}|^2 - \mathbb{P}_2^*\right) w\cdot n\, d\mathcal{H}^2 = 0. \end{aligned} \tag{4.15}$$

The term $\lim_{r\to 0}\int_{\partial\mathfrak{b}_r} \mathfrak{w} w\cdot n\ d\mathcal{H}^2$ is not present because the integrand is continuous over $\mathcal{B}_0$. Since $w \to \bar{w}$ as $x \to \bar{x}$, as assumed above, and $\bar{w}$ is arbitrary, (4.15) must be valid under the transformation

$$\bar{w} \mapsto -\bar{w} \tag{4.16}$$

so that, thanks to the arbitrariness of $\bar{w}$, (4.15) becomes

$$\mathfrak{f} = \lim_{r\to 0} \int_{\partial \mathfrak{b}_r} \left(\mathbb{P}_2^* n - \left(\frac{1}{2}\rho_0 |\dot{y}|^2 - \rho_0 \dot{y} \cdot \tilde{w} \right) n \right) d\mathcal{H}^2, \tag{4.17}$$

where $\tilde{w} := F\bar{w}$ is the limiting value of Fw as $x \to \bar{x}$. Equation (4.7) follows by taking into account that

$$\frac{1}{2}\rho_0 |\dot{y}|^2 - \rho_0 \dot{y} \cdot \tilde{w} = k_{\mathrm{rel}} - \frac{1}{2}\rho_0 |\tilde{w}|^2 \tag{4.18}$$

and

$$\lim_{r\to 0} \int_{\partial \mathfrak{b}_r} \rho_0 |\tilde{w}|^2 n \, d\mathcal{H}^2 = \rho_0 |\tilde{w}|^2 \lim_{r\to 0} \int_{\partial \mathfrak{b}_r} n \, d\mathcal{H}^2 = 0. \qquad \square \tag{4.19}$$

$\bar{w}$ is different from zero only when the driving force $\mathfrak{f}$ exceeds a certain threshold beyond which (4.7) becomes the evolution equation

$$g(|\bar{w}|)\bar{w} = \lim_{r\to 0} \int_{\partial \mathfrak{b}_r} (\mathbb{P}_2 - k_{\mathrm{rel}} I) n \, d\mathcal{H}^2. \tag{4.20}$$

A few remarks about the conditions under which the point defect evolves are added below. The issue is completely standard. Let $\mathbf{e}$ be a unit vector attached at $\bar{x}$ so that $\bar{w} = |\bar{w}|\mathbf{e}$. By varying $\mathbf{e}$ within S^2, one finds in principle different strength of the material around $\bar{x}$, due to possible inhomogeneity. A map $F = S^2 \to \mathbb{R}^+$ then describes the distribution of the resistance to the breaking of bonds around $\bar{x}$. Following the standard use one says that $\mathfrak{f}$ is *subcritical* when $\mathfrak{f} \cdot \mathbf{e} < F(\mathbf{e})$ for all $\mathbf{e} \in S^2$, *critical* when there exist some $\mathbf{e} \in S^2$ such that $\mathfrak{f} \cdot \mathbf{e} = F(\mathbf{e})$ while subcritical state is granted for all directions, and *supercritical* when there exists some $\mathbf{e} \in S^2$ such that $\mathfrak{f} \cdot \mathbf{e} > F(\mathbf{e})$. Supercritical behavior along a certain direction $\mathbf{e}$ implies the evolution of the point defect along $\mathbf{e}$. With $f := \mathfrak{f} \cdot \mathbf{e}$, the dissipation $\mathfrak{D}$ along $\mathbf{e}$ is given by $\mathfrak{D}(\mathfrak{f}, \mathbf{e}) = (\mathfrak{f} \cdot \mathbf{e})\bar{w} = g(\bar{w})|\bar{w}|^2$. Also, the amplitude $\bar{w}$ itself is determined by $\mathfrak{f}$ and $\mathbf{e}$ so that one gets $\bar{w} = \tilde{w}(f, \mathbf{e})$, with $\tilde{w}(\cdot, \mathbf{e})$ a strictly increasing function of f. In isotropic conditions $\bar{w} = \tilde{w}(f)$. Supercritical behavior may occur along 'many' $\mathbf{e}$'s. The direction along which the point defect evolves is selected by requiring that the dissipation is maximized, precisely one computes $\max_{\mathbf{e}\in S^2}\{\mathfrak{D}(\mathfrak{f}, \mathbf{e}) \mid \mathfrak{f} \cdot \mathbf{e} > F(\mathbf{e})\}$. In isotropic conditions, the direction along which the point defect evolves is then such that $\mathfrak{f} \cdot \mathbf{e} > \mathfrak{f} \cdot \mathbf{s}$ for any $\mathbf{s} \in S^2$, $\mathbf{s} \neq \mathbf{e}$.

5. Linear inclusions

Consider now a linear rigid inclusion (a reinforcement) across which velocity and stress fields might suffer discontinuities. Such an inclusion is represented in $\mathcal{B}_0$ by a single curve l parametrized by arc length $\mathfrak{s} \in [0, \bar{\mathfrak{s}}]$ and represented by a point valued C^2 map $Z : [0, \bar{\mathfrak{s}}] \to \mathcal{B}_0$ so that the derivative $Z_{,\mathfrak{s}}$ of Z with respect to $\mathfrak{s}$, calculated at $Z(\mathfrak{s})$, is the tangent vector $\mathsf{t}(\mathfrak{s})$ at $Z = Z(\mathfrak{s})$.

It is assumed that both the rate of relabeling w and the rate of change in observers v are continuous in space.

A special class of parts $\mathfrak{b}_{l,r}$ of $\mathcal{B}_0$ is helpful. Each representative $\mathfrak{b}_{l,r}$ of this class is a 'curved' cylinder wrapped around l, obtained by translating a disc D_r of radius r from $Z(\mathfrak{s}_1)$ to $Z(\mathfrak{s}_2)$, two arbitrary points of l with $\mathfrak{s}_1 < \mathfrak{s}_2$, maintaining D_r orthogonal to $\mathfrak{t}$ at each $Z(\mathfrak{s})$ and the center of D_r coinciding with $Z(\mathfrak{s})$.

Consider two (coaxial) 'curved' cylinders $\mathfrak{b}_{l,r_1}$ and $\mathfrak{b}_{l,r_2}$ with $r_1 > r_2$. Even in this case, for any field a which takes values in a linear space and is possibly singular at l, the integral of a over $\mathfrak{b}_{l,r_1}$ is understood here in the limit sense

$$\int_{\mathfrak{b}_{l,r_1}} a(x)\, dx = \lim_{r_2 \to 0} \int_{\mathfrak{b}_{l,r_1} \setminus \mathfrak{b}_{l,r_2}} a(x)\, dx. \tag{5.1}$$

A force field

$$[0, \bar{\mathfrak{s}}] \ni \mathfrak{s} \mapsto \mathfrak{f} = \mathfrak{f}(\mathfrak{s}) \in \mathbb{R}^3 \tag{5.2}$$

acts on l. It is configurational in the sense that it would be conjugated only with the velocity of the line inclusion, in material representation, if the line inclusion itself would move in the actual configuration relative to the rest of the body. By picturing in $\mathcal{B}_0$ a kinematics of this type by means of the inverse motion y^{-1}, in fact, an independent (actually, fictitious) kinematics would appear in $\mathcal{B}_0$ which, on the contrary, would remain fixed for all time. The situation of the evolving point defect would be exactly the same. Such a motion does not occur because the linear inclusion is fixed, but one may imagine that $\mathfrak{f}$ exists even when it is not strong enough to break the material bonds; in this sense $\mathfrak{f}$ develops power here only in the rate of change w of material labels in $\mathcal{B}_0$.

Below I consider $\mathfrak{F}$ to be div-regular in the sense that I presume that the limit

$$\int_{\mathfrak{b}_{\mathcal{J}}} \operatorname{Div} \mathfrak{F}\, dx = \lim_{r \to 0} \int_{\mathfrak{b}_{\mathcal{J}} \setminus \mathfrak{b}_r} \operatorname{Div} \mathfrak{F}\, dx \tag{5.3}$$

exists. This assumption is technical but crucial for the theorem below. In the same sense $\mathcal{Q}$ is assumed to be regular.

THEOREM 4. *Let $\mathfrak{b}_{l,R}$ and $\mathfrak{b}_{l,r}$ be arbitrary curved cylinders wrapped around l, with $R > r$, $\mathfrak{b}_{l,r} \subset \mathfrak{b}_{l,R}$ and $\partial \mathfrak{b}_{l,r} \cap \partial \mathfrak{b}_{l,R} = \emptyset$. Let the assumptions above be valid and also suppose that*

$$\frac{d}{dt} \int_{\mathfrak{b}_{l,r}} \mathcal{Q}\, dx + \int_{\partial \mathfrak{b}_{l,r}} \mathfrak{F} \cdot n\, d\mathcal{H}^2 + \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{f} \cdot w\, d\mathfrak{s} = 0 \tag{5.4}$$

for any possible choice of $\mathfrak{b}_l$. If $\mathcal{L}$ is invariant under changes in observers and relabeling, for any $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathbb{R}^+$, covariant pointwise balances for a point defect follow as in the list below: at each $\mathfrak{s} \in [0, \bar{\mathfrak{s}}]$,

$$\lim_{r \to 0} \int_{\partial D_r} (P - \operatorname{Div} \mathfrak{S}) n\, d\mathcal{H}^1 = - \lim_{r \to 0} \int_{\partial D_r} (\rho_0 \dot{y} \otimes \bar{w}) n\, d\mathcal{H}^1, \tag{5.5}$$

$$\mathfrak{f} = \lim_{r \to 0} \int_{\partial D_r} (\mathbb{P}_2 - k_{\mathrm{rel}} I) n\, d\mathcal{H}^1, \tag{5.6}$$

with $k_{\mathrm{rel}} = \frac{1}{2} \rho_0 |\dot{y} - F \bar{w}|^2$.

The proof follows the lines of the one of Theorem 3. Since here $\mathfrak{b}_{l,r} = D_r \times [\mathfrak{s}_1, \mathfrak{s}_2]$, one takes into account in addition just the arbitrariness of the interval $[\mathfrak{s}_1, \mathfrak{s}_2]$. Beyond thresholds for the driving force $\mathfrak{f}$, the motion of the line defect is activated as in the case of point defects and the evolution of the line defect may generate a crack.

6. Cracks

Within the setting of infinitesimal deformation regime, the influence of strain gradient effects on the propagation of cracks has been discussed in [41] (see also [21]). The relevant nonlinear theory has been developed in [23] as a special case of fracture mechanics in complex bodies because the presence of the second gradient of deformation can be linked to latent effects of complex material substructure in the sense introduced in [2].

In this section the matter is re-discussed: evolution equations of the crack tip are derived by using the results of Theorems 2 and 4 above, and so in an invariant way, completely different than the one in [23]. This follows the path indicated for field theories on the first jet bundle in [25].

A crack occurs in $\mathcal{B}$ while $\mathcal{B}_0$ is free of cracks. Since the crack may open and/or close, the placement map $x \mapsto y(x)$ fails to be one-to-one on a surface Σ in $\mathcal{B}_0$ assumed smooth for simplicity. Unlike the surface Σ described in Section 2, here Σ does not cut the body completely, rather it *ends* within $\mathcal{B}_0$ and its margin, the *tip*, is represented by a simple curve described by a differentiable map $Z : [0, \bar{\mathfrak{s}}] \to \mathcal{B}_0$.

The lateral margins of the crack are endowed with a *surface energy* ϕ which is assumed *constant*, a special case of the one in Section 2. Along $\mathcal{J}$ a vector field $\mathfrak{s} \mapsto \mathsf{n}(\mathfrak{s})$ is defined and is such that at each $\mathfrak{s}$ the vector n is normal to the tangent $\mathsf{t}(\mathfrak{s})$. The scalar curvature of $\mathcal{J}$ is $\mathfrak{k} := -\mathsf{t}_{,\mathfrak{s}\mathfrak{s}} \cdot \mathsf{n}$.

When the crack evolves in the current configuration, there is a monotone evolution (without normal motion) of Σ in time in $\mathcal{B}_0$. $\mathcal{J}$ moves relative to $\mathcal{B}_0$ and pieces of $\Sigma(t)$ far from $\mathcal{J}$ remain at rest. For instants $t_1, t_2 \in [0, \bar{t}]$ with $t_1 \le t_2$, one gets $\Sigma(t_1) \subseteq \Sigma(t_2)$.

During the evolution of the crack in the current place, the corresponding evolution of $\mathcal{J}(t)$ is described by $Z : [0, \bar{\mathfrak{s}}] \times [0, \bar{t}] \to \mathcal{B}_0$, and

$$v_{\text{tip}} = \frac{\partial Z(\mathfrak{s}, t)}{\partial t} \tag{6.1}$$

is the velocity of the tip. Precisely, here v_{tip} is of the form $v_{\text{tip}} = V\mathsf{n}$ with $V = v_{\text{tip}} \cdot \mathsf{n}$.

As the surface energy is constant, the surface vector density in Definition 3 reduces to

$$\mathfrak{X} := -\phi(I - m \otimes m)w. \tag{6.2}$$

Σ is not endowed with normal motion relative to the rest of the body so that the proposition below holds and is a special case of Theorem 2.

PROPOSITION 1. *Along the lateral margins of a closed crack in a second-grade material the following balance equations hold:*

$$[P - \operatorname{Div} \mathfrak{S}]m = 0, \tag{6.3}$$

$$m \cdot [\mathbb{P}_2]m + \phi(\mathcal{K} - m \cdot \mathsf{L}m) = 0. \tag{6.4}$$

To derive balance equations along the tip, Theorem 4 is helpful; however, here one must consider that the tip of the crack, a line in three dimensions, is the margin of a surface so that one must consider the contribution of the constant surface energy ϕ at the tip. Another difference from the treatment of linear inclusions presented above is that one has to consider an additional *line energy* along the tip, indicated by λ_{tip}, which is the energy of the material bonds at the tip and can be considered constant along the tip itself. The presence of the line energy suggests introducing the line counterpart of the vector densities $\mathfrak{F}$ and $\mathfrak{X}$ along the tip, namely the scalar density $\mathfrak{Y}$ defined by

$$\mathfrak{Y} = \lambda_{\mathrm{tip}} \mathfrak{t} \cdot v_{\mathrm{tip}}. \tag{6.5}$$

Special parts $\mathfrak{b}_{\mathcal{J},r}$ of $\mathcal{B}_0$ are helpful in analyzing the balance at the tip, each having the same geometry of $\mathfrak{b}_{l,r}$, defined in the previous section.

Additionally, when the crack evolves, its motion has a picture in $\mathcal{B}_0$ resulting in the evolution of Σ, which is an 'additional' (independent) kinematics in $\mathcal{B}_0$. A driving force $\mathfrak{f}$ is then power-conjugated with v_{tip}. In contrast to the previous section in which $\mathfrak{f}$ is postulated, now one can say that it exists a priori because $\mathfrak{f} \cdot v_{\mathrm{tip}}$ is at each point the power developed in breaking bonds. $\mathfrak{f}$ is intrinsically dissipative so that $\mathfrak{f} = g(|v_{\mathrm{tip}}|)v_{\mathrm{tip}}$, with g a positive definite function.

Below, two arbitrary curved cylinders $\mathfrak{b}_{\mathcal{J},R}$ and $\mathfrak{b}_{\mathcal{J},r}$, $R > r$, wrapped around the tip, are selected. Let also $\mathfrak{b}_{\mathcal{J},R}$ be fixed in time while $\mathfrak{b}_{\mathcal{J},r}$ be time-varying. At $t = 0$ they are co-axial. Attention is here focused on a time interval in which $\partial\mathfrak{b}_{\mathcal{J},R} \cap \partial\mathfrak{b}_{\mathcal{J},r}(t) = \emptyset$.

THEOREM 5 (balances at the tip). *Let the balance*

$$\frac{d}{dt}\int_{\mathfrak{b}_{\mathcal{J}}} \mathcal{Q}\, dx + \int_{\partial\mathfrak{b}_{\mathcal{J}}} \mathfrak{F} \cdot n\, d\mathfrak{H}^2 + \int_{\partial\mathfrak{b}_{\mathcal{J}}\cap\mathcal{C}} \mathfrak{X} \cdot \mathfrak{m}\, d\mathfrak{H}^1 + \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{f} \cdot v_{\mathrm{tip}}\, d\mathfrak{s} + \mathfrak{Y}(\mathfrak{s}_2) - \mathfrak{Y}(\mathfrak{s}_1) = 0 \tag{6.6}$$

be valid for any time interval in which $\partial\mathfrak{b}_{\mathcal{J},R} \cap \partial\mathfrak{b}_{\mathcal{J},r} = \emptyset$. If $\mathcal{L}$ is invariant under changes in observers and relabeling, pointwise balances along $\mathcal{J}$ follow as in the list below:

$$\lim_{r\to 0}\int_{\partial D_r} Pn\, d\mathfrak{H}^1 = -\lim_{r\to 0}\int_{\partial D_r} (\rho_0 \dot{y} \otimes v_{\mathrm{tip}})n\, d\mathfrak{H}^1, \tag{6.7}$$

where D_r is the cross-section, a disc, of $\partial\mathfrak{b}_{\mathcal{J},r}$, and

$$\mathfrak{f} = J - \phi - \lambda_{\mathrm{tip}}\mathfrak{k}, \tag{6.8}$$

where

$$J = \mathsf{n} \cdot \lim_{r\to 0}\int_{\partial D_r} (\mathbb{P}_2 - k_{\mathrm{rel}} I)n\, d\mathfrak{H}^1 \tag{6.9}$$

and

$$k_{\mathrm{rel}} = \frac{1}{2}\rho_0 |\dot{y} - F v_{\mathrm{tip}}|^2. \tag{6.10}$$

If $v_{\mathrm{tip}} \neq 0$, then (6.8) *becomes $g(|v_{\mathrm{tip}}|)v_{\mathrm{tip}} = J - \phi - \lambda_{\mathrm{tip}}\mathfrak{k}$.*

The proof follows the lines of the proofs of Theorems 3 and 4.

7. ADDITIONAL REMARKS

Second-grade elasticity and the related (in a sense subsequent) second-grade plasticity find natural applications in modeling materials in which internal lengths generating non-local effects play a role. A critical review by Hutchinson [13] lists prominent examples including nano-indentation, torsion of copper wires, dislocation clustering (especially in thin films), particle-reinforced composites. In particular, if one considers a composite made of a matrix reinforced by spherical particles, an energy including "strain gradients will involve the spacing between the spheres as a constitutive parameter. An elasticity theory based on this energy density necessarily involves higher order stresses, which are the stress-like quantities conjugated with strain gradients" (see [13, p. 234]). Another prominent case is the one of polymeric bodies, specifically nematic elastomers. As shown in [31] it is possible to establish a link between the gradient of the vector field describing locally the 'orientation' of the polymeric chains and the curvature tensor, giving rise to strain-gradient elasticity. In this case the elastomeric substructure is considered *latent* in the sense of Capriz [2].

To visualize a concrete example in which Theorem 2 applies, consider two composites reinforced by spherical particles and attach them along a surface Σ by means of a polymeric glue composing a thin film. The bulk hyperstress $\mathfrak{S}$ is generated in the bulk by the 'spherical' phase while the surface hyperstress $\mathfrak{T}$ is generated in the glue by the polymeric chains.

The introduction of the surface hyperstress and the covariant derivation of related surface balances of interactions is the main result of this paper. It is supplemented by the (covariant) analysis of the action of the bulk hyperstress on linear inclusions, point defects and cracks.

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