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Special functions. — *Remarkable identities related to the Riemann zeta function*, by GABRIELE DI CERBO, communicated on 11 May 2007.

ABSTRACT. — We derive some formulas and identities for the Riemann zeta function related to the celebrated Ramanujan formula.

KEY WORDS: Riemann zeta function; Ramanujan identities; Bernoulli numbers.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 11Y60, 11M99.

1. INTRODUCTION

In Ramanujan's Notebooks [1] we can find some identities for the Riemann zeta function at odd integers, e.g.

$$\zeta(3) = \frac{7}{180}\pi^3 - 2\sum_{k=1}^{\infty} \frac{1}{k^3(e^{2\pi k}-1)}. \tag{1}$$

The general expression is also due to Ramanujan and reads as follows:

THEOREM 1.1. *Let α and β be positive numbers such that $\alpha\beta = \pi^2$. Let n be a positive integer. Then*

$$\begin{aligned}\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{1}{k^{2n+1}(e^{2\alpha k}-1)}\right\} &= (-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{1}{k^{2n+1}(e^{2\beta k}-1)}\right\} \\ &\quad - 2^{2n}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k.\end{aligned} \tag{2}$$

From this identity we obtain the Riemann zeta function at odd integers as a rational multiple of π^{2n+1} plus, in general, two other series. This formula is the analogue of Euler's one at even integers. The most natural substitution is $\alpha = \pi$ and $\beta = \pi$; however, we obtain interesting results only if $n = 2m-1$. In this case we have a generalization of (1),

$$\zeta(4m-1) = -2\sum_{k=1}^{\infty}\frac{1}{k^{4m-1}(e^{2\pi k}-1)} - \frac{1}{2}(2\pi)^{4m-1}\sum_{k=0}^{2m}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{4m-2k}}{(4m-2k)!}.$$

On the other hand, the computation of (2) when n is even and $\alpha = \beta$ gives no information about the zeta function. It is easy to notice that all the terms with the zeta function and the other series vanish and we have only the sum with the Bernoulli numbers, so that

$$\sum_{k=0}^{2m+1} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{4m+2-2k}}{(4m+2-2k)!} = 0.$$

In order to obtain some identities when n is even we are forced to choose other substitutions. For example, if $n = 2$, $\alpha = \pi/2$ and $\beta = 2\pi$ we have

$$\zeta(5) = \frac{\pi^5}{270} - \frac{32}{15} \sum_{k=1}^{\infty} \frac{1}{k^5(e^{\pi k} - 1)} + \frac{2}{15} \sum_{k=1}^{\infty} \frac{1}{k^5(e^{4\pi k} - 1)}.$$

This relation links the zeta function to two series which differ only in exponents. Plouffe [2], [3], inspired by the identity (1), discovered several remarkable identities using high-precision software. For example,

$$\zeta(5) = \frac{\pi^5}{294} - \frac{72}{35} \sum_{k=1}^{\infty} \frac{1}{k^5(e^{2\pi k} - 1)} - \frac{2}{35} \sum_{k=1}^{\infty} \frac{1}{k^5(e^{2\pi k} + 1)}. \tag{3}$$

A generalization of this type of identities (see Theorem 2.1 below) has been proved by Vepstas [4] using complex-analytic techniques. In this paper we derive the same result as a direct consequence of Theorem 1.1.

Plouffe also gives other identities where the Riemann zeta function is not related to powers of π, e.g.

$$\begin{aligned} \zeta(5) = 24 \sum_{k=1}^{\infty} \frac{1}{k^5(e^{\pi k} - 1)} &- \frac{259}{10} \sum_{k=1}^{\infty} \frac{1}{k^5(e^{2\pi k} - 1)} \\ &- \frac{1}{10} \sum_{k=1}^{\infty} \frac{1}{k^5(e^{4\pi k} - 1)}. \end{aligned} \tag{4}$$

We show that these identities directly follow from Ramanujan's formula. The above results can be used to prove most of Plouffe's identities; we have not attacked here the case in which the series have a square root exponent.

2. Plouffe's identities

The main result of this paper is the following theorem which proves one set of Plouffe's identities; in particular, it is a generalization of (3).

THEOREM 2.1. *Let m be a positive integer. Then*

$$\begin{aligned}\zeta(4m+1)[1+(-4)^m-2^{4m+1}] &= 2\sum_{k=1}^{\infty}\frac{1}{k^{4m+1}(e^{2\pi k}+1)}\\ &\quad + 2[2^{4m+1}-(-4)^m]\sum_{k=1}^{\infty}\frac{1}{k^{4m+1}(e^{2\pi k}-1)}\\ &\quad + \frac{1}{2}(2\pi)^{4m+1}\sum_{k=0}^{2m+1}(-4)^k\frac{B_{2k}}{(2k)!}\frac{B_{4m+2-2k}}{(4m+2-2k)!}\\ &\quad + (2\pi)^{4m+1}\sum_{k=0}^{m}(-4)^{m+k}\frac{B_{4k}}{(4k)!}\frac{B_{4m+2-4k}}{(4m+2-4k)!}.\end{aligned}$$

PROOF. In order to simplify the computation we define the quantities

$$S_\lambda^{\pm}=\sum_{k=1}^{\infty}\frac{1}{k^{4m+1}(e^{\lambda\pi k}\pm 1)},$$

where λ is a complex number, and

$$A=\sum_{k=0}^{2m+1}(-4)^k\frac{B_{2k}}{(2k)!}\frac{B_{4m+2-2k}}{(4m+2-2k)!}.$$

Letting $n=2m$ and

$$\alpha=\frac{\pi}{2}(1-i), \qquad \beta=\pi(1+i)$$

in Theorem 1.1 we obtain

$$\begin{aligned}(5)\qquad \left(\frac{\pi}{2}(1-i)\right)^{-2m}&\left\{\frac{1}{2}\zeta(4m+1)+S_{1-i}^{-}\right\}\\ &= (-\pi(1+i))^{-2m}\left\{\frac{1}{2}\zeta(4m+1)+S_{2+2i}^{-}\right\}\\ &\quad - 2^{4m}\left(\frac{\pi}{2}(1-i)\right)^{2m+1}\sum_{k=0}^{2m+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{4m+2-2k}}{(4m+2-2k)!}(2i)^k.\end{aligned}$$

First we evaluate the series related to β and we note that

$$(6)\qquad S_{2+2i}^{-}=S_2^{-}.$$

The series related to α is real and it can be written as

$$\begin{aligned}(7)\qquad S_{1-i}^{-} &= \sum_{k=1}^{\infty}\frac{1}{k^{4m+1}((-1)^k e^{\pi k}-1)}\\ &= 2^{-4m-1}S_2^{-}-\sum_{k=1}^{\infty}\frac{1}{(2k-1)^{4m+1}(e^{\pi(2k-1)}+1)}\\ &= 2^{-4m-1}S_2^{-}-S_1^{+}+2^{-4m-1}S_2^{+}.\end{aligned}$$

We want to write S_1^+ as a linear combination of S_2^- and S_2^+. Using again the Ramanujan formula with the substitution $n = 2m$, $\alpha = \pi/2$ and $\beta = 2\pi$ it follows that

$$\left(\frac{2}{\pi}\right)^{2m}\left\{\frac{1}{2}\zeta(4m+1) + S_1^-\right\} = \left(\frac{1}{2\pi}\right)^{2m}\left\{\frac{1}{2}\zeta(4m+1) + S_4^-\right\} - 2^{2m-1}\pi^{2m+1}A.$$

Solving this identity for S_1^- we obtain

$$S_1^- = \zeta(4m+1)(2^{-4m-1} - 2^{-1}) + 2^{-4m}S_4^- - \frac{1}{2}\pi^{4m+1}A. \tag{8}$$

We observe that the series S_2^- and S_4^- have the property

$$2S_2^- = S_1^- - S_1^+, \quad 2S_4^- = S_2^- - S_2^+,$$

as follows from the identity $x^2 - 1 = (x-1)(x+1)$, respectively with $x = e^{\pi k}$ and $x = e^{2\pi k}$. The above considerations and identity (8) allow us to write

$$S_1^+ = \zeta(4m+1)(2^{-4m-1} - 2^{-1}) + (2^{-4m-1} - 2)S_2^- - 2^{-4m-1}S_2^+ - \frac{1}{2}\pi^{4m+1}A. \tag{9}$$

By (7) and (9) we get

$$S_{1-i}^- = \zeta(4m+1)(2^{-1} - 2^{-4m-1}) + 2^{-4m}S_2^+ + 2S_2^- + \frac{1}{2}\pi^{4m+1}A. \tag{10}$$

It remains to evaluate only the last term, i.e. the sum with the Bernoulli numbers

$$\begin{aligned} &-2^{4m}\left(\frac{\pi}{2}(1-i)\right)^{2m+1}\sum_{k=0}^{2m+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{4m+2-2k}}{(4m+2-2k)!}(2i)^k \\ &\quad = -2^{4m}\left(\frac{\pi}{2}(1-i)\right)^{2m+1}\sum_{k=0}^{m}(-1)^k 2^{2k}\frac{B_{4k}}{(4k)!}\frac{B_{4m+2-4k}}{(4m+2-4k)!} \\ &\quad\quad + 2^{4m}\left(\frac{\pi}{2}(1-i)\right)^{2m+1} i\sum_{k=0}^{m}(-1)^k 2^{2k+1}\frac{B_{4k+2}}{(4k+2)!}\frac{B_{4m-4k}}{(4m-4k)!} \\ &\quad = -2^{2m}\pi^{2m+1}(1-i)^{2m}\sum_{k=0}^{m}(-4)^k\frac{B_{4k}}{(4k)!}\frac{B_{4m+2-4k}}{(4m+2-4k)!} \end{aligned} \tag{11}$$

where the last identity is obtained via the formula

$$\sum_{k=0}^{m}(-1)^k 2^{2k+1}\frac{B_{4k+2}}{(4k+2)!}\frac{B_{4m-4k}}{(4m-4k)!} = -\sum_{k=0}^{m}(-1)^k 2^{2k}\frac{B_{4k}}{(4k)!}\frac{B_{4m+2-4k}}{(4m+2-4k)!}, \tag{12}$$

to be proved later. Substituting (6), (10) and (11) in (5) yields

$$\begin{aligned}\left(\frac{\pi}{2}(1-i)\right)^{-2m} &\left\{\zeta(4m+1)(1-2^{-4m-1})+2^{-4m}S_2^{+}+2S_2^{-}+\frac{1}{2}\pi^{4m+1}A\right\}\\ &= (-\pi(1+i))^{-2m}\left\{\frac{1}{2}\zeta(4m+1)+S_2^{-}\right\}\\ &\quad - 2^{2m}\pi^{2m+1}(1-i)^{2m}\sum_{k=0}^{m}(-4)^k\frac{B_{4k}}{(4k)!}\frac{B_{4m+2-4k}}{(4m+2-4k)!}.\end{aligned}$$

If we multiply both members of the latter expression by $(\frac{\pi}{2}(1-i))^{2m}$ we have

$$\begin{aligned}&\zeta(4m+1)(1-2^{-4m-1})+2^{-4m}S_2^{+}+2S_2^{-}+\frac{1}{2}\pi^{4m+1}A\\ &\quad= (-4)^{-m}\left\{\frac{1}{2}\zeta(4m+1)+S_2^{-}\right\}-\pi^{4m+1}\sum_{k=0}^{m}(-4)^{m+k}\frac{B_{4k}}{(4k)!}\frac{B_{4m+2-4k}}{(4m+2-4k)!}.\end{aligned}$$

Theorem 2.1 follows at once by solving the above equality for $\zeta(4m+1)$.

Thus, we only have to prove equality (12): let $\alpha = \pi/2(1-i)$ and $\beta = \pi(1+i)$ in (2), and again let $\alpha = \pi(1-i)$ and $\beta = (\pi/2)(1+i)$ in (2); this yields two different expressions,

$$\begin{aligned}\left(\frac{2}{\pi}\right)^{2m}\left(\frac{i}{2}\right)^{m}&\left\{\frac{1}{2}\zeta(4m+1)+S_{1-i}^{-}\right\}\\ &= \left(\frac{1}{\pi}\right)^{2m}\left(-\frac{i}{2}\right)^{m}\left\{\frac{1}{2}\zeta(4m+1)+S_2^{-}\right\}\\ &\quad - 2^{4m}\left(\frac{\pi}{2}(1-i)\right)^{2m+1}\sum_{k=0}^{2m+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{4m+2-2k}}{(4m+2-2k)!}(2i)^k\end{aligned}$$

and

$$\begin{aligned}\left(\frac{1}{\pi}\right)^{2m}\left(\frac{i}{2}\right)^{m}&\left\{\frac{1}{2}\zeta(4m+1)+S_2^{-}\right\}\\ &= \left(\frac{2}{\pi}\right)^{2m}\left(-\frac{i}{2}\right)^{m}\left\{\frac{1}{2}\zeta(4m+1)+S_{1-i}^{-}\right\}\\ &\quad - 2^{4m}(\pi(1-i))^{2m+1}\sum_{k=0}^{2m+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{4m+2-2k}}{(4m+2-2k)!}\left(\frac{i}{2}\right)^k.\end{aligned}$$

If we take the product of the second one by $(-1)^m$ we note that they differ only in the two sums with the Bernoulli numbers. Therefore, if we sum these two expressions, all the terms with $\zeta(4m+1)$, S_2^- and S_{1-i}^- cancel, so we have

$$-2^{4m}\left(\frac{\pi}{2}(1-i)\right)^{2m+1}\sum_{k=0}^{2m+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{4m+2-2k}}{(4m+2-2k)!}(2i)^k$$
$$-2^{4m}(-1)^m(\pi(1-i))^{2m+1}\sum_{k=0}^{2m+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{4m+2-2k}}{(4m+2-2k)!}\left(\frac{i}{2}\right)^k=0.$$

Considering only the real part or analogously the sums with even index only we get

$$2^{-2m-1}\sum_{k=0}^{m}(-4)^k\frac{B_{4k}}{(4k)!}\frac{B_{4m+2-4k}}{(4m+2-4k)!}=(-1)^{m+1}\sum_{k=0}^{m}(-1)^k\frac{B_{4k}}{(4k)!}\frac{B_{4m+2-4k}}{(4m+2-4k)!}2^{-2k}.$$

After the change of variable $k=m-j$ in the second sum we obtain

$$(-1)^{m+1}\sum_{k=0}^{m}(-1)^k\frac{B_{4k}}{(4k)!}\frac{B_{4m+2-4k}}{(4m+2-4k)!}2^{-2k}$$
$$=-2^{-2m-1}\sum_{j=0}^{m}(-1)^j\frac{B_{4m-4j}}{(4m-4j)!}\frac{B_{4j+2}}{(4j+2)!}2^{2j+1}.$$

By the above two identities, (12) holds. □

Theorem 2.1 allows us to derive the following result that generalizes identity (4). We use the same notation for the series $S_\lambda^\pm$ and for the quantity A as in the proof of Theorem 2.1; moreover, we define

$$B=\sum_{k=0}^{m}(-4)^{m+k}\frac{B_{4k}}{(4k)!}\frac{B_{4m+2-4k}}{(4m+2-4k)!}.$$

COROLLARY 2.2. *Let m be a positive integer. Then*

$$\zeta(4m+1)\left[\frac{1}{2}A((-4)^m-2^{4m})+B(4^{2m}-1)\right]$$
$$=-\left(\frac{1}{2}A+B\right)2^{4m+1}S_1^-+A(2^{4m+1}-(-4)^m-1)S_2^-+(2B-A)S_4^-.$$

PROOF. We rewrite the expression given by Theorem 2.1 using the equality $S_2^+=S_2^- - 2S_4^-$:

$$(2\pi)^{4m+1}\left(\frac{1}{2}A+B\right)=\zeta(4m+1)[1+(-4)^m-2^{4m+1}]-2[2^{4m+1}-(-4)^m+1]S_2^-+4S_4^-.$$

Analogously expression (8) can be written as

$$(2\pi)^{4m+1}\frac{1}{2}A=-\zeta(4m+1)(2^{4m}-1)-2^{4m+1}S_1^-+2S_4^-.$$

Combining the two relations we obtain the assertion. □

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