



**Differential geometry.** — *Conservative principle for differential forms*, by JUN MASAMUNE, communicated on 11 May 2007.

ABSTRACT. — Motivated by recent developments on the conservative principle for differential forms, we study sufficient conditions for a manifold to satisfy that principle.

KEY WORDS: Cauchy problem; conservative principle; differential forms.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 58J35, 58J65.

## 1. INTRODUCTION

Let  $M$  be a connected smooth Riemannian manifold without boundary. We say that  $M$  is *stochastically complete*, or  $M$  *satisfies the conservative principle*, if any constant function is stable under the action of the heat semigroup associated to the minimal self-adjoint Laplace–Beltrami operator, i.e., the equality

$$(1) \quad \langle e^{-t\Delta} f, h \rangle = \langle f, h \rangle$$

holds for every  $t > 0$ ,  $f \in C_0^\infty(M)$  and any constant function  $h$ . The Gaussian integral shows that a Euclidean space of any dimension is stochastically complete, but there exists a geodesically complete but stochastically incomplete Riemannian manifold [1]. In order to ensure the stochastic completeness, one needs to control the Brownian motion at infinity by imposing a condition either on the volume growth of geodesic balls or on the Ricci curvature (see e.g. [8] for an extensive overview of the theory).

Let  $A^q$  be the set of differential forms of degree  $q$  ( $A_0^q$  is the set of  $q$ -forms with compact support), and  $\Delta_q$  be the Hodge Laplacian acting on  $A^q$ . While both the stochastic completeness and Hodge theory have been studied with considerable efforts, there has been no notion of *conservative principle on  $A^q$* , until Vesentini’s recent work [19]. In that paper, he extended the notion of conservative principle from functions to  $A^q$  as follows:

DEFINITION 1. *We say the conservative principle holds on  $A^q$  if the equality*

$$(2) \quad \langle e^{-t\Delta_q} \psi, \eta \rangle = \langle \psi, \eta \rangle$$

*holds for every  $t > 0$ ,  $\psi \in A_0^q$ , and harmonic form  $\eta \in L^\infty(A^q)$ .*

This definition has two nice features: first, if  $q = 0$ , it reduces to the stochastic completeness (because by  $\alpha$  being harmonic, we mean  $d\alpha = 0$  and  $\delta\alpha = 0$ , and  $dh = 0$  implies that  $h$  is constant), and secondly, it is stated in terms of harmonic forms, so actually we will study the “conservativeness” of harmonic forms, which play the central role in

Hodge theory. Motivated by this background, in this article we will show the following results.

**THEOREM 1.** *Let  $M$  be an  $n$ -dimensional complete Riemannian manifold without boundary. If  $M$  is stochastically complete and the Weitzenböck tensor on  $A^q$  is bounded below for some  $0 \leq q \leq n$ , then the conservative principle holds on  $A^q$ .*

Since the Weitzenböck tensor on  $A^1$  coincides with the Ricci curvature, and the condition of Ricci curvature being bounded below implies the stochastic completeness [20], we have

**COROLLARY 1.** *If the Ricci curvature of a complete manifold  $M$  without boundary is bounded below, then the vector fields on  $M$  are conservative.*

The significant difference of the conservativeness for  $q = 0$  and for  $q > 0$  is the curvature condition. The reason why we need the curvature condition can be seen in Definition 1; indeed, in order to make the left hand side of (2) converge, we need

$$(3) \quad e^{-t\Delta_q}(A_0^q) \subset L^1,$$

which is true when  $q = 0$  but not clear for  $q > 0$ . Therefore, we will apply a semigroup domination theorem (e.g. [6], [9], [10], and [15]) to bound the heat kernel on  $A^q$  and ensure (3), provided the Weitzenböck tensor on  $A^q$  is bounded below.

We will also study these problems on incomplete manifolds. For that purpose, we extend the semigroup domination theorem to incomplete manifolds by studying the essential self-adjointness of the Hodge Laplacian, which is of interest in its own right:

**PROPOSITION 1.** *Let  $M$  be a complete manifold without boundary and  $\Sigma$  be its closed submanifold. The Hodge Laplacian with domain  $A_0(M \setminus \Sigma)$ , the set of smooth forms with compact support in  $M \setminus \Sigma$ , is essentially self-adjoint if and only if  $\Sigma$  has codimension greater than 3.*

This is a generalization of the corresponding result for the Laplace–Beltrami operator acting on functions [4], [12]. Since the semigroup domination theorem holds true if  $M$  is stochastically complete and if the Hodge Laplacian is essentially self-adjoint, we have

**COROLLARY 2.** *Let  $M$  be a complete Riemannian manifold without boundary and  $\Sigma \subset M$  be a closed submanifold with codimension greater than 3. If  $M$  satisfies the assumptions in Theorem 1 and Corollary 1, respectively, then the respective conclusions hold true for the incomplete manifold  $M \setminus \Sigma$ .*

Since the stochastic completeness of  $M$  is equivalent to the uniqueness of the bounded solution to the Cauchy problem for the heat equation if  $q = 0$ , it would be interesting to study the corresponding problem for  $q > 0$ . Regarding this problem, we will show

**THEOREM 2.** *Let  $M$  be a complete Riemannian manifold without boundary. Assume that there exists a point  $x_0 \in M$  so that*

$$(V) \quad \int^{\infty} \frac{r \, dr}{\log \mu(B(x_0, r))} = \infty.$$

Then every bounded solution  $\alpha(x, t)$  to the Cauchy problem

$$(4) \quad \begin{cases} \frac{\partial \alpha}{\partial t} + \Delta_q \alpha = 0, \\ \alpha|_{t=0_+} = 0 \quad (\text{in the sense of } L^2_{\text{loc}}(M)), \end{cases}$$

in  $M \times (0, T)$  is 0.

The volume growth condition (V) is sufficient for the stochastic completeness [8].

Finally, let us remark that all of our results extend to complete manifolds with boundary with the Neumann boundary condition (except Proposition 1, they fail for the Dirichlet boundary condition), and Theorems 1 and 2, and Corollary 1, extend to the Bismut–Witten Laplacian on a weighted manifold with associated Weitzenböck tensor [7].

## 2. NOTATIONS

We denote by  $\langle \alpha, \beta \rangle$  the inner product of  $q$ -forms  $\alpha, \beta$  with compact support, that is,

$$(5) \quad \langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta,$$

where  $*$  is the Hodge star operator (if  $M$  is not orientable, we consider its double covering). Denote by  $\mu$  the measure induced by the volume form  $*1$ . Then the pointwise inner product  $\langle \alpha, \beta \rangle(x)$  of  $\alpha$  and  $\beta$  at  $x \in M$  is the density with respect to  $\mu$ :

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha, \beta \rangle(x) \mu(dx).$$

We denote by  $A_0^q$  and  $L^2(A^q)$  the set of smooth  $q$ -forms with compact support and the set of square integrable  $q$ -forms, respectively. By the Stokes formula, the formal adjoint of  $d$  on  $L^2(A^q)$  is given by

$$\delta = (-1)^q *^{-1} d *.$$

We define the *Hodge Laplacian*  $\Delta_q$  on  $A^q$  as the Friedrichs extension of the quadratic form

$$\mathcal{E}(\alpha, \beta) = \langle (d + \delta)\alpha, (d + \delta)\beta \rangle, \quad D(\mathcal{E}) = A_0^q.$$

Since  $\Delta_q$  is self-adjoint, it generates a semigroup  $T_t := e^{-\Delta_q t}$ , whose kernel is a double form  $\vec{k}$ , called the *heat kernel* (e.g. [2], [14], and [16]). When  $q = 0$ , the heat kernel for functions is the smallest positive fundamental solution to the heat equation [5]. This characterization of the heat kernel is crucial in the proof of the equivalence of the stochastic completeness of  $M$  and the uniqueness of the bounded solution to the Cauchy problem for the heat equation if  $q = 0$  [8]. Of course, if  $q > 0$  there is no such characterization, and thus, the assumptions in Theorems 1 and 2 are different.

3. PROOF OF THE RESULTS

We start from the conservative principle, that is, Theorem 1.

PROOF OF THEOREM 1. Put

$$Gu := \int_0^\infty e^{-t} \int_M k(t, \cdot, y)u(y) \mu(dy) dt \quad \text{for every } u \in L^2,$$

where  $k$  is the heat kernel for functions. Let  $u_l \in C_0^\infty(M)$  be such that  $0 \leq u_l \leq u_{l+1} \leq 1$  for every  $l > 1$ , and  $u_l \rightarrow 1$  as  $l \rightarrow \infty$ . Since  $M$  is stochastically complete,  $v_l := Gu_l \in D(\Delta) \cap C^\infty(M)$  for every  $l > 0$  and

$$(6) \quad v_l \rightarrow 1 \quad \text{and} \quad \Delta v_l = v_l - u_l \rightarrow 0 \quad \text{as } l \rightarrow \infty, \mu\text{-a.e.}$$

For a bounded harmonic  $q$ -form  $\eta$ , put  $\eta_l := v_l \eta$ . Since  $\eta$  is harmonic,  $\Delta_q(\eta_l) = (\Delta v_l)\eta$  pointwise. Thus, noting that  $D(\Delta_q)$  is the set of  $L^2$  forms  $\alpha$  with  $\Delta_q \alpha \in L^2$  because Stampacchia's inequality holds on  $M$ , we have

$$\Delta_q \eta_l \in D(\Delta_q).$$

Denote by  $\vec{k}$  the heat kernel on  $A^q$ , and by  $T_t$  the corresponding semigroup. By the semigroup domination theorem for a complete manifold [15], if we assume that the Weitzenböck tensor on  $A^q$  is bounded below by  $c > -\infty$ , then

$$(7) \quad |\vec{k}(t, x, y)| \leq e^{-ct} |k(t, x, y)| \quad \text{for all } t > 0 \text{ and } x, y \in M.$$

Now we can proceed as follows:

$$(8) \quad \begin{aligned} |\langle T_\tau \psi, \eta_l \rangle - \langle \psi, \eta_l \rangle| &= \left| \int_0^\tau \langle \Delta_q T_t \psi, \eta_l \rangle dt \right| = \left| \int_0^\tau \langle T_t \psi, \Delta_q \eta_l \rangle dt \right| \\ &= \left| \int_0^\tau \langle T_t \psi, (\Delta v_l) \eta \rangle dt \right| \leq \|\Delta v_l\|_\infty \|\eta\|_\infty \int_0^\tau \|T_t \psi\|_1 dt \\ (9) \quad &\leq \|\Delta v_l\|_\infty \|\eta\|_\infty \int_0^\tau e^{-ct} \|\psi\|_1 dt. \end{aligned}$$

The left-hand side of (8) tends to  $|\langle T_\tau \psi, \eta \rangle - \langle \psi, \eta \rangle|$  as  $l \rightarrow \infty$ , because  $T_\tau \psi$  is integrable for every  $\tau > 0$  by (7). Finally, as (9) tends to 0 as  $l \rightarrow \infty$  by (6), we have completed the proof.  $\square$

We now prove the essential self-adjointness of the Hodge Laplacian on an incomplete manifold, namely, Proposition 1.

PROOF OF PROPOSITION 1. For a complete manifold  $M$ , consider the incomplete manifold  $N = M \setminus \Sigma$ , where  $\Sigma$  is a closed submanifold of  $M$ . Denote by  $\Delta_M$  and  $\Delta_N$  the Hodge Laplacians whose domains are  $A_0(M)$  and  $A_0(N)$ , respectively.

First we assume that the codimension of  $\Sigma$  is greater than 3, and show that  $\Delta_N$  is essentially self-adjoint. Since  $\Delta_M$  is essentially self-adjoint (e.g. [3], [17], [19]), and since  $\Delta_N \subset \Delta_M$ , it suffices to prove that

$$\overline{\Delta_M} \subset \overline{\Delta_N},$$

where the bar indicates the Hilbert closure of the operator. For that purpose, we wish to construct for any  $\alpha \in D(\overline{\Delta_M})$  a sequence  $\alpha_l \in D(\Delta_N)$  satisfying

$$\alpha_l \rightarrow \alpha \quad \text{in } \Delta\text{-graph norm as } l \rightarrow \infty.$$

By the essential self-adjointness of  $\Delta_M$ , we may assume that  $\alpha \in D(\Delta_M) = A_0(M)$  without loss of generality. Consider the cut-off function  $\chi_l$  [12] defined as

$$\chi_l := \theta(r^{1/l}), \quad l > 0,$$

where  $r$  is the distance function on  $M$  from  $\Sigma$ , and  $\theta \in C^\infty(\mathbb{R})$  satisfies:

$$\theta(r) = \begin{cases} 1 & \text{if } 3/4 \leq r, \\ 0 & \text{if } r \leq 1/2. \end{cases}$$

Because the codimension of  $\Sigma$  is greater than 3 and  $\Delta r$  can be estimated by  $cr^{-1}$  with some constant  $c > 0$  on a neighbourhood of  $\Sigma$  [11], the cut-off function  $\chi_l$  satisfies:

- $\lim_{l \rightarrow \infty} \chi_l = 1$  almost everywhere,
- both  $\|\Delta \chi_l\|_{L^2(K)}$  and  $\|d\chi_l\|_{L^2(K)}$  tend to 0 as  $l \rightarrow \infty$  for any compact set  $K \subset M$ .

Set

$$\alpha_l := \chi_l \alpha.$$

By the compactness of  $\text{supp}(\alpha)$ , the form  $\alpha_l$  belongs to  $D(\Delta_N)$  for sufficiently large  $l$ . By the formula (e.g. [14])

$$(\Delta(\chi_n \alpha))_{i_1 \dots i_p} = \chi_n (\Delta \alpha)_{i_1 \dots i_p} - 2(\nabla^i \chi_n)(\nabla_i \alpha_{i_1 \dots i_p}) + (\Delta \chi_n) \alpha_{i_1 \dots i_p},$$

and the fact that  $\|\nabla \alpha\|_{L^\infty}$  and  $\|\Delta \alpha\|_{L^\infty}$  are bounded, it follows that

$$\Delta(\chi_l \alpha) \rightarrow \Delta \alpha \quad \text{in } L^2(A(M)) \text{ as } l \rightarrow \infty.$$

By the Lebesgue theorem,  $\alpha_l \rightarrow \alpha$  in  $L^2(A)$  as  $l \rightarrow \infty$ . This shows that  $\overline{\Delta_M} = \overline{\Delta_N}$ , in particular  $\Delta_N$  is essentially self-adjoint.

Conversely, assume that the codimension of  $\Sigma$  is less than 4. Let  $f \in W^{2,2}(M) \cap C^\infty$  be the function which is greater than  $c > 0$  at some point  $x \in \Sigma$ . By the Sobolev theorem, if  $f_l \rightarrow f$  in  $W^{2,2}(M)$ , then there exists  $l_0$  such that  $f_l(x) > c/2$  for any  $l > l_0$ . This shows that  $f_l \notin W_0^{2,2}(N)$  for  $l > l_0$ , thus  $W_0^{2,2}(N) \subsetneq W^{2,2}(N)$ , and hence  $\overline{\Delta_N} \subsetneq \overline{\Delta_M}$ . We have completed the proof.  $\square$

Applying Proposition 1, we extend the semigroup domination theorem to the incomplete manifold  $M \setminus \Sigma$ , where the codimension of  $\Sigma$  is greater than 3. Then, by noting that  $M \setminus \Sigma$  is stochastically complete because  $\Sigma$  is almost polar in  $M$  [13], the proof of Theorem 1 applies to show Corollary 2.

Finally, we prove the uniqueness of bounded solutions of the Cauchy problem for the heat equation for  $q > 0$ , that is, Theorem 2.

PROOF OF THEOREM 2. Since the proof is similar to the case of  $q = 0$  (see Theorem 9.1 of [8]), we only demonstrate the different parts, and refer the reader to [8].

Set  $\xi(x, t) = \rho^2(x, t)/4(t - s)$  with fixed  $s$  and a Lipschitz function  $\rho$  satisfying  $\|d\rho\|_{L^\infty} \leq 1$ . Then

$$(10) \quad \dot{\xi} + |d\xi|^2 \leq 0,$$

where the dot stands for the partial derivative in  $t$ . For an arbitrary  $R > 0$ , let  $\chi$  be a Lipschitz function such that  $\chi = 0$  on  $(B_{2R})^c$  and  $\chi = 1$  in  $B_{3R/2}$ .

Let  $\alpha$  be a bounded  $q$ -form which is the solution to the Cauchy problem (4). By noting that  $*$  is isomorphic at each point, and that  $|\alpha \wedge \eta| \leq |\alpha| |\eta|$  pointwise, we may compute (we suppress  $M$  and  $d\mu$  for simplicity)

$$\begin{aligned} -\langle \delta\alpha, \delta(\chi^2 e^\xi \alpha) \rangle &= -\int \delta\alpha \wedge 2\chi e^\xi * d\chi \wedge \alpha - \int \delta\alpha \wedge \chi^2 e^\xi * d\xi \wedge \alpha \\ &\quad - \int \delta\alpha \wedge * \eta^2 e^\xi \delta\alpha \\ &\leq 2 \int |\delta\alpha| \chi e^\xi |\alpha| |d\chi| + \int \chi^2 e^\xi |\delta\alpha| |d\xi| |\alpha| - \int \chi^2 e^\xi |\delta\alpha|^2 \\ &\leq 2 \int \left[ \frac{1}{4} |\chi \delta\alpha|^2 + |d\chi|^2 |\alpha|^2 \right] e^\xi + \frac{1}{2} \int [|\delta\alpha|^2 + |d\xi|^2 |\alpha|^2] \chi^2 e^\xi \\ &\quad - \int \chi^2 e^\xi |\delta\alpha|^2 \\ &\leq 2 \int |d\chi|^2 |\alpha|^2 e^\xi + \frac{1}{2} \int |d\xi|^2 |\alpha|^2 \chi^2 e^\xi. \end{aligned}$$

In a similar way, we have

$$-\langle d\alpha, d(\chi^2 e^\xi \alpha) \rangle \leq 2 \int |d\chi|^2 |\alpha|^2 e^\xi + \frac{1}{2} \int |d\xi|^2 |\alpha|^2 \chi^2 e^\xi.$$

Therefore,

$$(11) \quad \int_{B_{2R}} \Delta\alpha \wedge * \alpha \chi^2 e^\xi \leq 2 \left[ 2 \int_{B_{2R}} |d\chi|^2 |\alpha|^2 e^\xi + \frac{1}{2} \int_{B_{2R}} |d\xi|^2 |\alpha|^2 \chi^2 e^\xi \right].$$

On the other hand,

$$(12) \quad \int_{\mathcal{C}} \frac{\partial(|\alpha|^2)}{\partial t} \chi^2 e^\xi d\mu dt = \int_{B_{2R}} |\alpha|^2 \chi^2 e^\xi \Big|_{\tau-\sigma}^{\tau} d\mu - \int_{\mathcal{C}} |\alpha|^2 \chi^2 \dot{\xi} e^\xi d\mu dt,$$

where  $\mathcal{C} = B_{2R} \times [\tau - \delta, \tau]$  is a cylinder. By (11) and by applying inequality (10) to (12), we deduce that

$$(13) \quad \int_{B_{2R}} |\alpha|^2 \chi^2 e^\xi \Big|_{\tau-\sigma}^{\tau} \leq 4 \int_{\mathcal{C}} |d\chi|^2 |\alpha|^2 e^\xi d\mu dt.$$

Now one may proceed to show that  $\alpha = 0$  on  $B_R$  by applying (13) and (V) as in the case of  $q = 0$  cited above. Since  $R > 0$  is arbitrary, this shows that  $\alpha = 0$  on  $M$ .  $\square$

REMARK 1. There exists a complete manifold satisfying (V) and on which

$$\lim_{t \rightarrow \infty} e^{-t\Delta_1}(A_0^1) \subsetneq L^1.$$

PROOF. Consider a Riemannian manifold

$$M = \{(x, y) \in \mathbb{R}^2 : r(x, y) := x^2 + y^2 < 1\}$$

with metric  $g = (1 - r^2)^{-1/2} dx dy$ . By a direct computation, we see that  $M$  is geodesically complete and that  $M$  satisfies (V). In Theorem 3.7 of [18], it is proved that there exists  $\alpha \in A_0^1$  such that  $\lim_{t \rightarrow \infty} e^{-t\Delta_1}\alpha \in L^2(A^1)$  is a non-integrable, non-trivial, harmonic form.  $\square$

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