Rend. Lincei Mat. Appl. 18 (2007), 365–389

Algebraic geometry. — *Characteristic varieties and constructible sheaves*,
by ALEXANDRU DIMCA.

ABSTRACT. — We explore the relation between the positive-dimensional irreducible components of the characteristic varieties of rank one local systems on a smooth surface and the associated (rational or irrational) pencils. Our study, which may viewed as a continuation of D. Arapura's paper [1], yields new geometric insight into the translated components relating them to the multiplicities of curves in the associated pencil, in a close analogy to the compact situation treated by A. Beauville [3]. The new point of view is the key role played by the constructible sheaves naturally arising from local systems.

KEY WORDS: Local system; constructible sheaf; twisted cohomology; characteristic variety; pencil of curves.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 14C21, 14F99, 32S22; Secondary 14E05, 14H50.

1. INTRODUCTION

Let M be a smooth complex quasi-projective variety. The first characteristic varieties $\mathcal{V}_m(M)$ describe the jumping loci for the dimension of the first twisted cohomology groups $H^1(M, \mathcal{L})$, where $\mathcal{L}$ is a rank one local system on M. These characteristic varieties, and their relative position inside the algebraic group $\mathbb{T}(M) = \mathrm{Hom}(H_1(M), \mathbb{C}^*)$ parametrizing the rank one local system on M, play a key role in understanding the fundamental group $\pi_1(M)$ (see [15], [16]).

Since we are interested here only in the first cohomology groups, we can replace M by a smooth quasi-projective surface, by taking a generic linear section and applying a general version of the Zariski theorem (see for instance [11, p. 25]). Therefore, to help the reader's intuition, we will assume in this paper that $\dim M = 2$, though the results and proofs hold in any dimension. This is a very interesting setting, as the opening lines of Catanese's Introduction in [6] tell us:

> "The study of fibrations of a smooth algebraic surface S over a smooth algebraic curve B lies at the heart of the classification theory and of the geometry of algebraic surfaces."

The *main aim* of this paper is to study the translated components of the characteristic variety $\mathcal{V}_1(M)$. According to Arapura's results [1], such a component $W = W(f, \rho)$ is described by a pair (f, ρ) where

(a) f is a surjective morphism $M \to S$, from the surface M to a smooth curve S, having a connected generic fiber F;
(b) $\rho \in \mathbb{T}(M)$ is a torsion character such that W is the translate by ρ of the subtorus $f^*(\mathbb{T}(S))$, i.e. $W = \rho \cdot f^*(\mathbb{T}(S))$.

Using the (Logarithmic) Isotropic Subspace Theorem (see Catanese [5] in the compact case and Bauer [2] and Catanese [6, Thm. 2.11] in the quasi-projective case), one can determine in many cases the various possible morphisms f from certain maximal isotropic subspaces in $H^1(M)$, relative to the cup-product

$$H^1(M) \times H^1(M) \to H^2(M).$$

A similar approach is provided by the study of the resonance varieties (see [15]).

In this paper we assume that the morphism f has already been determined and concentrate on the finite order character ρ above. Our results can be described briefly as follows. The characters ρ arising in (b) above for a given map f are parametrized by the Pontryagin dual $\hat{T}(f) = \mathrm{Hom}(T(f), \mathbb{C}^*)$ of a finite group $T(f)$ defined in terms of the topology of the mapping f. This group depends only on the multiple fibers in the pencil associated to f (see Theorem 5.3). When $\chi(S) < 0$, any character in $\hat{T}(f)$ actually gives rise to a component (see Proposition 4.3), while for $\chi(S) = 0$ (i.e. when S is an elliptic curve, a case treated by Beauville [3] when M is proper, or when $S = \mathbb{C}^*$) one should discard the trivial character in $\hat{T}(f)$ (see Corollary 5.8). Moreover, for a generic local system $\mathcal{L} \in W = W(f, \rho)$, the dimension of $H^1(M, \mathcal{L})$ is expressed in terms of the Euler characteristic $\chi(S)$ and the cardinality of the support of ρ (see Corollary 4.7).

The case $S = \mathbb{C}^*$ is the most mysterious, and Suciu's example of such a component for the deleted B_3-arrangement given in [24], [25] played a key role in our understanding of this question. We consider this component in detail in Examples 3.8 and 5.12, and for the generalization given by the $\mathcal{A}_m$-arrangements, discussed in [8] and [9], see Examples 5.13 and 5.14 below.

Our results are exemplified all along this paper on two types of situations:

CASE A: M is a curve arrangement complement in $\mathbb{P}^2$.

CASE B: M is a curve arrangement complement on a normal weighted homogeneous surface singularity, a case which includes the Seifert link complements discussed in Eisenbud and Neumann's book [17].

In fact, the reader interested only in Case A may refer to [13] for additional information.

In Section 2 we collect some basic facts on regular mappings $f : M \to S$ and the associated pencils. Lemma 2.2 intends to clarify the key notion of *admissible map* used by Arapura in [1].

In Section 3 we give the main definitions related to characteristic varieties. Theorem 3.6 collects some (more or less known) facts on irreducible components of characteristic varieties, which are derived by a careful reading of Arapura's paper [1]. The key topological property (ii) in Theorem 3.6 was not explicitly stated before (in the proper case, a related property is used in [3]). In Corollary 4.6 we give a purely topological proof of this property.

In Section 4 we emphasize the key role played in this setting by the constructible sheaves obtained as direct images of local systems on M under the mapping $f : M \to S$ (see for instance Propositions 4.3 and 4.5 and Lemmas 4.2 and 4.4). In particular, for a local system $\mathcal{L} \in W = W(f, \rho)$, the dimension of $H^1(M, \mathcal{L})$ is expressed in terms of the Euler characteristic $\chi(S)$ and the cardinality of the *singular support* $\Sigma(\mathcal{F})$ of the sheaf $\mathcal{F} = R^0 f_* \mathcal{L}_\rho$ (see Corollary 4.7).

In the final section we associate to a map $f : M \to S$ as above a finite abelian group $T(f)$ such that the torsion character ρ is determined by a character $\tilde{\rho}$ of $T(f)$ (see formula (5.7)). We compute this group $T(f)$ in terms of the multiplicities of the special fibers of f (see Theorem 5.3). The group $T(f)$ is intimately related to the *orbifold fundamental group* $\pi_1^{\text{orb}}(f)$ of the map f (and even more so to the corresponding *orbifold first homology group* $H_1^{\text{orb}}(f)$ of f), see Corollary 5.4.

In Theorem 5.7 we show that the character $\tilde{\rho} \in \hat{T}(f)$ is trivial if and only if the associated constructible sheaf $\mathcal{F} = R^0 f_*(\mathcal{L}_\rho)$ is a local system on S.

2. Generalities on pencils on a surface M

Let $\tilde{M}$ be a smooth compactification of a complex smooth quasi-projective surface M. Let $f : M \to S$ be a regular mapping, where S is a smooth curve. Then there is a minimal non-empty finite set $A \subset \tilde{M}$ such that f has an extension $\tilde{f}$ to $U = \tilde{M} \setminus A$ with values in the smooth projective model $\widehat{S}$ of S. By blowing up the points in A, we pass from $\tilde{M}$ to a new compactification $\widehat{M}$ of M such that f or $\tilde{f}$ is the restriction of a regular morphism $\widehat{f} : \widehat{M} \to \widehat{S}$.

We call any of the morphisms f, $\tilde{f}$ or $\widehat{f}$ above a *pencil of curves*. Such a pencil is *rational* if the curve S (or, equivalently, $\widehat{S}$) is rational, and it is *irrational* otherwise. For any $s \in \widehat{S}$, we denote by $\mathcal{C}_s$ the corresponding fiber in $\tilde{M}$ (obtained by taking the closure of $\tilde{f}^{-1}(s)$) or in $\widehat{M}$. The corresponding pencil will be denoted sometimes by $\mathcal{C} = (\mathcal{C}_s)_{s\in\widehat{S}}$.

We recall the following sufficient condition to have a rational pencil.

PROPOSITION 2.1. *If the complex smooth surface M satisfies the condition $W_1H^1(M, \mathbb{Q}) = 0$, where W is the weight filtration of the canonical mixed Hodge structure, and if $f : M \to S$ has a generic connected fiber, then S is a rational curve. Moreover, the condition $W_1H^1(M, \mathbb{Q}) = 0$ holds if the surface M admits a smooth compactification $\tilde{M}$ such that $H^1(\tilde{M}, \mathbb{Q}) = 0$.*

To prove this result, we need the following.

LEMMA 2.2. *Let X and S be smooth irreducible algebraic varieties with* $\dim S = 1$ *and let $f : X \to S$ be a non-constant morphism. Then for any compactification $\widehat{f} : \widehat{X} \to \widehat{S}$ of f with $\widehat{X}$, $\widehat{S}$ smooth, the following are equivalent.*

(i) *The generic fiber F of f is connected.*
(ii) *The generic fiber $\widehat{F}$ of $\widehat{f}$ is connected.*
(iii) *All the fibers of $\widehat{f}$ are connected.*

If these equivalent conditions hold, then $f_\sharp : \pi_1(X) \to \pi_1(S)$ and $\widehat{f}_\sharp : \pi_1(\widehat{X}) \to \pi_1(\widehat{S})$ are surjective.

PROOF. Note that $D = \widehat{X} \setminus X$ is a proper subvariety (not necessarily a normal crossing divisor) with finitely many irreducible components D_m. For each such component D_m, either $\widehat{f}(D_m)$ is a point, or $\widehat{f} : D_m \to \widehat{S}$ is surjective. In this latter case, it follows that $\dim(\widehat{F} \cap D_m) < \dim D_m \leq \dim \widehat{F}$. Since $\widehat{F}$ is smooth of pure dimension, it follows that $\widehat{F}$ is connected if and only if $F = \widehat{F} \setminus \bigcup_m (D_m \cap \widehat{F})$ is connected. To show that (ii) implies

(iii) it is enough to use the Stein factorization theorem (see for instance [19, p. 280]) and the fact that a morphism between two smooth projective curves which is of degree one (i.e. generically injective) is in fact an isomorphism.

To prove the last claim for f, note that there is a Zariski open and dense subset $S' \subset S$ such that f induces a locally trivial topological fibration $f : X' = f^{-1}(S') \to S'$ with fiber type F. Since F is connected, we get an epimorphism $f_\sharp : \pi_1(X') \to \pi_1(S')$. The inclusion of S' into S induces an epimorphism at the level of fundamental groups. Let $j : X' \to X$ be the inclusion. Then we have seen that $f \circ j$ induces an epimorphism at the level of fundamental groups. Therefore the same is true for f. The proof for $\widehat{f}$ is completely similar. □

PROOF OF PROPOSITION 2.1. From the surjectivity of $f_\sharp$ it follows that $f^* : H^1(S, \mathbb{Q}) \to H^1(M, \mathbb{Q})$ is injective. Since f^* preserves the weight filtration W, it follows that $W_1 H^1(S, \mathbb{Q}) = 0$, i.e. S is a rational curve.

If the surface M admits a smooth compactification $j : M \to \tilde{M}$ such that $H^1(\tilde{M}, \mathbb{Q}) = 0$, then $W_1 H^1(M, \mathbb{Q}) = j^* H^1(\tilde{M}, \mathbb{Q}) = 0$ as explained for instance in [11, p. 243]. □

EXAMPLE 2.3. The following classes of complex smooth surfaces M satisfy $W_1 H^1(M, \mathbb{Q}) = 0$ and will be used as test cases.

CASE A: complements of plane curve arrangements, i.e. $M = \mathbb{P}^2 \setminus C$ where C is a plane curve, usually with several irreducible components. One can then take $\tilde{M} = \mathbb{P}^2$. A more general situation is obtained by replacing $\mathbb{P}^2$ by any smooth simply connected surface $\tilde{M}$, e.g. a smooth complete intersection in some projective space.

CASE B: complements of curve arrangements on a weighted homogeneous isolated complete intersection $(X, 0)$ with $\dim X = 2$, whose link $\Sigma(X, 0)$ is a $\mathbb{Q}$-homology sphere. Such a singularity can by represented by an affine complete intersection surface X defined by some weighted homogeneous equations with respect to some positive integer weights $\mathbf{w} = (w_1, \dots, w_n)$,

$$f_1(x) = \cdots = f_{n-2}(x) = 0$$

in $\mathbb{C}^n$. Moreover the complex surface X is smooth away from the origin. Let $g \in \mathbb{C}[x_1, \dots, x_n]$ be another weighted homogeneous polynomial with respect to the weights $\mathbf{w}$ and set $C = g^{-1}(0)$, $M = X \setminus C$. Since the link of $(X, 0)$ is a $\mathbb{Q}$-homology sphere, one has $H^1(X \setminus 0, \mathbb{Q}) = H^2(X \setminus 0, \mathbb{Q}) = 0$. Note that each irreducible component C_j of C is a rational curve, since $C_j^* = C_j \setminus 0$ is exactly a $\mathbb{C}^*$-orbit of the $\mathbb{C}^*$-action on $\mathbb{C}^n$ associated to the given weights. Moreover, this shows that C_j^* is smooth. Let $C_1, \dots, C_r$ be the set of these components. The Gysin exact sequence

$$0 = H^1(X \setminus 0) \to H^1(M) \to \bigoplus_{j=1}^{r} H^0(C_j^*)(-1) \to H^2(X \setminus 0) = 0$$

with rational coefficients shows that $H^1(M, \mathbb{Q})$ is pure of type $(1, 1)$, in particular one has $W_1 H^1(M, \mathbb{Q}) = 0$. Moreover it gives $b_1(M) = r$.

The following explicit description of rational pencils is recalled for the reader's convenience.

PROPOSITION 2.4. *Let U be a smooth surface. If $f : U \to \mathbb{P}^1$ is a morphism, then $L = f^*\mathcal{O}(1)$ is a line bundle on U, generated by the two global sections $s_i = f^*(y_i)$, with y_1, y_2 a system of homogeneous coordinates on $\mathbb{P}^1$.*

Conversely, if L is a line bundle on U, generated by two global sections s_i for $i = 1, 2$, then there is a morphism $f : U \to \mathbb{P}^1$ such that $L = f^*\mathcal{O}(1)$ and $s_i = f^*(y_i)$, with y_1, y_2 a system of homogeneous coordinates on $\mathbb{P}^1$.

PROOF. It is well known (see for instance [19, p. 150]) that a morphism $f : U \to \mathbb{P}^1$ is given by a line bundle $\mathcal{L} \in \mathrm{Pic}(U)$ and two sections $s_1, s_2 \in \Gamma(U, \mathcal{L})$ which do not both vanish at any point in U. In fact $\mathcal{L} = f^*(\mathcal{O}(1))$ and $s_i = f^*(y_i)$, with y_1, y_2 a system of homogeneous coordinates on $\mathbb{P}^1$. With this notation, one has $f(x) = [a : b]$ where $[a : b] \in \mathbb{P}^1$ is such that $as_2(x) - bs_1(x) = 0$. □

REMARK 2.5. Since U is smooth, we have $\mathrm{Pic}(U) = C\ell(U)$ and similarly $\mathrm{Pic}(\tilde{M}) = C\ell(\tilde{M})$ (see for instance [19, p. 145]). On the other hand, the inclusion $j : U \to \tilde{M}$ induces an isomorphism $j^* : C\ell(\tilde{M}) \to C\ell(U)$, as $\mathrm{codim}\, A = 2$ (see [19, p. 133]). It follows that $j^* : \mathrm{Pic}(\tilde{M}) \to \mathrm{Pic}(U)$ is also an isomorphism, i.e. any line bundle $L \in \mathrm{Pic}(U)$ is the restriction to U of a line bundle $\tilde{L}$ on $\tilde{M}$. If $\tilde{M} = \mathbb{P}^2$, then $\tilde{L}$ has the form $\mathcal{O}(D)$ and the global sections of L are nothing else than the restrictions of global sections of the line bundle $\tilde{L} = \mathcal{O}(D)$, which are the degree D homogeneous polynomials. In general, the two sections s_i have natural extensions to $\tilde{M}$, and we may consider the divisors $C_i : s_i = 0$ on $\tilde{M}$ and the associated rational pencil $\mathcal{C}_f : \alpha_1 s_1 + \alpha_2 s_2$ of curves on $\tilde{M}$.

In the following we regard the difference $C = \tilde{M} \setminus M$ as a (reduced) curve and let $C = \bigcup_{j=1}^r C_j$ be the decomposition of C into irreducible components.

PROPOSITION 2.6. *Let $B \subset \widehat{S}$ be a finite set and denote by S the complement $\widehat{S} \setminus B$. For any surjective morphism $f : M \to S$ and any compactification $\tilde{M}$ of M as above, any irreducible component C_j of C is in one of the following cases.*

(1) *C_j is contained in a curve $\mathcal{C}_b$ in the pencil $\mathcal{C}$, corresponding to a point $b \in B$;*
(2) *C_j is strictly contained in a curve $\mathcal{C}_s$ in the pencil $\mathcal{C}$, corresponding to a point $s \in S$;*
(3) *C_j is a horizontal component, i.e. C_j intersects the generic fiber $\mathcal{C}_t$ of the pencil $\mathcal{C}$ outside the base locus.*

Moreover, if $|B| > 1$, then C_j is in the first case above if and only if the homology class γ_j of a small loop around C_j satisfies $H_1(f)(\gamma_j) \neq 0$ in $H_1(S, \mathbb{Z})$.

PROOF. Let C_j be an irreducible component of C. Then either $\tilde{f}(C_j)$ is a point, which leads to the first two cases, or $\tilde{f}(C_j)$ is dense in $\widehat{S}$, which leads to the last case. The strict inclusion in the second case comes from the surjectivity of f.

The last claim is obvious if we use the Mayer–Vietoris exact sequence of the covering $\widehat{S} = S \cup D$, where D is the union of small closed discs on $\widehat{S}$ centered at the points in B. For instance, in the first case, if δ_b is a small loop at b, then one has $H_1(f)(\gamma_j) = m_j \cdot \delta_b$, with $m_j > 0$ the multiplicity of the curve C_j in the divisor $f'^{-1}(b)$ (if the orientations of the loops γ_j and δ_b are properly chosen). See also equation (3.1) below. □

DEFINITION 2.7. *In the setting of Proposition* 2.6*, we say that the curve arrangement* C *is* minimal *with respect to the surjective mapping* $f : M \to S$ *if any component* C_j *of* C *is of type* (1)*, i.e.* C_j *is contained in a curve* $\mathcal{C}_b$ *in the pencil* $\mathcal{C}$*, corresponding to a point* $b \in B$*. We say that the curve arrangement* C *is* special *with respect to the surjective mapping* $f : M \to S$ *if some component* C_j *of* C *is of type* (2)*, i.e.* C_j *is strictly contained in a curve* $\mathcal{C}_s$ *in the pencil* $\mathcal{C}$*, corresponding to a point* $s \in S$*.*

REMARK 2.8. If $|B| > 1$, then the base locus A of the pencil $\mathcal{C}$ on the surface $\tilde{M}$ is just the intersection of any two distinct fibers $\mathcal{C}_b \cap \mathcal{C}_{b'}$ for b, b' distinct points in B. Note also that case (2) above cannot occur if all the fibers $\mathcal{C}_s$ for $s \in S$ are irreducible. The fibers $\mathcal{C}_s$ may be non-reduced, i.e. we consider them usually as divisors. Saying that C_j is contained in $\mathcal{C}_s$ means that $\mathcal{C}_s = m_j C_j + \cdots$ with $m_j > 0$.

3. LOCAL SYSTEMS AND CHARACTERISTIC VARIETIES

3.1. *Local systems on S*

We return to the notation $S = \widehat{S} \setminus B$, with $B = \{b_1, \ldots, b_k\}$ a finite set of cardinality $|B| = k \geq 0$. Let $g = g(\widehat{S})$ be the genus of the curve $\widehat{S}$ and denote by $\delta_1, \ldots, \delta_{2g}$ the usual $\mathbb{Z}$-basis of the first integral homology group $H_1(\widehat{S})$.

If δ_{2g+i} denotes an elementary loop based at some base point $b_i \in B$ and turning once around the point b_i, then with the usual choices, the first integral homology group of S is given by

$$H_1(S) = \mathbb{Z}\langle \delta_1, \ldots, \delta_{2g+k} \rangle / \langle \delta_{2g+1} + \cdots + \delta_{2g+k} \rangle. \tag{3.1}$$

Therefore, for $k > 0$, the rank one local systems on S are parametrized by the $(2g+k-1)$-dimensional algebraic torus $\mathbb{T}(S) = \mathrm{Hom}(H_1(S), \mathbb{C}^*)$ given by

$$\mathbb{T}(S) = \{\lambda = (\lambda_1, \ldots, \lambda_{2g+k}) \in (\mathbb{C}^*)^{2g+k} \mid \lambda_{2g+1} \cdots \lambda_{2g+k} = 1\}. \tag{3.2}$$

Here $\lambda_j \in \mathbb{C}^*$ is the monodromy along the loop δ_j. When $k = 0$, one has

$$\mathbb{T}(S) = \mathrm{Hom}(H_1(S), \mathbb{C}^*) = \{\lambda = (\lambda_1, \ldots, \lambda_{2g}) \in (\mathbb{C}^*)^{2g}\} = (\mathbb{C}^*)^{2g}. \tag{3.3}$$

Note that in both cases $\dim \mathbb{T}(S) = b_1(S)$. For $\lambda \in \mathbb{T}(S)$, we denote by $\mathcal{L}_\lambda$ the corresponding rank one local system on S.

The twisted cohomology groups $H^m(S, \mathcal{L}_\lambda)$ are easy to compute. There are two cases.

CASE 1: $\mathcal{L}_\lambda = \mathbb{C}_S$. Then we get the usual cohomology groups of S, namely for $k > 0$ we have $\dim H^0(S, \mathcal{L}_\lambda) = 1$, $\dim H^1(S, \mathcal{L}_\lambda) = 2g+k-1$ and $H^m(S, \mathcal{L}_\lambda) = 0$ for $m \geq 2$. And for $k = 0$ we have $\dim H^0(S, \mathcal{L}_\lambda) = \dim H^2(S, \mathcal{L}_\lambda) = 1$, $\dim H^1(S, \mathcal{L}_\lambda) = 2g$ and $H^m(S, \mathcal{L}_\lambda) = 0$ for $m \geq 3$.

CASE 2: $\mathcal{L}_\lambda$ nontrivial. This case corresponds to the case when at least one monodromy λ_j is not 1. In such a situation one has $2g + k \geq 2$. Then

$$\dim H^1(S, \mathcal{L}_\lambda) = 2g + k - 2 = -\chi(S) \tag{3.4}$$

and $H^m(S, \mathcal{L}_\lambda) = 0$ for $m \neq 1$. One has obviously $\dim H^0(S, \mathcal{L}_\lambda) = 0$ in this case. The vanishing of $H^2(S, \mathcal{L}_\lambda)$ follows by duality if $k = 0$, and since S is homotopically a bouquet of circles when $k > 0$.

3.2. *Local systems on M*

The rank one local systems on M are parametrized by the algebraic group

$$\mathbb{T}(M) = \operatorname{Hom}(H_1(M), \mathbb{C}^*) \tag{3.5}$$

which is an extension of the algebraic torus $(\mathbb{C}^*)^{b_1(M)}$ by the finite group Tors $H_1(M)$. This group can be described explicitly as soon as we know $H_1(M)$.

CASE A: complements of plane curve arrangements, i.e. $M = \mathbb{P}^2 \setminus C$ where C is a plane curve, with irreducible components C_j for $j = 1, \ldots, r$, $\deg C_j = d_j$. Let γ_j be an elementary loop around the irreducible component C_j, for $j = 1, \ldots, r$. Then it is known (see for instance [11, p. 102]) that

$$H_1(M) = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_r \rangle / \langle d_1\gamma_1 + \cdots + d_r\gamma_r \rangle \tag{3.6}$$

where d_j is the degree of the component C_j. It follows that the rank one local systems on M are parametrized by the algebraic group

$$\mathbb{T}(M) = \operatorname{Hom}(H_1(M), \mathbb{C}^*) = \{\rho = (\rho_1, \ldots, \rho_r) \in (\mathbb{C}^*)^r \mid \rho_1^{d_1} \cdots \rho_r^{d_r} = 1\}. \tag{3.7}$$

The connected component $\mathbb{T}^0(M)$ of the unit element $1 \in \mathbb{T}(M)$ is the $(r-1)$-dimensional torus given by

$$\mathbb{T}^0(M) = \{\rho = (\rho_1, \ldots, \rho_r) \in (\mathbb{C}^*)^r \mid \rho_1^{e_1} \cdots \rho_r^{e_r} = 1\} \tag{3.8}$$

with $D = \operatorname{GCD}(d_1, \ldots, d_r)$ and $e_j = d_j/D$ for $j = 1, \ldots, r$.

REMARK 3.3. If $d_1 = 1$, then $\{\gamma_2, \ldots, \gamma_r\}$ is a basis for $H_1(M)$ and the torus $\mathbb{T}(M)$ can be identified to $(\mathbb{C}^*)^{r-1}$ under the projection $\rho \mapsto (\rho_2, \ldots, \rho_r)$.

CASE B: complements of curve arrangements on a weighted homogeneous isolated complete intersection $(X, 0)$ with $\dim X = 2$, whose link $\Sigma(X, 0)$ is a $\mathbb{Z}$-homology sphere. Using the notation from Example 2.3, we see that $H_1(M) = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_r \rangle$, where γ_j is an elementary loop around the irreducible component C_j, for $j = 1, \ldots, r$. It follows that

$$\mathbb{T}(M) = \operatorname{Hom}(H_1(M), \mathbb{C}^*) = (\mathbb{C}^*)^r. \tag{3.9}$$

The computation of the twisted cohomology groups $H^m(M, \mathcal{L}_\rho)$ is one of the major problems. A simple situation is described in the following.

EXAMPLE 3.4. If $\mathcal{L}_\rho = \mathbb{C}_M$ and we are in one of the two cases above, this computation can be done as follows.

CASE A: The result depends on the local singularities of the plane curve C. In fact, $\dim H^0(M, \mathbb{C}) = 1$, $\dim H^1(M, \mathbb{C}) = r - 1$ and $H^m(M, \mathbb{C}) = 0$ for $m \geq 3$. To determine the remaining Betti number $b_2(M) = \dim H^2(M, \mathbb{C})$ amounts to determining the Euler characteristic $\chi(M) = 3 - \chi(C)$, and this can be done, e.g., by using the formula for $\chi(C)$ given in [11, p. 162].

CASE B: Here $\dim H^0(M, \mathbb{C}) = 1$, $\dim H^1(M, \mathbb{C}) = r$ and also $\dim H^2(M, \mathbb{C}) = r - 1$, as well as $H^m(M, \mathbb{C}) = 0$ for $m \geq 3$. To see this, note that M is an affine open subset in X (which yields $H^m(M, \mathbb{C}) = 0$ for $m \geq 3$), and there is a $\mathbb{C}^*$-action on M with finite isotropy groups (which yields $\chi(M) = 0$).

To study these cohomology groups $H^m(M, \mathcal{L}_\rho)$ in general, one idea is to study the characteristic varieties

$$\mathcal{V}_m(M) = \{\rho \in \mathbb{T}(M) \mid \dim H^1(M, \mathcal{L}_\rho) \geq m\}. \tag{3.10}$$

3.5. *Arapura's results*

We recall here some of the main results from [1], applied to the rank one local systems on M, with some additions from [20], [15] and some new consequences.

THEOREM 3.6. *Let W be an irreducible component of $\mathcal{V}_1(M)$ and assume that $d_W :=$ $\dim W \geq 1$. Then there is a surjective morphism $f_W : M \to S_W$ onto a smooth curve S_W, with a connected generic fiber $F(f_W)$, and a torsion character $\rho_W \in \mathbb{T}(M)$ such that*

$$W = \rho_W \otimes f_W^*(\mathbb{T}(S_W)).$$

More precisely, the following hold.

(i) *$S_W = \widehat{S}_W \setminus B_W$, with B_W a finite set satisfying $d_W = 2g(\widehat{S}_W) + k_W - 1 = -\chi(S_W) + 1$ if $k_W := |B_W| > 0$. If $B_W = \emptyset$, then $d_W = 2g(\widehat{S}_W) = -\chi(S_W) + 2$.*

(ii) *For any local system $\mathcal{L} \in W$, the restriction $\mathcal{L}|F(f_W)$ of $\mathcal{L}$ to the generic fiber of f_W is trivial, i.e. $\mathcal{L}|F(f_W) = \mathbb{C}_{F(f_W)}$.*

(iii) *If N_W is the order of the character ρ_W, then there is a commutative diagram*

$$\begin{array}{ccc} M' & \xrightarrow{p} & M \\ \downarrow{\scriptstyle f'_W} & & \downarrow{\scriptstyle f_W} \\ S'_W & \xrightarrow{q} & S_W \end{array}$$

where p is an unramified N_W-cyclic Galois covering, q is a possibly ramified N_W-cyclic Galois covering, f'_W is μ_{N_W}-equivariant in the obvious sense, has a generic fiber $F(f'_W)$ isomorphic to the generic fiber $F(f_W)$ of f_W, and $p^*\rho_W$ is trivial. Here μ_{N_W} denotes the cyclic group of the N_W-th roots of unity.

(iv) *If $1 \in W$ and $\mathcal{L} \in W$, then $\dim H^1(M, \mathcal{L}) \geq -\chi(S_W)$ and equality holds with finitely many exceptions.*

(v) *If $1 \in W$, then $d_W \geq 2$ for $k_W > 0$ and $d_W \geq 4$ for $k_W = 0$.*

(vi) *If $1 \notin W$ and either $d_W = 2$ and S_W is not an elliptic curve, or $d_W > 2$, then the subtorus $W' = f_W^*(\mathbb{T}(S_W))$ is another irreducible component of $\mathcal{V}_1(M)$.*

PROOF. The first assertion is just Thm. 1.6 in [1, Section V].

To prove claim (i) note that $d_W = \dim W = \dim \mathbb{T}(S_W) = b_1(S_W)$.

For (vi), consider now the situation $1 \notin W$. Note that $\chi(S_W) \geq 0$ if and only if either $g(\widehat{S}_W) = 0$ and $k_W \leq 2$, or $g(\widehat{S}_W) = 1$ and $k_W = 0$. The first possibility (which contains the trivial cases $S_W = \mathbb{P}^1$, $S_W = \mathbb{C}$ and the interesting case $\mathbb{C}^*$) is excluded, since then $\dim W = \dim \mathbb{T}(S_W) \leq 1$. The second case corresponds to S_W being an elliptic curve E, and can be excluded as above if we assume $d_W > 2$. The corresponding translated components in this case (assuming M proper) are described in [3]. A uniform treatment of the translated components in the only two interesting cases $\mathbb{C}^*$ and E is given below in Corollary 5.8.

With the exception of these special cases, it follows that $\chi(S_W) < 0$ and $W' = f_W^*(\mathbb{T}(S_W))$ is another irreducible component of $\mathcal{V}_1(M)$ by Prop. 1.7 in [1].

Now we prove claim (ii). Since $W = \rho_W \otimes f_W^*(\mathbb{T}(S_W))$, it is enough to prove the claim for $\mathcal{L}_1 = \mathcal{L}_{\rho_W}$. Since ρ_W is a torsion character, it follows that $\mathcal{L}_1$ is a unitary local system. Let $\overline{M}$ be a good compactification of M obtained by adding the normal crossing divisor D to M. Let (L_1, ∇_1) be the integrable flat connection on M corresponding to the local system $\mathcal{L}_1$ and let $(\overline{L}_1, \overline{\nabla}_1)$ be the Deligne extension of the connection (L_1, ∇_1) to $\overline{M}$ with residues having the real parts in $[0, 1)$. Then there is a Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(\overline{M}, \Omega^p_{\overline{M}}(\log D) \otimes \overline{L}_1) \;\Rightarrow\; H^{p+q}(M, \mathcal{L}_1).$$

Since by hypothesis $H^1(M, \mathcal{L}_1) \neq 0$, it follows that either $E_2^{1,0} \neq 0$, or $E_2^{0,1} \neq 0$. In the first case, we are exactly in the situation of [1, Prop. 1.3 in Section V] and our claim (ii) is proved in the final part of the proof. Just note that on the last line of that proof, one should replace "which forces $\psi|F$ to be trivial" by "which forces $\psi|(F \cap X)$ to be trivial". (This is due to the fact that F in [1] denotes the compactification of our affine fiber $F = F(f_W)$, and X in [1] corresponds to our M.) If we are in the latter case, then one can show that $\mathcal{L}_1^{-1}$ leads to the first case, exactly as in the second part of the proof of [1, Prop. 1.4 in Section V]. Since claim (ii) for $\mathcal{L}_1$ is equivalent to claim (ii) for $\mathcal{L}_1^{-1}$, we are done. Property (ii) corresponds to Prop. 1.2 in Beauville's paper [3]. As noted there, it is the same thing to ask triviality for the restriction to one generic fiber of f_W or to all generic fibers of f_W. See Corollary 4.6 below for a direct topological proof of (ii).

Claim (iii) is just the "untwisting" part of the proof of Thm. 1.6 in [1]. The existence of the diagram is explained there via the Stein factorization for $f_W \circ p$. However, the fact that the morphism q has degree N_W depends on the previous claim (ii), and this key point is not mentioned in [1].

The proof of (iv) is more technical. Using the projection formula

$$p_*(\mathbb{C}_{M'}) \otimes \mathcal{L} \simeq p_*(p^*(\mathcal{L})) \tag{3.11}$$

for $\mathcal{L} \in W$ (see for instance [12, p. 42]) and then the Leray spectral sequence for p (see for instance [12, p. 33]), one gets an isomorphism of μ_{N_W}-representations

$$H^1(M', p^*\mathcal{L}) = H^1(M, p_*(\mathbb{C}_{M'}) \otimes \mathcal{L}). \tag{3.12}$$

Following the argument in the proof of Thm. 1.6 in [1], we get

$$\dim H^1(M, \mathcal{L}) \geq -\chi(S_W).$$

The only point which deserves some attention is the fact that S_W and $\widehat{S}_W$ do not admit finite triangulations as required in [1], since they are not compact. However, we can replace them by finite simplicial complexes without changing the homotopy type, e.g. S_W can be replaced by the compact Riemann surface with boundary obtained from $\mathbb{P}^1$ by deleting small open discs centered at the points in B_W.

The fact that there are only finitely many local systems $\mathcal{L} \in W$ such that

$$\dim H^1(M, \mathcal{L}) > -\chi(S_W)$$

follows by an argument similar to the end of the proof of [1, Prop. 1.7, Section V]. For a different approach and a generalization to translated components, see Corollaries 4.7 and 5.9 below.

Finally, claim (v) follows directly from (iv). □

REMARK 3.7. Conversely, if $f : M \to S$ is a morphism with a generic connected fiber and with $\chi(S) < 0$, then $W_f = f^*(\mathbb{T}(S))$ is an irreducible component in $\mathcal{V}_1(M)$ such that $1 \in W_f$ and $\dim W_f \geq 2$ (see [1, Section V, Prop. 1.7]). Some basic situations of this general construction of irreducible components W_f are the following.

CASE A.

(i) *The local components* (see for instance [24, Subsection (2.3)] in the case of line arrangements). The case of curve arrangements in $\mathbb{P}^2$ runs as follows. Let $p \in \mathbb{P}^2$ be a point such that there is a degree d_p and an integer $k_p > 2$ such that

(1) the set $A_p = \{j \mid p \in C_j \text{ and } \deg C_j = d_p\}$ has cardinality k_p;
(2) $\dim \langle f_j \mid j \in A_p \rangle = 2$, with $f_j = 0$ being an equation for C_j.

If $\{P, Q\}$ is a basis of this 2-dimensional vector space, then the associated pencil induces a map

$$f_p : M \to S_p$$

where S_p is obtained from $\mathbb{P}^1$ by deleting the k_p points corresponding to the curves C_j, for $j \in A_p$. In this way, the point p produces an irreducible component in $\mathcal{V}_1(M)$, namely

$$W_p = f_p^*(\mathbb{T}(S_p))$$

of dimension $k_p - 1$, and which is called local because it depends only on the chosen point p. Note that in the case of line arrangements, p can be chosen to be any point of multiplicity at least 3.

(ii) *The components associated to neighborly partitions* (see [21]) correspond exactly to pencils associated to the line arrangement, as remarked in [18, proof of Theorem 2.4].

CASE B. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be as in Example 2.3. Let g_1 and g_2 be two weighted homogeneous polynomials of degree d with respect to the weights $\mathbf{w}$ such that

$$X \cap \{g_1 = 0\} \cap \{g_2 = 0\} = 0.$$

Define $g : X \setminus 0 \to \mathbb{P}^1$ by $x \mapsto (g_1(x) : g_2(x))$. Note that g is constant on the corresponding $\mathbb{C}^*$-orbits. Assume that the generic fiber of g is connected, i.e. it coincides with an orbit. Let $B \subset \mathbb{P}^1$ be a finite subset such that $k = |B| > 2$ and $\mathcal{C}_b = g^{-1}(b)$ is connected for any $b \in B$. Then if we set $S = \mathbb{P}^1 \setminus B$, $\mathcal{C} = \bigcup_{b\in B} \mathcal{C}_b$ and $M = (X \setminus 0) \setminus \mathcal{C}$, we have $H_1(M) = \mathbb{Z}^k$, $\mathbb{T}(M) = (\mathbb{C}^*)^k$, $\mathbb{T}(S) = (\mathbb{C}^*)^{k-1}$ and the subtorus $W = g^*(\mathbb{T}(S))$ is a $(k-1)$-dimensional irreducible component of $\mathcal{V}_1(M)$.

All these points in Case A are illustrated by the following beautiful example.

EXAMPLE 3.8. This is a key example discovered by A. Suciu (see Example 4.1 in [24] and Example 10.6 in [25]). Consider the line arrangement in $\mathbb{P}^2$ given by the equation

$$xyz(x-y)(x-z)(y-z)(x-y-z)(x-y+z) = 0.$$

We number the lines of the associated affine arrangement in $\mathbb{C}^2$ (obtained by setting $z = 1$) as follows: $L_1 : x = 0$, $L_2 : x - 1 = 0$, $L_3 : y = 0$, $L_4 : y - 1 = 0$, $L_5 : x - y - 1 = 0$, $L_6 : x - y = 0$ and $L_7 : x - y + 1 = 0$ (see the pictures in Example 4.1 in [24] and Example 10.6 in [25]). We also consider the line at infinity $L_8 : z = 0$. As stated in Example 4.1 in [24], there are:

(i) Seven local components: six of dimension 2, corresponding to the triple points, and one of dimension 3, for the quadruple point.

(ii) Five components of dimension 2, passing through the unit element and coming from the following neighborly partitions (of braid subarrangements): (15|26|38), (28|36|45), (14|23|68), (16|27|48) and (18|37|46). For instance, the pencil corresponding to the first partition is given by $P = L_1L_5 = x(x-y-z)$ and $Q = L_2L_6 = (x-z)(x-y)$. Note that $L_3L_8 = yz = Q - P$ is a decomposable fiber in this pencil.

(iii) A 1-dimensional component W in $\mathcal{V}_1(M)$ with

$$\rho_W = (1, -1, -1, 1, 1, -1, 1, -1) \in \mathbb{T}(M) \subset (\mathbb{C}^*)^8$$

and $f_W : M \to \mathbb{C}^*$ given by

$$f_W(x : y : z) = \frac{x(y-z)(x-y-z)^2}{(x-z)y(x-y+z)^2}$$

or, in affine coordinates,

$$f_W(x, y) = \frac{x(y-1)(x-y-1)^2}{(x-1)y(x-y+1)^2}.$$

Then $W \subset \mathcal{V}_1(M)$ and $W \cap \mathcal{V}_2(M)$ consists of two characters, ρ_W above and

$$\rho'_W = (-1, 1, 1, -1, 1, -1, 1, -1).$$

Note that this component W is a translated coordinate component. This is related to the fact that the associated pencil is special. For more on this arrangement see Example 5.12.

4. Translated components and constructible sheaves

We need the following version of the *projection formula*, which is used very often, e.g. [1], [20], but for which I was not able to find a reference.

LEMMA 4.1. *For any local system $\mathcal{L}_1$ on M and any local system $\mathcal{L}_2$ on S, one has*

$$(Rf_*\mathcal{L}_1) \otimes \mathcal{L}_2 = Rf_*(\mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2).$$

PROOF. To prove this lemma, we start with the usual projection formula, i.e., in the above notation,

$$(Rf_!\mathcal{L}_1) \otimes \mathcal{L}_2 = Rf_!(\mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) \tag{4.1}$$

(see Thm. 2.3.29, p. 42 in [12]). Let Z be a connected smooth complex algebraic variety of dimension m. Then the dualizing sheaf ω_Z is just $\mathbb{C}_Z[2m]$ and $D_Z\mathcal{L} = \mathcal{L}^\vee[2m]$ for any local system $\mathcal{L}$ on Z (see Example 3.3.8, p. 69 in [12]). Note also that for two bounded constructible complexes $\mathcal{A}^*$ and $\mathcal{B}^*$ in $D_c^b(Z, \mathbb{C})$ we have the isomorphisms

$$\begin{aligned} D_Z\mathcal{A}^* \otimes \mathcal{B}^* &= R\operatorname{Hom}(\mathcal{A}^*, \omega_Z) \otimes \mathcal{B}^* = R\operatorname{Hom}(\mathcal{A}^*, \omega_Z \otimes \mathcal{B}^*) \\ &= R\operatorname{Hom}(\mathcal{A}^*, \mathcal{B}^*)[2m]. \end{aligned} \tag{4.2}$$

It follows that

$$\begin{aligned} D_Z(\mathcal{A}^* \otimes \mathcal{B}^*) &= R\operatorname{Hom}(\mathcal{A}^* \otimes \mathcal{B}^*, \omega_Z) = R\operatorname{Hom}(\mathcal{A}^*, R\operatorname{Hom}(\mathcal{B}^*, \omega_Z)) \\ &= D_Z\mathcal{A}^* \otimes D_Z\mathcal{B}^*[-2m]. \end{aligned} \tag{4.3}$$

For the second isomorphism here we refer to Prop. 10.23, p. 175 in [4]. Apply now the duality functor D_S to the projection formula (4.1). On the left hand side we get

$$\begin{aligned} D_S((Rf_!\mathcal{L}_1) \otimes \mathcal{L}_2) &= D_S(Rf_!\mathcal{L}_1) \otimes D_S(\mathcal{L}_2)[-2] = Rf_*(D_M\mathcal{L}_1) \otimes D_S(\mathcal{L}_2)[-2] \\ &= Rf_*(\mathcal{L}_1^\vee) \otimes \mathcal{L}_2^\vee[4]. \end{aligned}$$

Except the isomorphisms explained above we have used here the isomorphism $D_SRf_! = Rf_*D_M$ (see Cor. 4.1.17, p. 90 in [12]). Similarly, on the right hand side we get $D_SRf_!(\mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) = Rf_*D_M(\mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) = Rf_*(\mathcal{L}_1^\vee \otimes (f^{-1}\mathcal{L}_2)^\vee)[4]$. Since $(f^{-1}\mathcal{L}_2)^\vee = f^{-1}(\mathcal{L}_2^\vee)$ and since any local system is the dual of its own dual, the proof is complete. □

Note that $\mathcal{F} = R^0f_*(\mathcal{L}_1)$ is in general no longer a local system on S, but a *constructible sheaf*. By definition, there exists a minimal finite set $\Sigma = \Sigma(\mathcal{F}) \subset S$, called the *singular support* of $\mathcal{F}$, such that $\mathcal{F}|(S \setminus \Sigma)$ is a local system (see [12, p. 87]). The main properties of this sheaf are given in the following result.

LEMMA 4.2. *Let $\mathcal{L}_1$ be a rank one local system on M, F the generic fiber of $f : M \to S$, and set $\mathcal{F} = R^0f_*(\mathcal{L}_1)$. Then either*

(i) *the restriction $\mathcal{L}_1|F$ is trivial, $\mathcal{F}|(S \setminus \Sigma)$ is a rank one local system and $\mathcal{F}_s = 0$ if and only if $s \in \Sigma$, or*
(ii) *the restriction $\mathcal{L}_1|F$ is non-trivial and $\mathcal{F} = 0$.*

PROOF. Consider first case (i). If $S' \subset S$ is a Zariski open subset such that the restriction $f' : M' \to S'$ with $M' = f^{-1}(S')$ is a topologically locally trivial fibration, then $\mathcal{F}|S'$ is a rank one local system. Indeed, for $s \in S'$ we have

$$\mathcal{F}_s = \lim_{s\in D} \mathcal{F}(D) = \lim_{s\in D} H^0(f^{-1}(D), \mathcal{L}_1) = \mathbb{C}.$$

Here the limit is taken over all the sufficiently small open discs D in S centered at s, and the last equality comes from the fact that the inclusion $F_s = f^{-1}(s) \to f^{-1}(D)$ is a homotopy equivalence and $\mathcal{L}_1|F_s = \mathbb{C}_{F_s}$ (recall that F_s is connected, and hence $f^{-1}(D)$ is connected as well). In particular $\Sigma \subset S \setminus S'$, and hence $\Sigma = \emptyset$ if $f : M \to S$ is a locally trivial fibration. The above argument also shows that $\mathcal{F}_s = 0$ if and only if $s \in \Sigma$.

In case (ii), assume that $\mathcal{F}_s \neq 0$ for some $s \in S$. Then there is a small open disc D in S centered at s such that $H^0(f^{-1}(D), \mathcal{L}_1) \neq 0$. This implies that the restriction $\mathcal{L}_1|f^{-1}(D)$ is trivial, and hence $\mathcal{L}_1|F$ is trivial as well, a contradiction. □

We have the following key result.

PROPOSITION 4.3. *Let $f : M \to S$ be a surjective morphism with a generic connected fiber F from the surface M onto the curve S. Then for any local system $\mathcal{L}_1$ on M and any local system $\mathcal{L}_2$ on S, one has the following exact sequence:*

$$0 \to H^1(S, R^0 f_*(\mathcal{L}_1) \otimes \mathcal{L}_2) \to H^1(M, \mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) \to H^0(S, R^1 f_*(\mathcal{L}_1) \otimes \mathcal{L}_2).$$

The last morphism is surjective in any of the following situations:

(i) *S is affine;*
(ii) *$\mathcal{L}_1|F$ is non-trivial;*
(iii) *$\mathcal{L}_1|F$ is trivial and $\mathcal{L}_2$ is generic, i.e. it is different from a finite set of local systems depending on f and $\mathcal{L}_1$.*

PROOF. We use the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q f_*(\mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2))$$

converging to $H^{p+q}(M, \mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2)$. By Lemma 4.1 we have

$$R^q f_*(\mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) = R^q f_*(\mathcal{L}_1) \otimes \mathcal{L}_2.$$

In particular, the above spectral sequence yields the exact sequence

$$0 \to H^1(S, R^0 f_*(\mathcal{L}_1) \otimes \mathcal{L}_2) \to H^1(M, \mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) \to K_2^{0,1} \to 0$$

where $K_2^{0,1}$ is the kernel of the differential $E_2^{0,1} \to E_2^{2,0}$.

When S is affine, this spectral sequence degenerates at E_2 since $E_2^{p,q} = 0$ for $p \notin \{0, 1\}$ by the Artin theorem (see Thm. 4.1.26, p. 95 in [12]), and this proves claim (i).

In case (ii) one has $E_2^{2,0} = H^2(S, R^0 f_*(\mathcal{L}_1) \otimes \mathcal{L}_2) = 0$ since $\mathcal{F} = R^0 f_*(\mathcal{L}_1) = 0$.

For case (iii), we use the exact sequence of cohomology with compact supports

$$0 = H^1(\Sigma, \mathcal{F} \otimes \mathcal{L}_2) \to H^2_c(U, \mathcal{F} \otimes \mathcal{L}_2) \to H^2(S, \mathcal{F} \otimes \mathcal{L}_2) \to H^2(\Sigma, \mathcal{F} \otimes \mathcal{L}_2) = 0$$

where $U = S \setminus \Sigma$ (note that S can be assumed to be compact, since otherwise we are in the affine case (i)); see for instance [12, p. 46]. Now $\mathcal{F}|U = \mathcal{L}_0$ is a rank one local system, and we can use duality to get

$$H_c^2(U, \mathcal{L}_0 \otimes \mathcal{L}_2) = H^0(U, \mathcal{L}_0^\vee \otimes \mathcal{L}_2^\vee)^\vee.$$

These cohomology groups are clearly trivial for $\mathcal{L}_2|U \neq \mathcal{L}_0^{-1}$. Since the restriction $\mathcal{L}_2|U$ determines the local system $\mathcal{L}_2$, this means that there is at most one local system $\mathcal{L}_2$ for which $E_2^{2,0} \neq 0$. □

To continue we need the following.

LEMMA 4.4. *The constructible sheaf $\mathcal{G} = R^1 f_*(\mathbb{C}_M)$ has no section with finite support.*

PROOF. This proof is given in D. Arapura [1, Proposition 1.7], but we repeat it here for the reader's convenience, and to clarify some points in Arapura's proof. Let D be a small disc in S centered at a bifurcation point $b \in S$, let $D^* = D \setminus \{b\}$ and choose a point $q \in D^*$. Set $M_D = f^{-1}(D)$, $M_D^* = f^{-1}(D^*)$ and $M_q = f^{-1}(q)$. The claim is equivalent to showing that the morphism

$$i_q^* : H^1(M_D, \mathbb{C}) \to H^1(M_q, \mathbb{C})$$

induced by the inclusion $i_q : M_q \to M_D$ is injective. Indeed, one has natural identifications $\mathcal{G}_b = H^1(M_D, \mathbb{C})$ and $\mathcal{G}_q = H^1(M_q, \mathbb{C})$ and i_q^* corresponds to the restriction morphism $\mathcal{G}_b \to \mathcal{G}_q$. The open inclusion $j_b : M_D^* \to M_D$ clearly induces a surjective morphism $H_1(M_D^*) \to H_1(M_D)$, and hence an injective morphism $j_b^* : H^1(M_D, \mathbb{C}) \to H^1(M_D^*, \mathbb{C})$.

Now, if the disc D was chosen small enough, the restriction of f over D^* is a locally trivial fibration with fiber type M_q and hence we get the following exact sequence (which is dual to an exact sequence similar to (5.2) below):

$$0 \to H^1(D^*, \mathbb{C}) \xrightarrow{f^*} H^1(M_D^*, \mathbb{C}) \xrightarrow{\iota_q^*} H^1(M_q, \mathbb{C}) \tag{4.4}$$

where $\iota_q : M_q \to M_D^*$ is the inclusion. It follows that $i_q^* : H^1(M_D, \mathbb{C}) \to H^1(M_q, \mathbb{C})$ is injective if and only if $I = \mathrm{im}(j_b^*) \cap \mathrm{im}\{H^1(D^*, \mathbb{C}) \xrightarrow{f^*} H^1(M_D^*, \mathbb{C})\} = 0$. Since $f : M \to S$ is surjective, it follows that $H = f^{-1}(b)$ is a hypersurface in M. Let p be a smooth point on the associated reduced hypersurface. It follows that there is an analytic curve germ $\phi : (\mathbb{C}, 0) \to (M, p)$ such that $f(\phi(t))$ has some order $d \geq 1$, where d is the multiplicity of H at p. Note that in D. Arapura's proof [1], the multiplicity d is supposed to be 1, which is not always the case.

Let $\sigma \in I$. Since $\sigma \in \mathrm{im}\{H^1(D^*, \mathbb{C}) \xrightarrow{f^*} H^1(M_D^*, \mathbb{C})\}$, it follows that there is a $\beta \in \mathrm{Hom}(H_1(D^*), \mathbb{C}) = H^1(D^*, \mathbb{C})$ such that $\sigma = \beta \circ f_*$. The germ ϕ induces a morphism $\phi_* : H_1(D^*) \to H_1(M_D^*)$ such that $f_* \circ \phi_*$ is multiplication by d on the group $H_1(D^*) = \mathbb{Z}$. It follows that $\sigma \circ \phi_* = d \cdot \beta$.

On the other hand, since $\sigma \in \mathrm{im}(j_b^*)$, there is $\sigma' \in \mathrm{Hom}(H_1(M_D), \mathbb{C})$ such that $\sigma = \sigma' \circ j_{b*}$. It follows that $\sigma \circ \phi_* = \sigma' \circ j_{b*} \circ \phi_*$ is trivial, since $j_b \circ \phi$ has an obvious extension ϕ from the punctured disc D^* to the disc D. In conclusion, $\sigma = 0$, and so $I = 0$, proving our claim. □

The above lemma can be generalized as follows.

PROPOSITION 4.5. *Let* $f : M \to S$ *be a surjective morphism with* $\dim S = 1$ *and a connected generic fiber* F. *If* $\mathcal{L}$ *is a rank one local system on* M, *then the constructible sheaf* $\mathcal{G} = R^1 f_*(\mathcal{L})$ *has no section with finite support. Equivalently,*

$$H^0(S, \mathcal{G} \otimes \mathcal{L}_2) = 0$$

for all but finitely many local systems $\mathcal{L}_2 \in \mathbb{T}(S)$.

PROOF. First we check that the last two claims are equivalent. Locally, the two sheaves $\mathcal{G}$ and $\mathcal{G} \otimes \mathcal{L}_2$ coincide, so they admit at the same time non-zero sections with finite support. If this is the case, then clearly

$$H^0(S, \mathcal{G} \otimes \mathcal{L}_2) \neq 0$$

for any local system $\mathcal{L}_2$. Suppose now that there are no such sections with finite support. Let $\Sigma' := \Sigma(\mathcal{G}) = \Sigma(\mathcal{G} \otimes \mathcal{L}_2)$ and note that in this case the restriction

$$H^0(S, \mathcal{G} \otimes \mathcal{L}_2) \to H^0(S \setminus \Sigma', \mathcal{G} \otimes \mathcal{L}_2)$$

is injective. Since $S \setminus \Sigma'$ is homotopically a bouquet of circles (or a compact curve if S is compact and $\Sigma' = \emptyset$), the last group is non-zero exactly when the monodromy of $\mathcal{L}_2$ along any of the loops forming a basis for the integral homology of S is the inverse of one of the eigenvalues of the monodromy of the local system $\mathcal{G}|(S \setminus \Sigma')$ along this loop, i.e. for a finite number of local systems $\mathcal{L}_2$.

With the notation from the proof of Lemma 4.4, we have to prove that the restriction morphism

$$i_q^* : H^1(M_D, \mathcal{L}) \to H^1(M_q, \mathcal{L}|M_q)$$

is injective.

The open inclusion $j_b : M_D^* \to M_D$ clearly induces an epimorphism $\pi_1(M_D^*) \to \pi_1(M_D)$, and hence an injective morphism $j_b^* : H^1(M_D, \mathcal{L}) \to H^1(M_D^*, \mathcal{L})$. This follows for instance by using the description of the first twisted cohomology groups $H^1(M, \mathcal{L})$ in terms of cross-homomorphisms (see [22]).

CASE 1: the restriction $\mathcal{L}|F$ is the trivial local system $\mathbb{C}_F$. To study the local system $\mathcal{L}' = \mathcal{L}|M_D^*$, note that it corresponds to a character

$$\rho : \pi_1(M_D^*) \to \mathbb{C}^*.$$

The exact sequence

$$1 \to \pi_1(M_q) \to \pi_1(M_D^*) \to \pi_1(D^*) \to 1$$

and the triviality of $\mathcal{L}|M_q$ (note that M_q is a generic fiber of f) imply that $\mathcal{L}' = f^*(\mathcal{L}_a)$, where $\mathcal{L}_a$ is the rank one local system on D^* with monodromy $a \in \mathbb{C}^*$. For this class of local systems we have a long exact cohomology sequence

$$\text{(4.5)} \qquad \to H^0(M_q, \mathbb{C}) \xrightarrow{h^0 - a^{-1} \cdot \mathrm{Id}} H^0(M_q, \mathbb{C}) \to H^1(M_D^*, \mathcal{L}') \xrightarrow{i_q^*} H^1(M_q, \mathbb{C})$$

(see [12, p. 212]). Here h^m are the monodromy operators of the fibration $M_q \to M_D^* \to D^*$ and clearly $h^0 = \mathrm{Id}$ since the fiber M_q is connected.

If $a = 1$, then locally at the bifurcation point $b \in S$ we have exactly the same situation as in Lemma 4.4, hence the result is already proven.

If $a \neq 1$, then the morphism $H^0(M_q, \mathbb{C}) \xrightarrow{h^0 - a^{-1}\cdot \mathrm{Id}} H^0(M_q, \mathbb{C})$ is an isomorphism, which yields an injection $H^1(M_D^*, \mathcal{L}') \xrightarrow{\iota_q^*} H^1(M_q, \mathbb{C})$. This gives the result in this case, since the composition of two injections is an injection.

CASE 2: the restriction $\mathcal{L}|F$ is a non-trivial local system. In this case $R^0 f_* \mathcal{L} = 0$ and the Leray spectral sequence of the fibration $M_q \to M_D^* \to D^*$ yields an isomorphism

$$H^1(M_D^*, \mathcal{L}) \to H^0(D^*, R^1 f_* \mathcal{L}).$$

Since $H^0(D^*, R^1 f_* \mathcal{L})$ is just the invariant part of $H^1(M_q, \mathcal{L}|M_q)$ under the monodromy of the local system $R^1 f_* \mathcal{L}$ on D^*, this gives rise to a natural injection

$$H^1(M_D^*, \mathcal{L}) \xrightarrow{\iota_q^*} H^1(M_q, \mathcal{L}|M_q).$$

which completes the proof in this case as well. □

The following corollary of the exact sequence in Proposition 4.3 and of Proposition 4.5 gives also a new, topological proof for the claim in Theorem 3.6(ii).

COROLLARY 4.6. *Let $f : M \to S$ be a surjective morphism with a generic connected fiber F from the surface M onto the curve S with $b_1(S) > 0$. Then for any local system $\mathcal{L}_1$ on M such that $\mathcal{L}_1|F$ is non-trivial, and for any generic local system $\mathcal{L}_2 \in \mathbb{T}(S)$, one has $H^1(M, \mathcal{L}_1 \otimes f^*(\mathcal{L}_2)) = 0$.*

As a consequence of Proposition 4.3, we get the following extension of Theorem 3.6(iv). (This special case corresponds to the case $\mathcal{L}_\rho = \mathbb{C}_M$, when $R^0 f_*(\mathcal{L}_\rho) = \mathbb{C}_S$ and hence $\Sigma = \emptyset$. For an illustration of the general case, see Example 5.14 below.)

COROLLARY 4.7. *If $\mathcal{L}_\rho$ is a rank one local system on M such that $\mathcal{L}_\rho|F$ is trivial, then*

$$\dim H^1(M, \mathcal{L}_\rho \otimes f^{-1}\mathcal{L}) \geq -\chi(S) + |\Sigma(R^0 f_*(\mathcal{L}_\rho))|$$

with equality for all but finitely many local systems $\mathcal{L} \in \mathbb{T}(S)$. In particular, if $W_{f,\rho} = \rho \otimes f^*(\mathbb{T}(S))$ is a positive-dimensional irreducible component of $\mathcal{V}_1(M)$, then $W_{f,\rho}$ is an irreducible component of $\mathcal{V}_q(M)$ for any $1 \leq q \leq q(f, \rho) := -\chi(S) + |\Sigma(R^0 f_*(\mathcal{L}_\rho))|$. Conversely, any positive-dimensional irreducible component of $\mathcal{V}_q(M)$ for $q \geq 1$ is of this type.

PROOF. To estimate $\dim H^1(S, \mathcal{F} \otimes \mathcal{L}_2)$ we compute

$$\chi(S, \mathcal{F} \otimes \mathcal{L}_2) = \dim H^0(S, \mathcal{F} \otimes \mathcal{L}_2) - \dim H^1(S, \mathcal{F} \otimes \mathcal{L}_2) = \chi(S \setminus \Sigma)$$

using Thm. 4.1.22, p. 93 in [12]. This yields

$$\dim H^1(S, \mathcal{F} \otimes \mathcal{L}_2) = \dim H^0(S, \mathcal{F} \otimes \mathcal{L}_2) - \chi(S) + |\Sigma| \geq -\chi(S). \tag{4.6}$$

In the case $\mathcal{L}_1 = \mathcal{L}_\rho$ such that $\mathcal{L}_1|F$ is trivial, Proposition 4.3 yields

$$H^1(M, \mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) = H^1(S, R^0 f_*(\mathcal{L}_1) \otimes \mathcal{L}_2)$$

for all but finitely many local systems $\mathcal{L}_2 \in \mathbb{T}(S)$. Similarly, the description of Σ given above shows that the group $H^0(S, \mathcal{F} \otimes \mathcal{L}_2)$ is zero unless $\Sigma = \emptyset$ and $\mathcal{L}_2 = \mathcal{F}^{-1}$.

The only thing to explain is the last claim in the case $q > 1$. Assume that W_q is a positive-dimensional irreducible component of $\mathcal{V}_q(M)$ for $q > 1$. Since $\mathcal{V}_q(M) \subset \mathcal{V}_1(M)$, there is an irreducible component W of $\mathcal{V}_1(M)$ such that $W_q \subset W$. Then the first claim in Corollary 4.7 implies that $W \subset \mathcal{V}_q(M)$, i.e. $W_q = W$. □

5. Translated components and multiple fibers

Let W be a translated irreducible component of $\mathcal{V}_1(M)$, i.e. $1 \notin W$. Then, as in Theorem 3.6, there is a torsion character $\rho \in \mathbb{T}(M)$ and a surjective morphism $f : M \to S$ with connected generic fiber F such that

$$W = \rho f^*(\mathbb{T}(S)). \tag{5.1}$$

We say in this situation that the component W is *associated* to the mapping f. In this section we give detailed information on the torsion character $\rho \in \mathbb{T}(M)$ in terms of the geometry of the associated mapping $f : M \to S$.

5.1. *The general setting*

Let F be the generic fiber of the mapping $f : M \to S$, i.e. F is the fiber of the topologically locally trivial fibration $f' : M' \to S'$ associated to f as in the previous section. Then we have an exact sequence

$$H_1(F) \xrightarrow{i'_*} H_1(M') \xrightarrow{f'_*} H_1(S') \to 0 \tag{5.2}$$

as well as a sequence

$$H_1(F) \xrightarrow{i_*} H_1(M) \xrightarrow{f_*} H_1(S) \to 0 \tag{5.3}$$

which is not necessarily exact in the middle, i.e. the group

$$T(f) = \ker f_*/\operatorname{im} i_* \tag{5.4}$$

is in general non-trivial. Here $i : F \to M$ and $i' : F \to M'$ denote the inclusions, and homology is taken with $\mathbb{Z}$-coefficients if not stated otherwise.

This group was studied in a compact (proper) setting by Serrano (see [23]), but no relation to local systems was considered there. On the other hand, this compact situation was also studied by A. Beauville in [3], with essentially the same aims as ours.

The sequence (5.3) induces an obvious exact sequence

$$0 \to T(f) \to H_1(M)/\operatorname{im} i_* \xrightarrow{f_*} H_1(S) \to 0. \tag{5.5}$$

Since $H_1(S)$ is a free $\mathbb{Z}$-module, applying the fuctor $\text{Hom}(-, \mathbb{C}^*)$ to the exact sequence (5.5), we get a new exact sequence

$$(5.6) \qquad 1 \to \mathbb{T}(S) \to \mathbb{T}(M)_F \to \text{Hom}(T(f), \mathbb{C}^*) \to 1.$$

Here $\mathbb{T}(M)_F$ is the subgroup in $\mathbb{T}(M)$ formed by all characters $\chi : H_1(M) \to \mathbb{C}^*$ such that $\chi \circ i_* = 0$. This means exactly that the associated local system $\mathcal{L}_\chi$ by restriction to F yields the trivial local system $\mathbb{C}_F$.

The torsion character $\rho \in \mathbb{T}(M)$ which occurs in (5.1) is in this subgroup $\mathbb{T}(M)_F$ (see Theorem3.6(ii)). Moreover, this character ρ is not unique, but its class

$$(5.7) \qquad \tilde{\rho} \in \mathbb{T}(M)_F/\mathbb{T}(S) \simeq \text{Hom}(T(f), \mathbb{C}^*) = \hat{T}(f)$$

is uniquely determined. From now on, we will regard $\tilde{\rho} \in \hat{T}(f)$. Hence, to understand the possible choices for $\tilde{\rho}$, we have to study the group $T(f)$ or, equivalently, its Pontryagin dual $\hat{T}(f)$.

5.2. *The computation of the group $T(f)$*

Let $f : M \to S$ be a surjective morphism with a generic connected fiber F as above. Let $C(f) \subset S$ be a finite, minimal subset such that if we put $S' = S \setminus C(f)$, $M' = f^{-1}(S')$, then the induced mapping $f : M' \to S'$ is a locally trivial fibration. For $c \in C(f)$ we denote by m_c the multiplicity of the divisor $F_c = f^{-1}(c)$. We have the following result, where the first claim is already in [3, the remarks after Proposition 1.19], and in Serrano [23]. However, this second author wrongly claims that the isomorphism in (i) holds for case (ii) as well. The mistake in [23] is in the proof of Thm. 1.3, Claim 1, where the relation between the γ_p's is incorrect. In the proof below, these 1-cycles γ_p's are denoted by δ_c and the correct relation is $\Delta = 0$.

THEOREM 5.3. (i) *If the curve S is proper, then*

$$T(f) = \Big(\bigoplus_{c \in C(f)} \mathbb{Z}/m_c\mathbb{Z}\Big)/(\hat{1}, \dots, \hat{1}).$$

(ii) *If the curve S is not proper, then*

$$T(f) = \bigoplus_{c \in C(f)} \mathbb{Z}/m_c\mathbb{Z}.$$

PROOF. The main ingredient to prove this theorem is Lemma 3 in [7], which yields the exact sequence

$$(5.8) \qquad \pi_1(F) \to \pi_1(M) \to \pi_1^{\text{orb}}(f) \to 1.$$

Here the *orbifold fundamental group* $\pi_1^{\text{orb}}(f)$ of the mapping f is the quotient of $\pi_1(S')$ by the normal subgroup generated by the elements $\delta_c^{m_c}$ for $c \in C(f)$, with δ_c a simple loop going once around the point c. Note that this result is stated in [7] under the assumption that the curve S is proper, but the proof given there works for S non-proper as well.

The exact sequence (5.8) yields, by passing to abelianizations, the exact sequence

$$H_1(F) \to H_1(M) \to H_1^{\mathrm{orb}}(f) \to 0. \tag{5.9}$$

We will denote by f_*^{orb} the epimorphism $H_1(M) \to H_1^{\mathrm{orb}}(f)$ in the exact sequence above.

Coming back to the notation from Subsection 3.1, we get the following presentation for the *orbifold first homology group* $H_1^{\mathrm{orb}}(f)$ of the mapping f:

$$H_1^{\mathrm{orb}}(f) = \mathbb{Z}\langle \delta_1, \dots, \delta_{2g+k}; \delta_c \text{ for } c \in C(f)\rangle / \langle \Delta, m_c\delta_c \text{ for } c \in C(f)\rangle \tag{5.10}$$

where $\Delta = \delta_{2g+1} + \cdots + \delta_{2g+k} + \sum_c \delta_c$. There is a natural surjective morphism

$$\theta : H_1^{\mathrm{orb}}(f) \to H_1(S) \tag{5.11}$$

given by $\delta_i \mapsto \delta_i$ for $i = 1, \dots, 2g + k$ and $\delta_c \mapsto 0$ for $c \in C(f)$. Here we use the presentation for $H_1(S)$ given in the formula (3.1). Comparing the exact sequence (5.9) to the sequence (5.3), we get an isomorphism

$$\ker(\theta) \simeq T(f). \tag{5.12}$$

When S is proper we have $k = 0$ and the group $\ker(\theta)$ is spanned by the loops δ_c for $c \in C(f)$, with the relations $m_c \cdot \delta_c = 0$ and $\Delta = \sum_c \delta_c = 0$. This yields claim (i), since clearly Δ corresponds to the element $(\hat{1}, \dots, \hat{1})$.

When S is not proper we have $k > 0$ and the group $\ker(\theta)$ is spanned by Δ and the loops δ_c for $c \in C(f)$, with the relations $m_c \cdot \delta_c = 0$ and $\Delta = 0$. Claim (ii) follows from this description. □

COROLLARY 5.4. *There is a non-canonical isomorphism*

$$H_1^{\mathrm{orb}}(f) \simeq H_1(S) \times T(f).$$

In particular,

$$\mathbb{T}^{\mathrm{orb}}(f) \simeq \mathbb{T}(S) \times \hat{T}(f)$$

where $\hat{T}(f) = \mathrm{Hom}(T(f), \mathbb{C}^*)$ *is the Pontryagin dual of the finite group* $T(f)$ *and* $\mathbb{T}^{\mathrm{orb}}(f) = \mathrm{Hom}(H_1^{\mathrm{orb}}(f), \mathbb{C}^*)$ *is the corresponding orbifold character group of* f.

EXAMPLE 5.5 (Computation of the group $T(f)$ in Case B, the Seifert links). Let $(X, 0)$ be a complex quasi-homogeneous normal surface singularity. Then the surface $X^* = X \setminus \{0\}$ is smooth and it has a $\mathbb{C}^*$-action with finite isotropy groups $\mathbb{C}^*_x$. These isotropy groups can be assumed to be trivial, except for those corresponding to finitely many orbits $p_1, \dots, p_s$ in $C(X) = X^*/\mathbb{C}^*$. We set $k_p = |\mathbb{C}^*_p|$ for $p \in P = \{p_1, \dots, p_s\}$.

The quotient $C(X)$ is a smooth projective curve. For any finite subset B in $C(X)$ we get a surjective mapping $f : M \to S$ induced by the quotient map $f_0 : X^* \to C(X)$, where $S = C(X) \setminus B$ and $M = f^{-1}(S)$.

In addition, the curve $C(X)$ is rational iff the link $L(X)$ of the singularity $(X, 0)$ is a $\mathbb{Q}$-homology sphere (use Cor. (3.7) on p. 53 and Thm. (4.21) on p. 66 in [11]). In particular, if the link $L(X)$ of the singularity $(X, 0)$ is a $\mathbb{Z}$-homology sphere, then $H_1(M) = \mathbb{Z}^q$ where

$q = |B|$, and a basis is provided by small loops γ_b around the fiber $F_b = f^{-1}(b)$ for $b \in B$, as explained in Subsection 3.2.

One has $f_*(\gamma_b) = k_b\delta_b$, with k_b the order of the isotropy groups of points x such that $f(x) = b$, and δ_b a small loop about $b \in \mathbb{P}^1$. The set of critical values of the map $f_0 : X^* \to C(X)$ is exactly P, and each fiber $F_p = f_0^{-1}(p)$ is smooth (isomorphic to $\mathbb{C}^*$), but of multiplicity $k_p > 1$. Writing down the map $f_{0*} : H_1(X^*) \to H_1(C(X))$ and using its surjectivity, we find that the integers k_p are pairwise coprime.

Let $(X, 0)$ be the germ of an isolated complex surface singularity such that the corresponding link L_X is an integral homology sphere. Let $(Y, 0)$ be a curve singularity on $(X, 0)$. Then using the conic structure of analytic sets, we see that the local complement $X \setminus Y$, with X and Y Milnor representatives of the singularities $(X, 0)$ and $(Y, 0)$, respectively, has the same homotopy type as the link complement $M = L_X \setminus L_Y$, where L_Y denotes the link of Y.

Moreover, if $(X, 0)$ and $(Y, 0)$ are quasi-homogeneous singularities at the origin of some affine space $\mathbb{C}^N$, with respect to the same weights, then the local complement can be globalized, i.e., replaced by the smooth quasi-projective variety $X \setminus Y$, where X and Y are this time affine varieties representing the germs $(X, 0)$ and $(Y, 0)$ respectively.

Using the well-known analytic description of the Seifert link $L = (\Sigma(k_1, \dots, k_n), S_1 \cup \dots \cup S_q)$ with $k_j \geq 1$ and $n \geq q \geq 2$ given in [17, p. 62] and the above notation, we see that the link complement $M(L) = \Sigma(k_1, \dots, k_n) \setminus (S_1 \cup \dots \cup S_q)$ has the homotopy type of the surface M obtained from the surface singularity X by deleting the orbits (regular for $k_j = 1$ and singular for $k_j > 1$) corresponding to the q knots S_j, $j = 1, \dots, q$. In other words, we have a finite set $B \subset \mathbb{P}^1$ with $|B| = q$ and a mapping $f : M \to S = \mathbb{P}^1 \setminus B$.

Let $N = k_1 \cdots k_q$, $N_j = N/k_j$ for $1 \leq j \leq q$, $N' = k_{q+1} \cdots k_n$, $N'_j = N'/k_j$ for $q + 1 \leq j \leq n$. We can assume that for $j > q$ one has $k_j = 1$ iff $j > q + s$, with s a positive integer. The above theorem implies in this case

$$T(f) = \mathbb{Z}/N'\mathbb{Z} = \bigoplus_{q+1 \leq j \leq n} \mathbb{Z}/k_j\mathbb{Z}.$$

For another way of computing the group $T(f)$ in some cases, we refer to [13, Section 6].

DEFINITION 5.6. *For a character $\tilde{\rho} : T(f) \to \mathbb{C}^*$, we define the* support $\operatorname{supp}(\tilde{\rho})$ *of $\tilde{\rho}$ to be the singular set $\Sigma(\mathcal{F})$ of the constructible sheaf $\mathcal{F} = R^0 f_*(\mathcal{L}_\rho)$ for some representative ρ of $\tilde{\rho}$.*

In other words, a critical value $c \in C(f)$ is in $\operatorname{supp}(\tilde{\rho})$ if for a small disc D_c centered at c, the restriction of the local system $\mathcal{L}_\rho$ to the associated tube $T(F_c) = f^{-1}(D_c)$ about the fiber F_c is non-trivial. Since two such representatives ρ differ by a local system in $f^*(\mathbb{T}(S))$, it follows from Lemma 4.1 that this support is correctly defined.

THEOREM 5.7. *Let $f : M \to S$ be a surjective morphism, with connected generic fiber F, and let $\tilde{\rho} : T(f) \to \mathbb{C}^*$ be a character. Then* $\operatorname{supp}(\tilde{\rho})$ *is empty if and only if the character $\tilde{\rho}$ is trivial.*

PROOF. If the character $\tilde{\rho}$ is trivial, we can represent it by $\rho = 1$ and clearly in this case $\operatorname{supp}(\tilde{\rho}) = \Sigma(\mathbb{C}_S) = \emptyset$.

Conversely, assume that $\mathrm{supp}(\tilde{\rho}) = \emptyset$. It follows that for any special value $c \in C(f)$ and any small tube $T(F_c)$ about the fiber F_c, the restriction $\mathcal{L}_\rho | T(F_c)$ is trivial. We know in addition that $\mathcal{L}_\rho | F$ is trivial for any generic fiber F of f.

Let as before $f' : M' \to S'$ denote the maximal locally trivial fibration associated to f, and recall that $S' = S \setminus C(f)$. Let $\rho' : H_1(M') \to \mathbb{C}^*$ be the composition of the character $\rho : H_1(M) \to \mathbb{C}^*$ with the morphism $H_1(M') \to H_1(M)$ induced by the inclusion $M' \to M$. Using the exact sequence (5.2), it follows that there is a unique character $\alpha' : H_1(S') \to \mathbb{C}^*$ such that $\rho' = f'^*(\alpha')$.

Let $c \in C(f)$ be any bifurcation value for f and let δ_c be the cycle in $H_1(S')$ given by a small loop around c. Then, using the fact that f' is a locally trivial fibration with a connected fiber F, it follows that the cycle $\delta_c \in H_1(S')$ has a lifting to a cycle $\tilde{\delta}_c \in H_1(M')$ such that $f'_*(\tilde{\delta}_c) = \delta_c$ and with the support of $\tilde{\delta}_c$ contained in the tube $T(F_c)$. It follows that

$$\rho'(\tilde{\delta}_c) = 1 = \alpha'(\delta_c).$$

As a result there is a unique character $\alpha : H_1(S) \to \mathbb{C}^*$ such that α' is the composition of α with the morphism $H_1(S') \to H_1(S)$ induced by the inclusion $S' \to S$.

Now we replace the representative ρ for $\tilde{\rho}$ by the character $\rho_1 = \rho \cdot f^*(\alpha^{-1})$. It follows that the restriction of ρ_1 to $H_1(M')$ is the trivial character. Using the Mayer–Vietoris sequence to express $H_1(M)$ in terms of the covering $M = M' \cup \bigcup_{c \in C(f)} T(F_c)$ we deduce that the character ρ_1 itself is trivial. This clearly implies that the character $\tilde{\rho}$ is trivial. □

The following result, based on Corollaries 4.7, 5.4 and Theorem 5.7, clarifies the case of translated components; see also [10] in the proper case.

COROLLARY 5.8. *Let $f : M \to S$ be a surjective morphism, with connected generic fiber F.*

(i) *If $\chi(S) < 0$, then the irreducible components in $\mathcal{V}_1(M)$ associated to f form a subgroup in $\mathbb{T}(M)$, isomorphic to the orbifold character group $\mathbb{T}^{\mathrm{orb}}(f)$. More precisely, they are given by $\hat{f}_*^{\mathrm{orb}}(\mathbb{T}^{\mathrm{orb}}(f))$, where the injective morphism $\hat{f}_*^{\mathrm{orb}} : \mathbb{T}^{\mathrm{orb}}(f) \to \mathbb{T}(M)$ is the dual of the epimorphism f_*^{orb}.*

(ii) *If $\chi(S) = 0$, then the irreducible components in $\mathcal{V}_1(M)$ associated to f are given by $\hat{f}_*^{\mathrm{orb}}(\mathbb{T}^{\mathrm{orb}}(f)^*)$, where $\mathbb{T}^{\mathrm{orb}}(f)^*$ is obtained from the orbifold character group $\mathbb{T}^{\mathrm{orb}}(f)$ by deleting the identity connected component.*

The same proof as above yields the following result, to be compared with Theorem 3.6(iv).

COROLLARY 5.9. *Let $f : M \to S$ be a surjective morphism, with connected generic fiber F, such that $\chi(S) \le 0$. Then, for any character $\tilde{\rho} : T(f) \to \mathbb{C}^*$,*

$$\dim H^1(M, \mathcal{L}_\rho \otimes f^*\mathcal{L}) \ge -\chi(S) + |\mathrm{supp}(\tilde{\rho})|$$

for any local system $\mathcal{L} \in \mathbb{T}(S)$, and the above inequality is an equality for all except finitely many local systems $\mathcal{L}$.

For the proofs of the following related two results we refer to [14].

PROPOSITION 5.10. *For $f : M \to S$ a surjective morphism, with connected generic fiber F, and for a non-trivial element $\tilde{\rho}$ in the Pontryagin dual $\widehat{T}(f)$, one has a natural adjunction isomorphism*

$$\mathcal{F} = Rj_* j^{-1}\mathcal{F}$$

where $\mathcal{F} = R^0 f_*(\mathcal{L}_\rho)$ and $j : S \setminus \Sigma(\mathcal{F}) \to S$ is the inclusion. In particular, the local system $j^{-1}\mathcal{F}$ on $S \setminus \Sigma(\mathcal{F})$ is non-trivial.

COROLLARY 5.11. *With the above notation, if S is a compact curve, then $|\Sigma(\mathcal{F})| \neq 1$.*

EXAMPLE 5.12 (The deleted B_3-arrangement). We return to Example 3.8 and apply the above discussion to this test case. The corresponding mapping $f : M \to \mathbb{C}^*$ has $B = \{0, \infty\}$ and $C(f) = \{1\}$. Indeed, with obvious notation, we get the following divisors: $D_0 = L_1 + L_4 + 2L_5$, $D_\infty = L_2 + L_3 + 2L_7$ and $D_1 = L_6 + 2L$ where $L : x + y - 1 = 0$ is exactly the line from the B_3-arrangement that was deleted in order to get Suciu's arrangement. Moreover, the associated fibration $f' : M' \to S'$ in this case is just the fibration of the B_3-arrangement discussed in [18, Example 4.6].

The line L is the only multiple component and $m_1 = 2$. Then Theorem 5.3 implies that

$$T(f) = \mathbb{Z}/2\mathbb{Z}.$$

Let $\gamma_i = \gamma(L_i)$. We know that $\rho(\gamma_i) = \pm 1$ and to get the exact values we proceed as follows. First note that we can choose $\rho(\gamma_1) = 1$, since the associated torus is

$$f^*(\mathbb{T}(\mathbb{C}^*)) = \{(t, t^{-1}, t^{-1}, t, t^2, 1, t^{-2}, 1) \mid t \in \mathbb{C}^*\}.$$

(In fact the choice $\rho(\gamma_1) = -1$ produces the character ρ'_W introduced in Example 3.8.) Next let $\alpha = \sum_{i=1}^{7} \alpha_i \gamma_i \in H_1(M)$. Then $\alpha \in \ker f_*$ if and only if

$$\alpha_1 + \alpha_4 + 2\alpha_5 = \alpha_2 + \alpha_3 + 2\alpha_7. \tag{5.13}$$

In our case, the canonical projection $\theta : \ker f_* \to \mathbb{Z}/2\mathbb{Z}$ is given by $\alpha \mapsto \alpha_2 + \alpha_3 - \alpha_6$ (for details see [13, Theorem 6.3]). It follows that $\gamma_6 \in \ker f_*$ and $\theta(\gamma_6) = 1 \in \mathbb{Z}/2\mathbb{Z}$. Hence $\rho(\gamma_6) = -1$.

Next $\gamma_1 + \gamma_2 \in \ker f_*$ and $\theta(\gamma_1 + \gamma_2) = 1 \in \mathbb{Z}/2\mathbb{Z}$. It follows that $\rho(\gamma_1)\rho(\gamma_2) = -1$, i.e. $\rho(\gamma_2) = -1$. The reader can continue in this way and get the value of $\rho = \rho_W$ given above in Example 3.8.

EXAMPLE 5.13 (A more general example: the $\mathcal{A}_m$-arrangement). Let $\mathcal{A}_m$ be the line arrangement in $\mathbb{P}^2$ defined by the equation

$$x_1 x_2 (x_1^m - x_2^m)(x_1^m - x_3^m)(x_2^m - x_3^m) = 0.$$

This arrangement is obtained by deleting the line $x_3 = 0$ from the complex reflection arrangement associated to the full monomial group $G(3, 1, m)$ and was studied in [8] and in [9]. The celebrated deleted B_3-arrangement studied above is obtained by taking $m = 2$.

Consider the associated pencil

$$(P, Q) = (x_1^m(x_2^m - x_3^m), x_2^m(x_1^m - x_3^m)).$$

Then the set B consists of two points, namely $(0 : 1)$ and $(1 : 0)$, and the set $C(f)$ is the singleton $(1 : 1)$ (see for instance [18, Example 4.6]). It follows that $m_{(1:1)} = m$ and hence via Theorem 5.3 we get

$$T(f) = \mathbb{Z}/m\mathbb{Z}.$$

Using Corollary 5.8, we expect $(m - 1)$ 1-dimensional components in $\mathcal{V}_1(M)$, and this is precisely what has been proved in [8], or in Thm. 5.7 of [9]. There are $r = 2 + 3m$ lines in the arrangement, and to describe these components we use the coordinates

$$(z_1, z_2, z_{12:1}, \dots, z_{12:m}, z_{13:1}, \dots, z_{13:m}, z_{23:1}, \dots, z_{23:m})$$

on the torus $(\mathbb{C}^*)^r$ containing $\mathbb{T}(M)$. Here z_j is associated to the line $x_j = 0$ for $j = 1, 2$, and $z_{ij:k}$ is associated to the line $x_i - w^k x_j$, where $i, j = 1, 3$, $k = 1, \dots, m$, and $w = \exp(2\pi\sqrt{-1}/m)$. All the above 1-dimensional components have the same associated 1-dimensional subtorus

$$\mathbb{T} = f^*(\mathbb{T}(\mathbb{C}^*)) = \{(u^m, u^{-m}, 1, \dots, 1, u^{-1}, \dots, u^{-1}, u, \dots, u) \mid u \in \mathbb{C}^*\}$$

where $f : M \to \mathbb{C}^*$ is the morphism associated to the pencil (P, Q), and each element 1, u^{-1} and u is repeated m times. Let γ_c be an elementary loop about one line L in the fiber $\mathcal{C}_c$, with multiplicity 1, e.g. $L : x_1 - x_2 = 0$. Similarly, let γ_b be an elementary loop about one line L' in the fiber $\mathcal{C}_b$, with multiplicity 1, where $b = \infty = (0 : 1)$, e.g. $L' : x_2 - x_3 = 0$. And let γ_0 be an elementary loop about one line L_0 in the fiber $\mathcal{C}_0$, with multiplicity 1, where $0 = (1 : 0)$, e.g. $L_0 : x_1 - x_3 = 0$. One can show easily that

(i) the classes $[\gamma_c]$ and $[\gamma_b + \gamma_0]$ in the group $T(f)$ are independent of the choices made;
(ii) $[\gamma_c] = -[\gamma_b + \gamma_0]$ is a generator of $T(f)$.

It follows that a torsion character $\rho \in \mathbb{T}(M)$ such that $\mathcal{L}_\rho|F = \mathbb{C}_F$ and inducing a non-trivial character $\tilde{\rho} : T(f) \to \mathbb{C}^*$ is given by

$$\rho = (1, 1, w^k, \dots, w^k, w^{-k}, \dots, w^{-k}, 1, \dots, 1)$$

for $k = 1, \dots, m - 1$. Here $\tilde{\rho}([\gamma_c]) = w^k$ and ρ is normalized by setting the last m components equal to 1.

EXAMPLE 5.14 (A non-linear arrangement). Consider again the pencil $\mathcal{C} : (P, Q) = (x_1^m(x_2^m - x_3^m), x_2^m(x_1^m - x_3^m))$ associated above to the $\mathcal{A}_m$-arrangement, for $m \geq 2$. We introduce the following new notation: $C = \{(0 : 1), (1 : 0), (1 : 1)\}$. Let $B \subset \mathbb{P}^1$ be a finite set such that $|B| = k \geq 2$ and $B \cap C = \emptyset$. Consider the curve arrangement in $\mathbb{P}^2$ obtained by taking the union of the $3m$ lines given by

$$(x_1^m - x_2^m)(x_1^m - x_3^m)(x_2^m - x_3^m) = 0$$

with the k fibers $\mathcal{C}_b$ for $b \in B$. Let M be the corresponding complement and $f : M \to S := \mathbb{P}^1 \setminus B$ be the map induced by the pencil $\mathcal{C}$. Then one has the following.

(i) $T(f) = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$. Let e_j for $j = 1, 2, 3$ denote the canonical basis of $T(f)$ as a $\mathbb{Z}/m\mathbb{Z}$-module.

(ii) For a character $\tilde{\rho} : T(f) \to \mathbb{C}^*$, let $W_\rho = \mathcal{L}_\rho \otimes f^*(\mathbb{T}(S))$ be the associated component. Then $\dim W_\rho = k - 1$ and for a local system $\mathcal{L} \in W_\rho$ one has

$$\dim H^1(M, \mathcal{L}) \geq k - 2 + \epsilon(\rho)$$

where equality holds for all but finitely many $\mathcal{L} \in W_\rho$ and

$$\epsilon(\rho) = |\{j \mid \tilde{\rho}(e_j) \neq 1\}| \in \{0, 1, 2, 3\}.$$

Indeed, the set $\{j \mid \tilde{\rho}(e_j) \neq 1\}$ can be identified with $\mathrm{supp}(\tilde{\rho})$ and the claim follows from Corollaries 5.8 and 5.9. This shows that the various translates W_ρ of the subtorus $W' = \mathbb{T}_W = f^*(\mathbb{T}(S))$ all have the same dimension, but they are irreducible components of various characteristic varieties $\mathcal{V}_q(M)$, with $q = q(f, \rho) = k - 2 + \epsilon(\rho)$ as in Corollary 4.7, a fact apparently not noticed before.

ACKNOWLEDGMENTS. We would like to thank Alexander Suciu for interesting and stimulating discussions on the subject of this paper, and for suggesting several improvements of the presentation.

REFERENCES

[1] D. ARAPURA, *Geometry of cohomology support loci for local systems. I*. J. Algebraic Geom. 6 (1997), 563–597.

[2] I. BAUER, *Irrational pencils on non-compact algebraic manifolds*. Internat. J. Math. 8 (1997), 441–450.

[3] A. BEAUVILLE, *Annulation du H^1 pour les fibrés en droites plats*. In: Complex Algebraic Varieties (Bayreuth, 1990), Lecture Notes in Math. 1507, Springer, Berlin, 1992, 1–15.

[4] A. BOREL et al., *Intersection Cohomology*. Progr. Math. 50, Birkhäuser, 1984.

[5] F. CATANESE, *Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations*. Invent. Math. 104 (1991), 263–289 (with an appendix by A. Beauville).

[6] F. CATANESE, *Fibered surfaces, varieties isogeneous to a product and related moduli spaces*. Amer. J. Math. 122 (2000), 1–44.

[7] F. CATANESE - J. KEUM - K. OGUISO, *Some remarks on the universal cover of an open K3 surface*. Math. Ann. 325 (2003), 279–286.

[8] D. C. COHEN, *Triples of arrangements and local systems*. Proc. Amer. Math. Soc. 130 (2002), 3025–3031.

[9] D. C. COHEN - G. DENHAM - A. I. SUCIU, *Torsion in Milnor fiber homology*. Algebr. Geom. Topol. 3 (2003), 511–535.

[10] T. DELZANT, *Trees, valuations and Green–Lazarsfeld set*. math.AG/0702477.

[11] A. DIMCA, *Singularities and Topology of Hypersurfaces*. Universitext, Springer, 1992.

[12] A. DIMCA, *Sheaves in Topology*. Universitext, Springer, 2004.

[13] A. DIMCA, *Pencils of plane curves and characteristic varieties*. math.AG/0606442.

[14] A. DIMCA, *On the irreducible components of characteristic varieties*. An. Ştiinţ. Univ. Ovidius Constanţa 15 (2007), 67–74.

[15] A. DIMCA - S. PAPADIMA - A. SUCIU, *Formality, Alexander invariants, and a question of Serre*. math.AT/0512480.

[16] A. DIMCA - S. PAPADIMA - A. SUCIU, *Finiteness properties of fundamental groups of smooth projective varieties*. math.AG/0609456.

[17] D. EISENBUD - W. NEUMANN, *Three-Dimensional Link Theory and Invariants of Plane Curve Singularities*. Ann. of Math. Stud. 110, Princeton Univ. Press, Princeton, NJ, 1985.

[18] M. FALK - S. YUZVINSKY, *Multinets, resonance varieties, and pencils of plane curves*. Compos. Math. 143 (2007), 1069–1088.

[19] R. HARTSHORNE. *Algebraic Geometry*. Grad. Texts in Math. 52, Springer, 1977.

[20] A. LIBGOBER, *Characteristic varieties of algebraic curves*. In: C. Ciliberto et al. (eds.), Applications of Algebraic Geometry to Coding Theory, Physics and Computation, Kluwer, 2001, 215–254.

[21] A. LIBGOBER - S. YUZVINSKY, *Cohomology of the Orlik–Solomon algebras and local systems*. Compos. Math. 121 (2000), 337–361.

[22] C. T. MCMULLEN, *The Alexander polynomial of a* 3*-manifold and the Thurston norm on cohomology*. Ann. Sci. École Norm. Sup. 35 (2002), 153–171.

[23] F. SERRANO, *Multiple fibres of a morphism*. Comment. Math. Helv. 65 (1990), 287–298.

[24] A. SUCIU, *Translated tori in the characteristic varieties of complex hyperplane arrangements*. In: Arrangements in Boston: a Conference on Hyperplane Arrangements (1999), Topology Appl. 118 (2002), 209–223.

[25] A. SUCIU, *Fundamental groups of line arrangements: enumerative aspects*. In: Advances in Algebraic Geometry Motivated by Physics (Lowell, MA, 2000), Contemp. Math. 276, Amer. Math. Soc., Providence, RI, 2001, 43–79.

Received 10 May 2007,
and in revised form 3 August 2007.

Laboratoire J. A. Dieudonné
UMR du CNRS 6621
Université de Nice Sophia-Antipolis
Parc Valrose
06108 NICE Cedex 02, France
dimca@math.unice.fr