Partial differential equations. — *Some existence results for the Toda system on closed surfaces*, by ANDREA MALCHIODI and CHEIKH BIRAHIM NDIAYE, communicated on 11 May 2007.

ABSTRACT. — Given a compact closed surface Σ, we consider the generalized Toda system of equations on Σ: $-\Delta u_i = \sum_{j=1}^{2} \rho_j a_{ij} \left(\frac{h_j e^{u_j}}{\int_\Sigma h_j e^{u_j} \, dV_g} - 1 \right)$ for $i = 1, 2$, where ρ_1, ρ_2 are real parameters and h_1, h_2 are smooth positive functions. Exploiting the variational structure of the problem and using a new min-max scheme we prove existence of solutions for generic values of ρ_1 and for $\rho_2 < 4\pi$.

1. INTRODUCTION

The system

$$(1) \qquad -\Delta u_i = \sum_{j=1}^{N} a_{ij} e^{u_j}, \quad i = 1, \ldots, N,$$

defined on a domain $\Omega \subseteq \mathbb{R}^2$, where $A = (a_{ij})_{ij}$ is the *Cartan matrix* of $SU(N+1)$,

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & -1 & 2 & -1 \\ 0 & \ldots & \ldots & 0 & -1 & 2 \end{pmatrix},$$

is known as the *Toda system*, and it arises in the study of non-abelian Chern–Simons theory (see for example [16] or [31]).

In this paper we consider a generalized version of (1) on a closed surface Σ (which from now on we assume with total volume 1), namely

$$(2) \qquad -\Delta u_i = \sum_{j=1}^{N} \rho_j a_{ij} \left(\frac{h_j e^{u_j}}{\int_\Sigma h_j e^{u_j} \, dV_g} - 1 \right), \quad i = 1, \ldots, N,$$

where h_i are smooth and positive functions on Σ. We specialize here to the case $N = 2$,

so the system becomes

$$(3) \quad \begin{cases} -\Delta u_1 = 2\rho_1 \left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\, dV_g} - 1 \right) - \rho_2 \left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\, dV_g} - 1 \right) \\[4mm] -\Delta u_2 = 2\rho_2 \left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\, dV_g} - 1 \right) - \rho_1 \left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\, dV_g} - 1 \right) \end{cases} \quad \text{on } \Sigma.$$

Problem (3) is variational, and solutions can be found as critical points of a functional $J_\rho : H^1(\Sigma) \times H^1(\Sigma) \to \mathbb{R}$, $\rho = (\rho_1, \rho_2)$, defined as

$$J_\rho(u_1, u_2) = \left[\frac{1}{2} \sum_{i,j=1}^2 \int_\Sigma a^{ij} \nabla u_i \cdot \nabla u_j\, dV_g \right] + \sum_{i=1}^2 \rho_i \int_\Sigma u_i\, dV_g - \sum_{i=1}^2 \rho_i \log \int_\Sigma h_i e^{u_i}\, dV_g.$$

Here a^{ij} are the entries of the inverse matrix A^{-1}.

The structure of the functional J_ρ strongly depends on the values of ρ_1 and ρ_2. For example, the condition $\rho_i \le 4\pi$ for both $i = 1, 2$ has been proven in [18] to be necessary and sufficient for J_ρ to be bounded from below (see Theorem 2.1; we also refer to [27] and [28]). In particular, for $\rho_1, \rho_2 < 4\pi$, J_ρ becomes coercive (once we factor out the constants, since J_ρ is invariant under the transformation $u_i \mapsto u_i + c_i$, $c_i \in \mathbb{R}$) and solutions of (3) can be found as global minima.

The case in which one of the ρ_i's is 4π (or both are) is more subtle since the functional is still bounded from below but not coercive anymore. In [17] and [21] some conditions for existence are given in this case, and the proofs involve a delicate analysis of the limiting behavior of the solutions when the ρ_i's converge to 4π from below.

On the other hand, when one of the ρ_i's is greater than 4π, J_ρ is unbounded from below and solutions should be found as saddle points. In [23], [25] and [26] some existence results are given and it is proved that if $h_i \equiv 1$ and if some additional assumptions are satisfied, then $(0, 0)$ is a local minimizer for J_ρ, so the functional has a mountain pass structure and some corresponding critical points. Furthermore in [17] a very refined blow-up behavior of solutions is given (Theorem 2.4 below is a consequence of that analysis) and existence is proved if Σ has positive genus and if ρ_1, ρ_2 satisfy either (i) $\rho_1 < 4\pi$, $\rho_2 \in (4\pi, 8\pi)$ (or vice versa), or (ii) $\rho_1, \rho_2 \in (4\pi, 8\pi)$.

Our goal here is to give a general existence result when one of the coefficients ρ_i can be arbitrarily large. We have indeed the following theorem.

THEOREM 1.1. *Suppose m is a positive integer, and let $h_1, h_2 : \Sigma \to \mathbb{R}$ be smooth positive functions. Then for $\rho_1 \in (4\pi m, 4\pi(m+1))$ and $\rho_2 < 4\pi$ problem (3) is solvable.*

REMARK 1.2. By Proposition 2.5 below, if we also assume that $\int_\Sigma u_i\, dV_g = 0$ for $i = 1, 2$, the solutions of (3) stay bounded in $C^l(\Sigma)$ for any integer l.

To give an idea of the proof of Theorem 1.1 we first recall the analogy of (3) with some nonlinear scalar equations. First of all we should mention that (2) for $N = 1$,

$$(4) \quad -\Delta u = 2\rho \left(\frac{h e^u}{\int_\Sigma h e^u\, dV_g} - 1 \right) \quad \text{on } \Sigma,$$

arises in the study of mean field limit of point vortices of Euler flows, spherical Onsager vortex theory and condensates in some Chern–Simons–Higgs models (see for example the papers [3], [4], [7]–[10], [12], [20], [30] and the references therein).

We also mention the similarity of the scalar equation (4) with the geometric equations

$$(5) \qquad -\Delta_g u + K_g = K_{\tilde{g}} e^{2u} \quad \text{on } \Sigma, \qquad P_g u + 2Q_g = 2Q_{\tilde{g}} e^{4u} \quad \text{on } M.$$

Here K_g is the Gauss curvature of Σ, Δ_g the Laplace–Beltrami operator, $\tilde{g} = e^{2u} g$ a conformal metric and $K_{\tilde{g}}$ the Gauss curvature of $\tilde{g}$. The second equation in (5) is the transformation law of the *Q-curvature* on a four-dimensional manifold M under a similar conformal change of metric, and P_g is the *Paneitz operator* associated to (M, g) (see for example [15], [24] and the references therein).

We next recall the ideas used in [15] to find conformal metrics of constant Q-curvature. For the reader's convenience we transpose the discussion to equation (4), for which analogous considerations hold. Actually the method in [15] has been used in [13] to study (4) as well, in order to obtain existence results on surfaces of arbitrary genus.

Equation (4) also has variational structure and is the Euler equation of the functional

$$I_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 \, dV_g - 2\rho \int_\Sigma u \, dV_g - 2\rho \log \int_\Sigma h e^u \, dV_g, \quad u \in H^1(\Sigma),$$

which, as before, is bounded from below if and only if $\rho \leq 4\pi$ by the Moser–Trudinger inequality (see (6) below). For $\rho > 4\pi$, instead of using degree theory as in [9] and [10], one can indeed employ directly a min-max scheme based on improvements of (6). In fact, if the integral of e^u is distributed into ℓ different distinct regions, then (naively) the coefficient of the right-hand integral in (6) reduces by a factor ℓ. For a precise statement see Proposition 2.2 below. As a consequence, if $\rho \in (4k\pi, 4(k+1)\pi))$ and $I_\rho(u_l) \to -\infty$ along a sequence u_l, then e^{u_l} has to concentrate near at most k points in Σ. For such a result we refer to Lemma 2.4 in [15] or in [13]. Assuming that $\int_\Sigma e^{u_l} \, dV_g = 1$, we have $e^{u_l} \rightharpoonup \sum_{i=1}^k t_i \delta_{x_i}$ for some non-negative coefficients t_i such that $\sum_{i=1}^k t_i = 1$. This family of formal convex combinations of Dirac deltas is known as the set of *formal barycenters* of Σ (see Section 2), and we denote it by Σ_k. We notice that for $k = 1$ the set Σ_1 is simply homeomorphic to Σ but for larger k the t_i's do not have any bound from below or the x_i's could collapse onto each other, so the set could be degenerate near some of its points. In fact, Σ_k is a *stratified manifold*, that is, a union of sets of different dimensions. Nevertheless, since $e^{u_l} \rightharpoonup \sum_{i=1}^k t_i \delta_{x_i} \in \Sigma_k$, with some work it is possible to build a continuous and non-trivial map Π_k from sublevels $\{I_\rho \leq -L\}$ (with L large) into Σ_k. By non-triviality we mean that this map is homotopically non-trivial, and indeed for any $L > 0$ there exists a map $\varphi : \Sigma_k \to \{I_\rho \leq -L\}$ (see (22) for the explicit formula, and Proposition 4.1 in [13] for the evaluation of I_ρ) such that $\Pi_k \circ \varphi$ is homotopic to the identity on Σ_k, which is non-contractible. This allows us then to define a min-max scheme using maps from the topological cone over Σ_k with values into $H^1(\Sigma)$ (see e.g. [13, Section 5]) which coincide with φ on Σ_k (the boundary of the cone).

Having sketched this argument for the scalar equation (4), we can now describe our approach to study system (3). First of all we prove a compactness result under the assumptions of Theorem 1.1 (see Proposition 2.5). This result exploits the blow-up analysis in [17] when ρ_2 stays positive and away from zero. On the other hand, for $\rho_2 \in (-\infty, \delta]$

with δ positive and small, we use an argument inspired by Brezis and Merle [6], combined with a compactness result in [22] (see Theorem 2.3).

Next, a main ingredient in our proof is again an improved version of the Moser–Trudinger inequality for systems, which was given in [18] (see Theorem 2.1). In Proposition 3.1 we see that, in analogy with the scalar case, if e^{u_1} is distributed among disjoint sets, then the Moser–Trudinger inequality improves, and the bigger the spreading, the better the improvement. The argument relies on both Theorem 2.1 and Proposition 2.2. The way we use them is the following. Assuming e^{u_1} spreads into ℓ sets $S_1, \ldots, S_\ell$, we can find another ℓ-tuple $\tilde{S}_1, \ldots, \tilde{S}_\ell \subseteq \Sigma$ such that each of these sets contains a fixed portion of the integral of e^{u_1}, and $\tilde{S}_1$ also contains a fixed portion of the integral of e^{u_2} (see Lemma 3.2). Then, by a localization argument through some cutoff functions $g_1, \ldots, g_\ell$, we use the Moser–Trudinger inequality for systems near $\tilde{S}_1$, and the improved scalar inequality near $\tilde{S}_2, \ldots, \tilde{S}_\ell$. In this step we employ some interpolation inequalities and cutoffs in the Fourier modes of u_1, u_2 to deal with lower order terms.

From the improved inequality we derive the following consequence. If $\rho_1 \in (4\pi m, 4\pi(m + 1))$, if $\rho_2 < 4\pi$ and if $J_\rho(u_{1,l}, u_{2,l}) \to -\infty$ along a sequence $(u_{1,l}, u_{2,l})$, then $e^{u_{1,l}}$ has to concentrate near at most m points of Σ. Therefore, as for the scalar equation, we can map $e^{u_{1,l}}$ onto Σ_m for l large. Precisely, for $L \gg 1$ we can define a continuous projection $\Psi : \{J_\rho \leq -L\} \to \Sigma_m$ which is homotopically non-trivial. Indeed, recalling that Σ_m is non-contractible (see Lemma 2.6), there exists a map Φ such that $\Psi \circ \Phi$ is homotopic to the identity and $J_\rho(\Phi(\Sigma_m))$ can become arbitrarily large negative, so that Ψ is well-defined on its image.

Some comments on the construction of the map Φ are in order. If we want to obtain low values of J_ρ on a couple (u_1, u_2), since e^{u_1} has necessarily to concentrate near at most m points of Σ, a natural choice of the test functions (u_1, u_2) is $(\varphi_{\lambda,\sigma}, -\frac{1}{2}\varphi_{\lambda,\sigma})$, where σ is any element of Σ_m, and where $\varphi_{\lambda,\sigma}$ is given in (22) in Section 4. In fact, as λ tends to infinity, $e^{\varphi_{\lambda,\sigma}}$ converges to σ in the weak sense of distributions, while u_2 is chosen so as to obtain the best possible cancelation in the quadratic part of the functional (see Remark 4.3). We notice that this kind of function (for the case $m = 1$ only) was used in [18] to prove unboundedness of J_ρ from below if some ρ_i is greater than 4π. Letting σ vary, we get a full embedding of Σ_m into low sublevels of J_ρ through the map Φ.

At this point we are in a position to run a min-max scheme similar to that described above, based on the topological cone over Σ_m. The scheme yields a Palais–Smale sequence for J_ρ, but since we cannot ensure convergence directly, following Struwe [29] we introduce an auxiliary functional $J_{t\rho}$ ($t\rho = (t\rho_1, t\rho_2)$) where t belongs to a small neighborhood of 1. Running the same scheme on the functional $J_{t\rho}$, via a monotonicity argument, yields existence of critical points for almost every value of t, and in particular along a sequence $t_k \to 1$. To conclude, it is sufficient to apply the compactness result of Proposition 2.5.

The plan of the paper is the following. In Section 2 we collect some preliminary results regarding the Moser–Trudinger inequality, the barycentric sets Σ_k and the proof of Proposition 2.5. In Section 3 we give an improved version of the inequality for systems, and we apply it to characterize the low sublevels of J_ρ in terms of the concentration of the function e^{u_1} (see Corollary 3.5). Then in Section 4 we introduce the topological argument to study (3). We first define the global projection Ψ onto Σ_m (where m is the integer of

Theorem 1.1) and then we also define the map $\Phi : \Sigma_m \to H^1(\Sigma) \times H^1(\Sigma)$, proving that $\Psi \circ \Phi$ is homotopic to the identity on Σ_m. Finally, we run the min-max scheme based on the topological cones over Σ_m.

2. NOTATION AND PRELIMINARIES

In this section we collect some useful preliminary facts. For $x, y \in \Sigma$ we denote by $d(x, y)$ the metric distance between x and y on Σ, and by $\mathrm{dist}(S_1, S_2)$ the distance between two sets $S_1, S_2 \subseteq \Sigma$,

$$\mathrm{dist}(S_1, S_2) = \inf\{d(x, y) : x \in S_1, \ y \in S_2\}.$$

Recalling that we are assuming $\mathrm{Vol}_g(\Sigma) := \int_\Sigma 1 \, dV_g = 1$, given a function $u \in L^1(\Sigma)$, we denote its average (or integral) as

$$\bar{u} = \int_\Sigma u \, dV_g.$$

Below, C denotes large constants which are allowed to vary in different formulas or even within lines. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C, C_δ, etc. Also constants with subscripts are allowed to vary.

We now recall some Moser–Trudinger type inequalities and compactness results. The functional under interest is the following:

$$J_\rho(u_1, u_2) = \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_\Sigma a^{ij} \nabla u_i \cdot \nabla u_j \, dV_g\right] + \sum_{i=1}^{2} \rho_i \int_\Sigma u_i \, dV_g - \sum_{i=1}^{2} \rho_i \log \int_\Sigma h_i e^{u_i} \, dV_g,$$

which, for large values of ρ_1 and ρ_2, will in general be unbounded from below. In fact, there is a precise criterion for J_ρ to be bounded, proved by Jost and Wang.

THEOREM 2.1 ([18]). *For $\rho = (\rho_1, \rho_2)$ the functional $J_\rho : H^1(\Sigma) \times H^1(\Sigma) \to \mathbb{R}$ is bounded from below if and only if $\rho_i \leq 4\pi$ for $i = 1, 2$.*

Concerning the scalar Moser–Trudinger inequality

$$(6) \qquad \log \int_\Sigma e^{u-\bar{u}} \, dV_g \leq C + \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 \, dV_g,$$

we have the following improvement which occurs if the integral of $e^{u-\bar{u}}$ is distributed among different sets of positive mutual distance.

PROPOSITION 2.2. *Let $S_1, \ldots, S_\ell$ be subsets of Σ satisfying $\mathrm{dist}(S_i, S_j) \geq \delta_0$ for $i \neq j$, and let $\gamma_0 \in (0, 1/\ell)$. Then for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\tilde{\varepsilon}, \delta_0, \gamma_0)$ such that*

$$\log \int_\Sigma e^{u-\bar{u}} \, dV_g \leq C + \frac{1}{16\ell\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla u|^2 \, dV_g$$

for all $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{S_i} e^u \, dV_g}{\int_\Sigma e^u \, dV_g} \geq \gamma_0, \quad i \in \{1, \dots, \ell\}.$$

For the proof in the case $\ell = 2$ see [11]. We also refer the reader to [15].

We now recall the following compactness results from [22] and [17].

THEOREM 2.3 ([22]). *Let $(u_k)_k$ be a sequence of solutions of the equations*

$$-\Delta u_k = \lambda_k \left(\frac{V_k e^{u_k}}{\int_\Sigma V_k e^{u_k} \, dV_g} - W_k \right),$$

where $(V_k)_k$ and $(W_k)_k$ satisfy

$$\int_\Sigma W_k \, dV_g = 1, \quad \|W_k\|_{C^1(\Sigma)} \leq C, \quad |\log V_k| \leq C, \quad \|\nabla V_k\|_{L^\infty(\Sigma)} \leq C,$$

and where $\lambda_k \to \lambda_0 > 0$, $\lambda_0 \neq 8k\pi$ for $k = 1, 2, \dots$. Then, under the additional constraint $\int_\Sigma u_k \, dV_g = 1$, $(u_k)_k$ stays uniformly bounded in $L^\infty(\Sigma)$.

THEOREM 2.4 ([17]). *Let m_1, m_2 be non-negative integers, and suppose Λ_1, Λ_2 are compact subsets of the intervals $(4\pi m_1, 4\pi(m_1+1))$ and $(4\pi m_2, 4\pi(m_2+1))$ respectively. If $\rho_1 \in \Lambda_1$ and $\rho_2 \in \Lambda_2$ and if we impose $\int_\Sigma u_i \, dV_g = 0$, $i = 1, 2$, then the solutions of (3) stay uniformly bounded in $L^\infty(\Sigma)$ (actually in every $C^l(\Sigma)$ with $l \in \mathbb{N}$).*

This theorem, as stated in [17], requires m_1 and m_2 to be positive. However, it is clear from the blow-up analysis there that one can also allow zero values of m_1 or m_2. Combining Theorems 2.3 and 2.4 we obtain another compactness result which includes all the possibilities of Theorem 1.1.

PROPOSITION 2.5. *Suppose h_1, h_2 are smooth positive functions on Σ, and consider a sequence $(u_{1,k}, u_{2,k})$ of solutions of the system*

$$(7) \quad \begin{cases} -\Delta u_{1,k} = 2\rho_{1,k} \left(\dfrac{h_1 e^{u_{1,k}}}{\int_\Sigma h_1 e^{u_{1,k}} \, dV_g} - 1 \right) - \rho_{2,k} \left(\dfrac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} \, dV_g} - 1 \right) \\ -\Delta u_{2,k} = 2\rho_{2,k} \left(\dfrac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} \, dV_g} - 1 \right) - \rho_{1,k} \left(\dfrac{h_1 e^{u_{1,k}}}{\int_\Sigma h_1 e^{u_{1,k}} \, dV_g} - 1 \right) \end{cases} \quad on \; \Sigma.$$

Suppose $(\rho_{1,k})_k$ lie in a compact subset K_1 of $\bigcup_{i=1}^\infty (4i\pi, 4(i+1)\pi)$, and $(\rho_{2,k})_k$ lie in a compact subset K_2 of $(-\infty, 4\pi)$. If $\int_\Sigma u_{i,k} \, dV_g = 0$ for $i = 1, 2$ and $k \in \mathbb{N}$, then the functions $(u_{1,k}, u_{2,k})$ of (7) stay uniformly bounded in $L^\infty(\Sigma) \times L^\infty(\Sigma)$.

PROOF. First we claim that for any $p > 1$ there exists $\overline{\rho} > 0$ (depending on p, K_1, K_2, h_1 and h_2) such that for $\rho_{2,k} \leq \overline{\rho}$ the solutions of $(e^{u_{2,k}})_k$ are uniformly bounded in $L^p(\Sigma)$.

The proof of this claim follows an argument in [6]: using the Green representation formula and the fact that $\rho_1 > 0$ we find (recall that $\int_\Sigma u_{2,k} \, dV_g = 0$)

$$u_{2,k}(x) \leq C + \int_\Sigma G(x, y) \left(2\rho_{2,k} \frac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} \, dV_g} \right) dV_g(y),$$

where $G(x, y)$ is the Green function of $-\Delta$ on Σ. Using the Jensen inequality we then find

$$e^{pu_{2,k}(x)} \le C \int_{\Sigma} \exp(2p\rho_{2,k} G(x, y)) \frac{h_2 e^{u_{2,k}}}{\int_{\Sigma} h_2 e^{u_{2,k}} dV_g} dV_g(y).$$

Recalling that $G(x, y) \simeq \frac{1}{2\pi} \log\left(\frac{1}{d(x,y)}\right)$ and using also the Fubini theorem we get

$$\int_{\Sigma} e^{pu_{2,k}} dV_g \le C \sup_{x \in \Sigma} \int_{\Sigma} \frac{1}{d(x, y)^{p\rho_{2,k}/\pi}} dV_g(y).$$

Now it is sufficient to take $\overline{\rho} = \pi/2p$ in order to obtain the claim.

To prove the proposition, in the case $\rho_{2,k} \ge \overline{\rho}$ we simply use Theorem 2.4, while for $\rho_{2,k} \le \overline{\rho}$ we employ the above claim. In fact, from uniform L^p bounds on $e^{u_{2,k}}$ and from elliptic regularity theory, we obtain uniform $W^{2,p}$ bounds on the sequence $(v_k)_k$, where v_k is defined as the unique (we can assume that every v_k has zero average) solution of

$$-\Delta v_k = -\rho_{2,k}\left(\frac{h_2 e^{u_{2,k}}}{\int_{\Sigma} h_2 e^{u_{2,k}} dV_g} - 1\right).$$

Taking p sufficiently large, by the Sobolev embedding, we also obtain uniform $C^{1,\alpha}$ bounds on $(v_k)_k$ (and hence on $(e^{v_k})_k$). Now we write $u_{1,k} = w_{1,k} + v_k$, so that $w_{1,k}$ satisfies

$$-\Delta w_{1,k} = 2\rho_{1,k}\left(\frac{h_1 e^{v_k} e^{w_{1,k}}}{\int_{\Sigma} h_1 e^{v_k} e^{w_{1,k}} dV_g} - 1\right).$$

Moreover, since we are assuming $\int_{\Sigma} u_{1,k} dV_g = 0$ and since $\int_{\Sigma} v_k dV_g = 0$ as well, we also have $\int_{\Sigma} w_{1,k} dV_g = 0$. Hence, applying Theorem 2.3 with $u_k = w_{1,k}$, $\lambda_k = 2\rho_{1,k}$, $V_k = h_1 e^{v_k}$ and $W_k \equiv 1$, we obtain uniform bounds on $w_{1,k}$ in $L^\infty(\Sigma)$. Since $(v_k)_k$ stays uniformly bounded in $L^\infty(\Sigma)$, we also get uniform bounds on $u_{1,k}$ in $L^\infty(\Sigma)$. Then, from the second equation in (7) we also achieve uniform bounds on $u_{2,k}$ in $W^{2,p}(\Sigma)$ (and hence in $L^\infty(\Sigma)$, by taking p large enough). This concludes the proof. $\square$

At this point some notation is in order. For $k \in \mathbb{N}$, we let Σ_k denote the family of formal sums

$$(8) \qquad \Sigma_k = \left\{ \sum_{i=1}^{k} t_i \delta_{x_i} : t_i \ge 0, \sum_{i=1}^{k} t_i = 1, x_i \in \Sigma \right\},$$

where δ_x stands for the Dirac delta at the point $x \in \Sigma$. We endow this set with the weak topology of distributions. This is known in the literature as the *formal set of barycenters* of Σ (of order k, see [1], [2], [5]). Although this is not in general a smooth manifold (except for $k = 1$), it is a *stratified set*, a union of cells of different dimensions. The maximal dimension is $3k - 1$, when all the points x_i are distinct and all the t_i's belong to the open interval $(0, 1)$.

Next we recall the following result from the last references (see also Lemma 3.7 in [15]), which is necessary in order to carry out the topological argument below.

LEMMA 2.6 (well-known). *For any $k \ge 1$ one has $H_{3k-1}(\Sigma_k; \mathbb{Z}_2) \ne 0$. As a consequence, Σ_k is non-contractible.*

If $\varphi \in C^1(\Sigma)$ and $\sigma \in \Sigma_k$, we denote the action of σ on φ as

$$\langle \sigma, \varphi \rangle = \sum_{i=1}^{k} t_i \varphi(x_i), \qquad \sigma = \sum_{i=1}^{k} t_i \delta_{x_i}.$$

Moreover, if f is a non-negative L^1 function on Σ with $\int_\Sigma f \, dV_g = 1$, we can define a distance of f from Σ_k in the following way:

$$(9) \qquad \operatorname{dist}(f, \Sigma_k) = \inf_{\sigma \in \Sigma_k} \sup \left\{ \left| \int_\Sigma f\varphi \, dV_g - \langle \sigma, \varphi \rangle \right| : \|\varphi\|_{C^1(\Sigma)} = 1 \right\}.$$

We also let

$$\mathcal{D}_{\varepsilon,k} = \{ f \in L^1(\Sigma) : f \geq 0, \|f\|_{L^1(\Sigma)} = 1, \operatorname{dist}(f, \Sigma_k) < \varepsilon \}.$$

From a straightforward adaptation of the arguments of Proposition 3.1 in [15], we obtain the following result.

PROPOSITION 2.7. *Let k be a positive integer, and for $\varepsilon > 0$ let $\mathcal{D}_{\varepsilon,k}$ be as above. Then there exists $\varepsilon_k > 0$, depending on k and Σ, such that for $\varepsilon \leq \varepsilon_k$ there exists a continuous map $\psi : \mathcal{D}_{\varepsilon,k} \to \Sigma_k$.*

Now we introduce some more notation. For any positive integer m, we let K_m denote the topological cone over Σ_m,

$$(10) \qquad\qquad K_m = (\Sigma_m \times [0, 1])/\sim,$$

where the equivalence relation collapses the set $\Sigma_m \times \{1\}$ to a single point.

3. An improved Moser–Trudinger inequality with applications

In this section we present an improvement of the Moser–Trudinger type inequality for the Toda system given in [18]. To get this improvement, we assume that the integral of the functions e^{u_1} is distributed among different sets with positive mutual distance. Our proof relies heavily on the main result in [18], and is combined with some arguments in [11] and [15]. As an application (see Corollary 3.5), we derive a characterization of the sublevels $\{J_\rho \leq -L\}$, for $L > 0$ large, in terms of the concentration of e^{u_1}.

3.1. *The improved inequality*

In this subsection we analyze the Moser–Trudinger inequality, depending on the *distribution* of the function e^{u_1}. A consequence of this inequality is an upper bound (depending on ρ_1) on the number of concentration points of e^{u_1}.

PROPOSITION 3.1. *Let $\delta_0 > 0$, $\ell \in \mathbb{N}$, and let $S_1, \ldots, S_\ell$ be subsets of Σ satisfying $\operatorname{dist}(S_i, S_j) \geq \delta_0$ for $i \neq j$. Let $\gamma_0 \in (0, 1/\ell)$. Then, for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\tilde{\varepsilon}, \delta_0, \gamma_0, \ell, \Sigma)$ such that*

$$\ell \log \int_\Sigma e^{u_1 - \overline{u}_1} \, dV_g + \log \int_\Sigma e^{u_2 - \overline{u}_2} \, dV_g \leq C + \frac{1}{8\pi - 2\tilde{\varepsilon}} \left[\sum_{i,j=1}^{2} \int_\Sigma a^{ij} \nabla u_i \cdot \nabla u_j \, dV_g \right]$$

provided

$$\frac{\int_{S_i} e^{u_1} dV_g}{\int_{\Sigma} e^{u_1} dV_g} \geq \gamma_0, \quad i \in \{1, \dots, \ell\}. \tag{11}$$

Before proving the proposition, we state a preliminary lemma, which will be proved later on.

LEMMA 3.2.   *Under the assumptions of Proposition 3.1, there exist $\tilde{\gamma}_0, \tilde{\delta}_0 > 0$, depending only on $\gamma_0, \delta_0, \Sigma$, and ℓ sets $\tilde{S}_1, \dots, \tilde{S}_\ell$ such that $\mathrm{dist}(\tilde{S}_i, \tilde{S}_j) \geq \tilde{\delta}_0$ for $i \neq j$ and such that*

$$\frac{\int_{\tilde{S}_1} e^{u_1} dV_g}{\int_{\Sigma} e^{u_1} dV_g} \geq \tilde{\gamma}_0, \quad \frac{\int_{\tilde{S}_1} e^{u_2} dV_g}{\int_{\Sigma} e^{u_2} dV_g} \geq \tilde{\gamma}_0, \quad \frac{\int_{\tilde{S}_i} e^{u_1} dV_g}{\int_{\Sigma} e^{u_1} dV_g} \geq \tilde{\gamma}_0, \quad i \in \{2, \dots, \ell\}.$$

PROOF OF PROPOSITION 3.1.   We modify the argument in [11] and [15]. Let $\tilde{S}_1, \dots, \tilde{S}_\ell$ be given by Lemma 3.2. Assuming without loss of generality that $\bar{u}_1 = \bar{u}_2 = 0$, we can find ℓ functions $g_1, \dots, g_\ell$ satisfying the properties

$$\begin{cases} g_i(x) \in [0, 1] & \text{for every } x \in \Sigma; \\ g_i(x) = 1 & \text{for every } x \in \tilde{S}_i, i = 1, \dots, \ell; \\ \mathrm{supp}(g_i) \cap \mathrm{supp}(g_j) = \emptyset & \text{for } i \neq j; \\ \|g_i\|_{C^2(\Sigma)} \leq C_{\tilde{\delta}_0}, \end{cases} \tag{12}$$

where $C_{\tilde{\delta}_0}$ is a positive constant depending only on $\tilde{\delta}_0$. We decompose the functions u_1 and u_2 in the following way:

$$u_1 = \hat{u}_1 + \tilde{u}_1, \quad u_2 = \hat{u}_2 + \tilde{u}_2, \quad \hat{u}_1, \hat{u}_2 \in L^\infty(\Sigma). \tag{13}$$

The explicit decomposition (via some truncation in the Fourier modes) will be chosen later on. Using Lemma 3.2, for any $b \in 2, \dots, \ell$ we can write

$$\begin{aligned} \ell \log \int_{\Sigma} e^{u_1} dV_g + \log \int_{\Sigma} e^{u_2} dV_g &= \log\left[\int_{\Sigma} e^{u_1} dV_g \int_{\Sigma} e^{u_2} dV_g \left(\int_{\Sigma} e^{u_1} dV_g\right)^{\ell-1}\right] \\ &\leq \log\left(\int_{\tilde{S}_1} e^{u_1} dV_g \int_{\tilde{S}_1} e^{u_2} dV_g\right) \\ &\quad + \log\left[\left(\int_{\tilde{S}_b} e^{u_1} dV_g\right)^{\ell-1}\right] - \ell \log \tilde{\gamma}_0 \\ &\leq \log\left(\int_{\Sigma} e^{g_1 u_1} dV_g \int_{\Sigma} e^{g_1 u_2} dV_g\right) \\ &\quad + \log\left[\left(\int_{\Sigma} e^{g_b u_1} dV_g\right)^{\ell-1}\right] - \ell \log \tilde{\gamma}_0. \end{aligned}$$

Now, using the fact that $\hat{u}_1$ and $\hat{u}_2$ belong to $L^\infty(\Sigma)$, we also write

$$\ell \log \int_\Sigma e^{u_1} \, dV_g + \log \int_\Sigma e^{u_2} \, dV_g \leq \log\left(\int_\Sigma e^{g_1 \tilde{u}_1} \, dV_g \int_\Sigma e^{g_1 \tilde{u}_2} \, dV_g\right)$$
$$+ \log\left[\left(\int_\Sigma e^{g_b \tilde{u}_1} \, dV_g\right)^{\ell-1}\right] - \ell \log \tilde{\gamma}_0$$
$$+ \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}).$$

Therefore we get

$$(14) \quad \ell \log \int_\Sigma e^{u_1} \, dV_g + \log \int_\Sigma e^{u_2} \, dV_g \leq \log \int_\Sigma e^{g_1 \tilde{u}_1} \, dV_g + \log \int_\Sigma e^{g_1 \tilde{u}_2} \, dV_g$$
$$+ (\ell - 1) \log \int_\Sigma e^{g_b \tilde{u}_1} \, dV_g - \ell \log \tilde{\gamma}_0$$
$$+ \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}).$$

At this point we can use Theorem 2.1 with parameters $(4\pi, 4\pi)$, applied to the couple $(g_1 \tilde{u}_1, g_1 \tilde{u}_2)$, and the standard Moser–Trudinger inequality (6) to $g_b \tilde{u}_1$ to get the estimates

$$\log \int_\Sigma e^{g_1 \tilde{u}_1} \, dV_g + \log \int_\Sigma e^{g_1 \tilde{u}_2} \, dV_g \leq \frac{1}{4\pi}\left[\frac{1}{2} \sum_{i,j=1}^2 \int_\Sigma a^{ij} \nabla(g_1 \tilde{u}_i) \cdot \nabla(g_1 \tilde{u}_j) \, dV_g\right]$$
$$+ (\overline{g_1 \tilde{u}_1} + \overline{g_1 \tilde{u}_2}) + C,$$

$$(15)$$

$$(\ell - 1) \log \int_\Sigma e^{g_b \tilde{u}_1} \, dV_g \leq \frac{\ell - 1}{16\pi} \int_\Sigma |\nabla(g_b \tilde{u}_1)|^2 \, dV_g + (\ell - 1)\overline{g_b \tilde{u}_1} + (\ell - 1)C.$$

Now we notice that for $N = 2$ one has

$$a^{ij} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Therefore, using elementary inequalities (completion of squares) one can check that for every point $x \in \Sigma$,

$$(16) \qquad \frac{1}{2} \sum_{i,j} a^{ij} g(\xi_i, \xi_j) \geq \frac{1}{4} g(\xi_1, \xi_1) \quad \text{for every couple } (\xi_1, \xi_2) \in T_x\Sigma \times T_x\Sigma.$$

This can be checked for example by using orthonormal coordinates at x, so that the metric g just becomes the identity at this point. Applying this inequality to the couple $(\nabla(g_b \tilde{u}_1), \nabla(g_b \tilde{u}_2))$ and integrating one finds

$$(17) \qquad \frac{\ell - 1}{16\pi} \int_\Sigma |\nabla(g_b \tilde{u}_1)|^2 \, dV_g \leq \frac{\ell - 1}{4\pi}\left[\frac{1}{2} \sum_{i,j=1}^2 \int_\Sigma a^{ij} \nabla(g_b \tilde{u}_i) \cdot \nabla(g_b \tilde{u}_j) \, dV_g\right].$$

Putting together (14)–(17) we obtain

$$(18) \quad \ell \log \int_{\Sigma} e^{u_1} \, dV_g + \log \int_{\Sigma} e^{u_2} \, dV_g \leq \frac{1}{4\pi} \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} a^{ij} \nabla(g_1 \tilde{u}_i) \cdot \nabla(g_1 \tilde{u}_j) \, dV_g \right]$$

$$+ \frac{\ell - 1}{8\pi} \left[\sum_{i,j=1}^{2} \int_{\Sigma} a^{ij} \nabla(g_b \tilde{u}_i) \cdot \nabla(g_b \tilde{u}_j) \, dV_g \right]$$

$$+ (\overline{g_1 \tilde{u}_1} + \overline{g_1 \tilde{u}_2}) + (\ell - 1)\overline{g_b \tilde{u}_1} + \ell C$$

$$- \ell \log \tilde{\gamma}_0 + \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}).$$

Now we notice that, by interpolation, for any $\varepsilon > 0$ there exists $C_{\varepsilon, \tilde{\delta}_0}$ (depending only on ε and $\tilde{\delta}_0$) such that

$$\left[\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} a^{ij} \nabla(g_1 \tilde{u}_i) \cdot \nabla(g_1 \tilde{u}_j) \, dV_g \right] \leq \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} g_1^2 a^{ij} \nabla \tilde{u}_i \cdot \nabla \tilde{u}_j \, dV_g \right]$$

$$+ \varepsilon \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} a^{ij} \nabla \tilde{u}_i \cdot \nabla \tilde{u}_j \, dV_g \right]$$

$$+ C_{\varepsilon, \tilde{\delta}_0} \int_{\Sigma} (\tilde{u}_1^2 + \tilde{u}_2^2) \, dV_g.$$

Inserting this inequality into (18) we get

$$\ell \log \int_{\Sigma} e^{u_1} \, dV_g + \log \int_{\Sigma} e^{u_2} \, dV_g \leq \frac{1}{4\pi} \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} g_1^2 a^{ij} \nabla \tilde{u}_i \cdot \nabla \tilde{u}_j \, dV_g \right]$$

$$+ \frac{\ell - 1}{4\pi} \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} g_b^2 a^{ij} \nabla \tilde{u}_i \cdot \nabla \tilde{u}_j \, dV_g \right]$$

$$+ \frac{\ell}{4\pi} \varepsilon \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} a^{ij} \nabla \tilde{u}_i \cdot \nabla \tilde{u}_j \, dV_g \right]$$

$$+ \ell C_{\varepsilon, \tilde{\delta}_0} \int_{\Sigma} (\tilde{u}_1^2 + \tilde{u}_2^2) \, dV_g$$

$$+ (\overline{g_1 \tilde{u}_1} + \overline{g_1 \tilde{u}_2}) + (\ell - 1)\overline{g_b \tilde{u}_1}$$

$$+ \ell C - \ell \log \tilde{\gamma}_0 \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)})$$

for $b = 2, \dots, \ell$.

We now choose $b \in \{2, \dots, \ell\}$ such that

$$\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} g_b^2 a^{ij} \nabla u_i \cdot \nabla \tilde{u}_j \, dV_g \leq \frac{1}{\ell - 1} \frac{1}{2} \sum_{i,j=1}^{2} \int_{\bigcup_{s=2}^{\ell} \text{supp}(g_s)} a^{ij} \nabla u_i \cdot \nabla \tilde{u}_j \, dV_g.$$

Since the g_i's have disjoint supports (see (12)), the last formula yields

$$\ell \log \int_{\Sigma} e^{u_1} \, dV_g + \log \int_{\Sigma} e^{u_2} \, dV_g \leq \frac{1}{4\pi} (1 + \ell\varepsilon) \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} a^{ij} \nabla \tilde{u}_i \cdot \nabla \tilde{u}_j \, dV_g \right]$$
$$+ \ell C_{\varepsilon,\tilde{\delta}_0} \int_{\Sigma} (\tilde{u}_1^2 + \tilde{u}_2^2) \, dV_g + (\overline{g_1 \tilde{u}_1} + \overline{g_1 \tilde{u}_2})$$
$$+ (\ell - 1) \overline{g_b \tilde{u}_1} \ell C - \ell \log \tilde{\gamma}_0$$
$$+ \ell (\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}).$$

Now, by elementary estimates we find

$$\ell \log \int_{\Sigma} e^{u_1} \, dV_g + \log \int_{\Sigma} e^{u_2} \, dV_g \leq \frac{1}{4\pi} (1 + \ell\varepsilon) \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} a^{ij} \nabla \tilde{u}_i \cdot \nabla \tilde{u}_j \, dV_g \right]$$
$$+ C_{\varepsilon,\tilde{\delta}_0,\ell} \int_{\Sigma} (\tilde{u}_1^2 + \tilde{u}_2^2) \, dV_g$$
$$+ C_{\varepsilon,\tilde{\delta}_0,\ell,\tilde{\gamma}_0} + \ell (\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}).$$

Now comes the choice of $\hat{u}_1$ and $\hat{u}_2$ (see (13)). We choose $\tilde{C}_{\varepsilon,\tilde{\delta}_0,\ell}$ so large that

$$C_{\varepsilon,\tilde{\delta}_0,\ell} \int_{\Sigma} (v_1^2 + v_2^2) \, dV_g < \frac{\varepsilon}{2} \int_{\Sigma} a^{ij} \nabla v_i \cdot \nabla v_j \, dV_g, \quad \forall v_1, v_2 \in V_{\varepsilon,\tilde{\delta}_0,\ell},$$

where $V_{\varepsilon,\tilde{\delta}_0,\ell}$ denotes the span of the eigenfunctions of the Laplacian on Σ corresponding to eigenvalues greater than $\tilde{C}_{\varepsilon,\tilde{\delta}_0,\ell}$.

Then we set
$$\hat{u}_i = P_{V_{\varepsilon,\tilde{\delta}_0,\ell}} u_i, \quad \tilde{u}_i = P_{V_{\varepsilon,\tilde{\delta}_0,\ell}^\perp} u_i,$$

where $P_{V_{\varepsilon,\tilde{\delta}_0,\ell}}$ (resp. $P_{V_{\varepsilon,\tilde{\delta}_0,\ell}^\perp}$) stands for the orthogonal projection onto $V_{\varepsilon,\tilde{\delta}_0,\ell}$ (resp. $V_{\varepsilon,\tilde{\delta}_0,\ell}^\perp$). Since $\overline{u}_i = 0$, the H^1-norm and the L^∞-norm on $V_{\varepsilon,\tilde{\delta}_0,\ell}$ are equivalent (with a constant which depends on ε, $\tilde{\delta}_0$ and ℓ), hence by our choice of u_1 and u_2,

$$\|\hat{u}_i\|_{L^\infty(\Sigma)}^2 \leq \hat{C}_{\varepsilon,\tilde{\delta}_0,\ell} \frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} a^{ij} \nabla u_i \cdot \nabla u_j \, dV_g,$$

and

$$C_{\varepsilon,\tilde{\delta}_0,\ell} \int_{\Sigma} (\tilde{u}_1^2 + \tilde{u}_2^2) \, dV_g < \frac{\varepsilon}{2} \sum_{i,j=1}^{2} \int_{\Sigma} a^{ij} \nabla v \cdot \nabla v_j \, dV_g.$$

Hence the last formulas imply

$$\ell \log \int_{\Sigma} e^{u_1} \, dV_g + \log \int_{\Sigma} e^{u_2} \, dV_g \leq \frac{1}{4\pi} (1 + 3\ell\varepsilon) \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Sigma} a^{ij} \nabla \tilde{u}_i \cdot \nabla \tilde{u}_j \, dV_g \right] + \hat{C}_{\varepsilon,\tilde{\delta}_0,\ell,\tilde{\gamma}_0}.$$

This concludes the proof.   □

PROOF OF LEMMA 3.2. First of all we fix a number $r_0 < \delta_0/80$. Then we cover Σ with a finite union of metric balls $(B_{r_0}(x_l))_l$. The number of these balls can be bounded by an integer N_{r_0} which depends only on r_0 (and Σ).

Next we cover the closure $\overline{S}_i$ of every set S_i by a finite number of these balls, and we choose a point $y_i \in \bigcup_l \{x_l\}$ such that

$$\int_{B_{r_0}(y_i)} e^{u_1} \, dV_g = \max\left\{\int_{B_{r_0}(x_l)} e^{u_1} \, dV_g : B_{r_0}(x_l) \cap \overline{S}_i \neq \emptyset\right\}.$$

We also choose $y \in \bigcup_l \{x_l\}$ such that

$$\int_{B_{r_0}(y)} e^{u_2} \, dV_g = \max_l \int_{B_{r_0}(x_l)} e^{u_2} \, dV_g.$$

Since the total number of balls is bounded by N_{r_0} and since by our assumption the integral of e^{u_1} over S_i is greater than or equal to γ_0, it follows that

$$\frac{\int_{B_{r_0}(y_i)} e^{u_1} \, dV_g}{\int_\Sigma e^{u_1} \, dV_g} \geq \frac{\gamma_0}{N_{r_0}}, \qquad \frac{\int_{B_{r_0}(y)} e^{u_2} \, dV_g}{\int_\Sigma e^{u_2} \, dV_g} \geq \frac{1}{N_{r_0}}.$$

By the properties of the sets S_i, we have

$$B_{20r_0}(y_i) \cap B_{r_0}(y_j) = \emptyset \quad \text{for } i \neq j, \quad \operatorname{card}\{y_s : B_{r_0}(y_s) \cap B_{20r_0}(y) \neq \emptyset\} \leq 1.$$

In other words, if we fix y_i, the ball $B_{20r_0}(y_i)$ intersects none of the balls $B_{r_0}(y_j)$ except $B_{r_0}(y_i)$, and given y, $B_{20r_0}(y)$ intersects at most one of the balls $B_{r_0}(y_i)$.

Now, by relabeling the points, we can assume that one of the following two possibilities occurs:

(a) $B_{20r_0}(y) \cap B_{r_0}(y_1) \neq \emptyset$ (and hence $B_{20r_0}(y) \cap B_{r_0}(y_i) = \emptyset$ for $i > 1$);
(b) $B_{20r_0}(y) \cap B_{r_0}(y_i) = \emptyset$ for every $i = 1, \ldots, \ell$.

In case (a) we define
$$\tilde{S}_i = B_{30r_0}(y_i) \quad \text{for } i = 1, \ldots, \ell.$$

while in case (b) we define

$$\tilde{S}_i = \begin{cases} B_{10r_0}(y_1) \cup B_{10r_0}(y) & \text{for } i = 1, \\ B_{10r_0}(y_i) & \text{for } i = 2, \ldots, \ell, \end{cases}$$

We also set $\tilde{\gamma}_0 = \gamma_0/N_{r_0}$ and $\tilde{\delta}_0 = 5r_0$. We notice that $\tilde{\gamma}_0$ and $\tilde{\delta}_0$ depend only on γ_0, δ_0 and Σ, as claimed, and that the sets $\tilde{S}_i$ satisfy the required conditions. This concludes the proof of the lemma. $\square$

3.2. *Application to the study of J_ρ*

In this subsection we apply the improved inequality in order to understand the structure of the sublevels of J_ρ. Our main result here is Corollary 3.5.

In the next lemma we show a criterion which implies the situation described by (11). The result is proven in [15, Lemma 2.3], but we repeat here the argument for the reader's convenience.

LEMMA 3.3. *Let $f \in L^1(\Sigma)$ be a non-negative function with $\|f\|_{L^1(\Sigma)} = 1$. Also fix an integer ℓ and suppose that there exist $\varepsilon > 0$ and $r > 0$ such that*

$$\int_{\bigcup_{i=1}^{\ell} B_r(p_i)} f \, dV_g < 1 - \varepsilon \quad \text{for all the } \ell\text{-tuples } p_1, \ldots, p_\ell \in \Sigma.$$

Then there exist $\overline{\varepsilon} > 0$ and $\overline{r} > 0$, depending only on ε, r, ℓ and Σ (and not on f), and $\ell + 1$ points $\overline{p}_1, \ldots, \overline{p}_{\ell+1} \in \Sigma$ (which depend on f) satisfying

$$\int_{B_{\overline{r}}(\overline{p}_1)} f \, dV_g > \overline{\varepsilon}, \; \ldots, \; \int_{B_{\overline{r}}(\overline{p}_{\ell+1})} f \, dV_g > \overline{\varepsilon}, \quad B_{2\overline{r}}(\overline{p}_i) \cap B_{2\overline{r}}(\overline{p}_j) = \emptyset \quad \text{for } i \neq j.$$

PROOF. Suppose by contradiction that for every $\overline{\varepsilon}, \overline{r} > 0$ and for any $\ell + 1$ points $p_1, \ldots, p_{\ell+1} \in \Sigma$,

$$(19) \qquad \int_{B_{\overline{r}}(p_1)} f \, dV_g \geq \overline{\varepsilon}, \; \ldots, \; \int_{B_{\overline{r}}(p_{\ell+1})} f \, dV_g \geq \overline{\varepsilon}$$
$$\Rightarrow \; B_{2\overline{r}}(p_i) \cap B_{2\overline{r}}(p_j) \neq \emptyset \text{ for some } i \neq j.$$

We let $\overline{r} = r/8$, where r is given in the statement. We can find $h \in \mathbb{N}$ and h points $x_1, \ldots, x_h \in \Sigma$ such that Σ is covered by $\bigcup_{i=1}^{h} B_{\overline{r}}(x_i)$. If ε is as above, we also set $\overline{\varepsilon} = \varepsilon/2h$. We point out that the choice of $\overline{r}$ and $\overline{\varepsilon}$ depends on r, ε and Σ only, as required.

Let $\{\tilde{x}_1, \ldots, \tilde{x}_j\} \subseteq \{x_1, \ldots, x_h\}$ be the points for which $\int_{B_{\overline{r}}(\tilde{x}_i)} f \, dV_g \geq \overline{\varepsilon}$. We define $\tilde{x}_{j_1} = \tilde{x}_1$, and set

$$A_1 = \bigcup_i \{B_{\overline{r}}(\tilde{x}_i) : B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_1}) \neq \emptyset\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_1}).$$

If there exists $\tilde{x}_{j_2}$ such that $B_{2\overline{r}}(\tilde{x}_{j_2}) \cap B_{2\overline{r}}(\tilde{x}_{j_1}) = \emptyset$, we define

$$A_2 = \bigcup_i \{B_{\overline{r}}(\tilde{x}_i) : B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_2}) \neq \emptyset\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_2}).$$

Proceeding in this way, we define recursively some points $\tilde{x}_{j_3}, \ldots, \tilde{x}_{j_s}$ satisfying

$$B_{2\overline{r}}(\tilde{x}_{j_s}) \cap B_{2\overline{r}}(\tilde{x}_{j_a}) = \emptyset \quad \forall 1 \leq a < s,$$

and

$$A_s = \bigcup_i \{B_{\overline{r}}(\tilde{x}_i) : B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_s}) \neq \emptyset\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_s}).$$

By (19), the process cannot go further than $\tilde{x}_{j_\ell}$, and hence using the definition of $\overline{r}$ we obtain

$$(20) \qquad \bigcup_{i=1}^{j} B_{\overline{r}}(\tilde{x}_i) \subseteq \bigcup_{i=1}^{\ell} B_{4\overline{r}}(\tilde{x}_{j_i}) \subseteq \bigcup_{i=1}^{\ell} B_{r/2}(\tilde{x}_{j_i}).$$

Then by our choice of $h, \overline{\varepsilon}, \{\tilde{x}_1, \ldots, \tilde{x}_j\}$ and by (20),

$$\int_{\Sigma \setminus \bigcup_{i=1}^{\ell} B_r(\tilde{x}_{j_i})} f \, dV_g \leq \int_{\Sigma \setminus \bigcup_{i=1}^{j} B_{\overline{r}}(\tilde{x}_i)} f \, dV_g \leq (h - j)\overline{\varepsilon} \leq \varepsilon/2.$$

Finally, if we choose $\overline{p}_i = \tilde{x}_{j_i}$, $i = 1, \ldots, \ell$, we get a contradiction to the assumptions. $\square$

Next we characterize the functions in $H^1(\Sigma) \times H^1(\Sigma)$ for which the value of J_ρ is large negative.

LEMMA 3.4. *Suppose $\rho_1 \in (4\pi m, 4\pi(m+1))$ and that $\rho_2 < 4\pi$. Then for any $\varepsilon > 0$ and $r > 0$ there exists a large positive $L = L(\varepsilon, r)$ such that for every $(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ with $J_\rho(u) \leq -L$ and with $\int_\Sigma e^{u_i} dV_g = 1$, $i = 1, 2$, there exist m points $p_{1,u_1}, \ldots, p_{m,u_1} \in \Sigma$ such that*

$$\text{(21)} \qquad \int_{\Sigma \setminus \bigcup_{i=1}^m B_r(p_{i,u_1})} e^{u_1} dV_g < \varepsilon.$$

PROOF. Suppose by contradiction that the statement is not true. Then we can apply Lemma 3.3 with $\ell = m + 1$ and $f = e^{u_1}$ to obtain $\hat{\delta}_0, \hat{\gamma}_0$ and sets $\hat{S}_1, \ldots, \hat{S}_{m+1}$ such that

$$\text{dist}(\hat{S}_i, \hat{S}_j) \geq \hat{\delta}_0, \quad i \neq j,$$

$$\int_{\hat{S}_i} e^{u_1} dV_g > \hat{\gamma}_0 \int_\Sigma e^{u_1} dV_g, \quad i = 1, \ldots, m + 1.$$

Now we notice that, by the Jensen inequality, $\int_\Sigma u_i dV_g \leq 0$ for $i = 1, 2$, and that two cases may occur:

(a) $\rho_2 \leq 0$;
(b) $\rho_2 > 0$.

In case (a) we have $\rho_2 \int_\Sigma u_2 dV_g \geq 0$. Using also inequality (16) we find

$$J_\rho(u_1, u_2) \geq \frac{1}{4} \int_\Sigma |\nabla u_1|^2 dV_g + \rho_1 \int_\Sigma u_1 dV_g - C.$$

Now it is sufficient to use Proposition 2.2 with $\ell = m + 1$, $\delta_0 = \hat{\delta}_0$, $\gamma_0 = \hat{\gamma}_0$, $S_j = \hat{S}_j$, $j = 1, \ldots, m + 1$ and $\tilde{\varepsilon} \in (0, 16\pi(m+1) - 4\rho_1)$ to get

$$J_\rho(u_1, u_2) \geq \frac{1}{4} \int_\Sigma |\nabla u_1|^2 dV_g - \frac{\rho_1}{16\pi(m+1) - \tilde{\varepsilon}} \int_\Sigma |\nabla u_1|^2 dV_g - C$$

$$\geq \frac{16\pi(m+1) - 4\rho_1 - \tilde{\varepsilon}}{4[16\pi(m+1) - \tilde{\varepsilon}]} \int_\Sigma |\nabla u_1|^2 dV_g - \tilde{C},$$

where $\tilde{C}$ is independent of (u_1, u_2).

In case (b) we use Proposition 3.1 with $\delta_0 = \hat{\delta}_0$, $\gamma_0 = \hat{\gamma}_0$, $\ell = m + 1$, $S_j = \hat{S}_j$ and $\tilde{\varepsilon}$ such that $(4\pi - \tilde{\varepsilon})(m+1) > \rho_1$ and $4\pi - \tilde{\varepsilon} > \rho_2$ (recall that $\rho_1 < 4\pi(m+1)$ and that $\rho_2 < 4\pi$) to deduce that

$$J_\rho(u_1, u_2) \geq (4\pi - \tilde{\varepsilon})[-(m+1)\bar{u}_1 - \bar{u}_2] + \rho_1 \bar{u}_1 + \rho_2 \bar{u}_2$$

$$= (\rho_1 - (m+1)(4\pi - \tilde{\varepsilon}))\bar{u}_1 + (\rho_2 - 4\pi + \tilde{\varepsilon})\bar{u}_2 - C \geq -C,$$

by the Jensen inequality, where again $\tilde{C}$ is independent of (u_1, u_2). This concludes the proof. $\square$

As a consequence of Lemma 3.4 we have the following result, regarding the distance of the functions e^{u_1} (suitably normalized) from Σ_m (see (9)).

COROLLARY 3.5. *Let $\bar{\varepsilon}$ be a (small) arbitrary positive real number, and let $\rho_1 \in (4\pi m, 4\pi(m+1))$, $\rho_2 < 4\pi$. Then there exists $L > 0$ such that, if $J_\rho(u_1, u_2) \leq -L$ and $\int_\Sigma e^{u_i} \, dV_g = 1$, we have $\mathrm{dist}(e^{u_1}, \Sigma_m) < \bar{\varepsilon}$.*

PROOF. We consider ε and r small and positive (to be fixed later), and we let L be the corresponding constant given by Lemma 3.4. We let $p_1, \ldots, p_m$ denote the corresponding points. Now we define $\sigma \in \Sigma_m$ by

$$\sigma = \sum_{j=1}^{m} t_j \delta_{p_j}, \quad \text{where} \quad t_i = \int_{A_{r,i}} e^{u_1} \, dV_g, \quad A_{r,i} := B_r(p_i) \setminus \bigcup_{s=1}^{i-1} B_r(p_s),$$

for every $i = 1, \ldots, m-1$, and

$$A_{r,m} = \Sigma \setminus \bigcup_{s=1}^{m-1} B_r(p_s).$$

Notice that all the $A_{r,j}$'s are disjoint by construction. Now, given $\varphi \in C^1(\Sigma)$ with $\|\varphi\|_{C^1(\Sigma)} = 1$, (using also (21)) we have $\Sigma = \bigcup_{j=1}^{m} A_{r,j}$ and

$$\left| \int_{\Sigma \setminus \bigcup_{j=1}^{m} B_r(p_j)} e^{u_1} (\varphi - \varphi(p_m)) \, dV_g \right| < 2\varepsilon,$$

$$\left| \int_{A_{r,j}} \varphi e^{u_1} \, dV_g - t_j \varphi(p_j) \right| \leq C_\Sigma r \|\varphi\|_{C^1(\Sigma)} \leq C_\Sigma r$$

for $j = 1, \ldots, m-1$, and

$$\left| \int_{B_r(p_m)} e^{u_1} (\varphi - \varphi(p_m)) \, dV_g \right| \leq C_\Sigma r \|\varphi\|_{C^1(\Sigma)} \leq C_\Sigma r.$$

By (9) it then follows that

$$\mathrm{dist}(e^{u_1}, \Sigma_m) \leq \sup \left\{ \left| \int_\Sigma e^{u_1} \varphi \, dV_g - \langle \sigma, \varphi \rangle \right| \, \Big| \, \|\varphi\|_{C^1(\Sigma)} = 1 \right\} \leq 2\varepsilon + m C_\Sigma r.$$

Now it is sufficient to choose ε and r such that $2\varepsilon + m C_\Sigma r < \bar{\varepsilon}$. This concludes the proof. $\square$

4. THE MIN-MAX ARGUMENT

In this section we perform the topological construction to be used in order to produce solutions of (3). First of all, Corollary 3.5 allows us to construct a projection Ψ from suitable sublevels of J_ρ onto Σ_m. Next, the main idea is to use for the min-max some maps

from the cone K_m over Σ_m (see (10)) into $H^1(\Sigma) \times H^1(\Sigma)$. We require that these maps at the boundary all coincide with a given function Φ, which is defined in the next subsection.

The map Φ is chosen so that $\Psi \circ \Phi$ is homotopic to the identity on Σ_m (see Proposition 4.2), and the functional J_ρ on the image is very large negative. Considering then the image of K_m with respect to the above maps (with fixed boundary datum), in Proposition 2.7 we will verify that the maximal value of J_ρ on the image will be strictly greater than the maximum on the boundary. By standard arguments (considering a pseudo-gradient flow for J_ρ), we conclude that the functional has a Palais–Smale sequence at some level α_ρ.

At this point, in order to prove boundedness of the Palais–Smale sequences, we employ crucially a method due to Struwe. We introduce a modified functional $J_{t\rho}$ and we prove a monotonicity of $\alpha_{t\rho}$ with respect to t. This allows us to prove existence of solutions of (3) with ρ replaced by $t_k\rho$ where $t_k \to 1$ as $k \to \infty$. Finally, we apply the compactness result of Proposition 2.5 to achieve existence for $t = 1$ as well.

4.1. *Construction of the maps Ψ and Φ*

PROPOSITION 4.1. *Suppose m is a positive integer, $\rho_1 \in (4\pi m, 4\pi(m + 1))$, and $\rho_2 < 4\pi$. Then there exists a large $L > 0$ and a continuous projection Ψ from $\{J_\rho \leq -L\} \cap \{\int_\Sigma e^{u_1} dV_g = 1\}$ (with the natural topology of $H^1(\Sigma) \times H^1(\Sigma)$) onto Σ_m which is homotopically non-trivial.*

PROOF. We fix ε_m so small that Proposition 2.7 applies with $k = m$. Then we apply Corollary 3.5 with $\overline{\varepsilon} = \varepsilon_m$. We let L be the corresponding large number, so that if $J_\rho(u) \leq -L$, then $\mathrm{dist}(e^{u_1}, \Sigma_m) < \varepsilon_m$. Hence for these ranges of u_1 and u_2, since the map $u \mapsto e^u$ is continuous from $H^1(\Sigma)$ into $L^1(\Sigma)$, the projection Π_m from $H^1(\Sigma)$ onto Σ_m is well defined and continuous. The non-triviality of this map is a consequence of Proposition 4.2(ii) below.   □

The next step consists in mapping Σ_m into arbitrarily negative sublevels of J_ρ. In order to do this, we need some preliminary notation. Given $\sigma \in \Sigma_m$, $\sigma = \sum_{i=1}^m t_i \delta_{x_i}$, and $\lambda > 0$, we define the function $\varphi_{\lambda,\sigma} : \Sigma \to \mathbb{R}$ by

$$(22) \qquad \varphi_{\lambda,\sigma}(y) = \log \sum_{i=1}^m t_i \left(\frac{\lambda}{1 + \lambda^2 d_i^2(y)} \right)^2,$$

where we have set

$$d_i(y) = d(y, x_i), \quad x_i, y \in \Sigma.$$

We point out that, since the distance from a fixed point of Σ is a Lipschitz function, $\varphi_{\lambda,\sigma}(y)$ is also Lipschitz in y, and hence it belongs to $H^1(\Sigma)$.

PROPOSITION 4.2. *Suppose m is a positive integer, $\rho_1 \in (4\pi m, 4\pi(m + 1))$, and $\rho_2 < 4\pi$. For $\lambda > 0$ and $\sigma \in \Sigma_m$, define $\Phi : \Sigma_m \to H^1(\Sigma) \times H^1(\Sigma)$ as*

$$(\Phi(\sigma))(\cdot) = (\Phi(\sigma)_1(\cdot), \Phi(\sigma)_2(\cdot)) := \left(\varphi_{\lambda,\sigma}(\cdot), -\frac{1}{2}\varphi_{\lambda,\sigma}(\cdot) \right),$$

where $\varphi_{\lambda,\sigma}$ is given in (22). Then for L sufficiently large there exists $\lambda > 0$ such that

(i) $J_\rho(\Phi(\sigma)) \leq -L$ uniformly in $\sigma \in \Sigma_m$;

(ii) $\Psi \circ \Phi$ is homotopic to the identity on Σ_m,

where Ψ is defined in Proposition 4.1, and where we assume L to be so large that Ψ is well defined on $\{J_\rho \leq -L\}$.

PROOF. The main ideas follow the strategy in [15], but for the reader's convenience we present here a simplified argument (for the H^2 setting in [15] it was necessary to introduce a cutoff function on the distances d_i, which made the computations more involved).

The proof of (i) relies on showing the following two pointwise estimates on the gradient of $\varphi_{\lambda,\sigma}$:

$$(23) \qquad |\nabla \varphi_{\lambda,\sigma}(y)| \leq C\lambda \quad \text{for every } y \in \Sigma,$$

where C is a constant independent of σ and λ, and

$$(24) \qquad |\nabla \varphi_{\lambda,\sigma}(y)| \leq \frac{4}{d_{\min}(y)} \quad \text{where} \quad d_{\min}(y) = \min_{i=1,\ldots,m} d(y, x_i).$$

To prove (23) we notice that

$$(25) \qquad \frac{\lambda^2 d(y, x_i)}{1 + \lambda^2 d^2(y, x_i)} \leq C\lambda, \quad i = 1, \ldots, m,$$

where C is a fixed constant (independent of λ and x_i). Moreover, we have

$$(26) \qquad \nabla \varphi_{\lambda,\sigma}(y) = -2\lambda^2 \frac{\sum_i t_i (1 + \lambda^2 d_i^2(y))^{-3} \nabla_y(d_i^2(y))}{\sum_j t_j (1 + \lambda^2 d_j^2(y))^{-2}}.$$

Using the fact that $|\nabla_y(d_i^2(y))| \leq 2d_i(y)$ and inserting (25) into (26) we obtain (23) immediately. Similarly we find

$$|\nabla \varphi_{\lambda,\sigma}(y)| \leq 4\lambda^2 \frac{\sum_i t_i (1 + \lambda^2 d_i^2(y))^{-3} d_i(y)}{\sum_j t_j (1 + \lambda^2 d_j^2(y))^{-2}} \leq 4\lambda^2 \frac{\sum_i t_i (1 + \lambda^2 d_i^2(y))^{-2} \frac{d_i(y)}{\lambda^2 d_i^2(y)}}{\sum_j t_j (1 + \lambda^2 d_j^2(y))^{-2}}$$

$$\leq 4 \frac{\sum_i t_i (1 + \lambda^2 d_i^2(y))^{-2} \frac{1}{d_{\min}(y)}}{\sum_j t_j (1 + \lambda^2 d_j^2(y))^{-2}} \leq \frac{4}{d_{\min}(y)},$$

which is (24).

Now, using (23), (24) and the fact that $\nabla \Phi(\sigma)_2 = -\frac{1}{2} \nabla \Phi(\sigma)_1$, one easily finds that

$$\frac{1}{2} \sum_{i,j=1}^2 \int_\Sigma a^{ij} (\nabla \Phi(\sigma)_i) \cdot (\nabla \Phi(\sigma)_j) \, dV_g \leq C + 4 \int_{\Sigma \setminus \bigcup_i B_{1/\lambda}(x_i)} \frac{1}{d_{\min}^2(y)} \, dV_g(y).$$

Reasoning as in [15] one can show that

$$\int_{\Sigma \setminus \bigcup_i B_{1/\lambda}(x_i)} \frac{1}{d_{\min}^2(y)} \, dV_g(y) \leq 8\pi m (1 + o_\lambda(1)) \log \lambda \quad (o_\lambda(1) \to 0 \text{ as } \lambda \to +\infty),$$

and that

$$\int_\Sigma \varphi_{\lambda,\sigma}\, dV_g = -2(1 + o_\lambda(1)) \log \lambda, \quad \log \int_\Sigma e^{\varphi_{\lambda,\sigma}}\, dV_g = O(1),$$

$$\log \int_\Sigma e^{-\frac{1}{2}\varphi_{\lambda,\sigma}}\, dV_g = (1 + o_\lambda(1)) \log \lambda.$$

Using the last four inequalities one then obtains

$$J_\rho(\Phi(\sigma)) \le (8m\pi - 2\rho_1 + o_\lambda(1)) \log \lambda + C,$$

where C is independent of λ and σ. Since we are assuming that $\rho_1 > 4m\pi$, we achieve (i).

To prove (ii) it is sufficient to consider the family of maps $T_\lambda : \Sigma_m \to \Sigma_m$ defined by

$$T_\lambda(\sigma) = \Psi(\Phi_\lambda(\sigma)), \quad \sigma \in \Sigma_m.$$

We recall that when λ is sufficiently large this composition is well defined. Therefore, since $e^{\varphi_{\lambda,\sigma}} / \int_\Sigma e^{\varphi_{\lambda,\sigma}}\, dV_g \rightharpoonup \sigma$ in the weak sense of distributions, letting $\lambda \to \infty$ we obtain a homotopy between $\Psi \circ \Phi$ and Id_{Σ_m}. This concludes the proof. $\square$

REMARK 4.3.   We point out that, fixing $\xi_1 \in \mathbb{R}^2$, the choice of ξ_2 which minimizes the quadratic form $\sum_{i,j} a^{ij}\xi_1 \cdot \xi_j$ is $\xi_2 = -\frac{1}{2}\xi_1$. This motivates the coefficient $-\frac{1}{2}$ in the second component of Φ.

4.2. *The min-max scheme: proof of Theorem 1.1*

In this section we prove Theorem 1.1 employing a min-max scheme based on the construction of the above set Σ_m (see Lemma 4.4). As anticipated in the introduction, we then define a modified functional $J_{t\rho_1,t\rho_2}$ for which we can prove existence of solutions in a dense set of values of t. Following an idea of Struwe, this is done by proving the a.e. differentiability of the map $t \mapsto \alpha_{t\rho}$, where $\alpha_{t\rho}$ is the min-max value for the functional $J_{t\rho_1,t\rho_2}$ given by the scheme.

Let K_m be the topological cone over Σ_m (see (10)). First, let L be so large that Proposition 4.1 applies with $L/4$, and then choose Φ such that Proposition 4.2 applies for L. Fixing L and Φ, we define the class of maps

$$(27) \quad \Pi_\Phi = \{\pi : K_m \to H^1(\Sigma) \times H^1(\Sigma) : \pi \text{ is continuous and } \pi|_{\Sigma_m(=\partial K_m)} = \Phi\}.$$

Then we have the following properties.

LEMMA 4.4. *The set Π_Φ is non-empty and*

$$\alpha_\rho = \inf_{\pi \in \Pi_\Phi} \sup_{\tilde{\sigma} \in K_m} J_{\rho_1,\rho_2}(\pi(\tilde{\sigma})) \quad satisfies \quad \alpha_\rho > -L/2.$$

PROOF.   To prove that $\Pi_\Phi \ne \emptyset$, we just notice that the map

$$\overline{\pi}(\sigma, t) = t\Phi(\sigma), \quad \sigma \in \Sigma_m, \, t \in [0, 1]\, ((\sigma, t) \in K_m),$$

belongs to Π_Φ. Assuming by contradiction that $\alpha_\rho \leq -L/2$, there would exist a map $\pi \in \Pi_\Phi$ with $\sup_{\tilde{\sigma} \in K_m} J_{\rho_1,\rho_2}(\pi(\tilde{\sigma})) \leq -\frac{3}{8}L$. Then, since Proposition 4.1 applies with $L/4$, writing $\tilde{\sigma} = (\sigma, t)$, with $\sigma \in \Sigma_m$, the map

$$t \mapsto \Psi \circ \pi(\cdot, t)$$

would be a homotopy in Σ_m between $\Psi \circ \Phi$ and a constant map. But this is impossible since Σ_m is non-contractible (see Lemma 2.6) and since $\Psi \circ \Phi$ is homotopic to the identity, by Proposition 4.2. Therefore we deduce $\alpha_\rho > -L/2$. $\quad\square$

PROOF OF THEOREM 1.1.   We introduce a variant of the above min-max scheme, following [29] and [12]. For t close to 1, we consider the functional

$$J_{t\rho_1,t\rho_2}(u) = \frac{1}{2} \sum_{i,j} \int_\Sigma a^{ij} \nabla u_i \cdot \nabla u_j \, dV_g + t\rho_1 \int_\Sigma u_1 \, dV_g + t\rho_2 \int_\Sigma u_2 \, dV_g$$
$$- t\rho_1 \log \int_\Sigma h_1 e^{u_1} \, dV_g - t\rho_2 \log \int_\Sigma h_2 e^{u_2} \, dV_g.$$

Repeating the estimates of the previous sections, one easily checks that the above min-max scheme applies uniformly for $t \in [1 - t_0, 1 + t_0]$ with t_0 sufficiently small. More precisely, given $L > 0$ as before, for t_0 sufficiently small we have

$$\sup_{\pi \in \Pi_\Phi} \sup_{\tilde{\sigma} \in \partial K_m} J_{t\rho_1,t\rho_2}(\pi(\tilde{\sigma})) < -2L$$

and

$$\alpha_{t\rho} := \inf_{\pi \in \Pi_\Phi} \sup_{\tilde{\sigma} \in K_m} J_{t\rho_1,t\rho_2}(\pi(\tilde{\sigma})) > -L/2$$

for every $t \in [1 - t_0, 1 + t_0]$ (where Π_Φ is defined in (27)).

Next we notice that for $t' \geq t$,

$$\frac{J_{t\rho_1,t\rho_2}(u)}{t} - \frac{J_{t'\rho_1,t'\rho_2}(u)}{t'} = \frac{1}{2}\left(\frac{1}{t} - \frac{1}{t'}\right) \int_\Sigma a^{ij} \nabla u_i \cdot \nabla u_j \, dV_g \geq 0$$

for every $u \in H^1(\Sigma) \times H^1(\Sigma)$. Therefore it follows easily that also

$$\frac{\alpha_{t\rho}}{t} - \frac{\alpha_{t'\rho}}{t'} \geq 0,$$

that is, the function $t \mapsto \alpha_{t\rho}/t$ is non-increasing, and hence is almost everywhere differentiable. Using Struwe's monotonicity argument (see for example [12]), one can see that at the points where $\alpha_{t\rho}/t$ is differentiable $J_{t\rho_1,t\rho_2}$ admits a bounded Palais–Smale sequence at level $\alpha_{t\rho}$, which converges to a critical point of $J_{t\rho_1,t\rho_2}$. Therefore, since the points with differentiability fill densely the interval $[1 - t_0, 1 + t_0]$, there exists $t_k \to 1$ such that the following system has a solution $(u_{1,k}, u_{2,k})$:

$$-\Delta u_{i,k} = \sum_{j=1}^N t_k \rho_j a_{ij}\left(\frac{h_j e^{u_{j,k}}}{\int_\Sigma h_j e^{u_{j,k}} \, dV_g} - 1\right), \quad i = 1, 2.$$

Now it is sufficient to apply Proposition 2.5 to obtain a limit (u_1, u_2) which is a solution of (3). This concludes the proof. $\qquad\square$

REFERENCES

[1] A. BAHRI, *Critical Points at Infinity in Some Variational Problems*. Res. Notes Math. 182, Longman-Pitman, London, 1989.

[2] A. BAHRI - J. M. CORON, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*. Comm. Pure Appl. Math. 41 (1988), 253–294.

[3] D. BARTOLUCCI - G. TARANTELLO, *The Liouville equation with singular data: a concentration-compactness principle via a local representation formula*. J. Differential Equations 185 (2002), 161–180.

[4] D. BARTOLUCCI - C. C. CHEN - C. S. LIN - G. TARANTELLO, *Profile of blow-up solutions to mean field equations with singular data*. Comm. Partial Differential Equations 29 (2004), 1241–1265.

[5] G. E. BREDON, *Topology and Geometry*. Grad. Texts in Math. 139, Springer, 1997.

[6] H. BREZIS - F. MERLE, *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions*. Comm. Partial Differential Equations 16 (1991), 1223–1253.

[7] E. CAGLIOTI - P.-L. LIONS - C. MARCHIORO - M. PULVIRENTI, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description*. Comm. Math. Phys. 143 (1992), 501–525.

[8] E. CAGLIOTI - P. L. LIONS - C. MARCHIORO - M. PULVIRENTI, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. Part II*. Comm. Math. Phys. 174 (1995), 229–260.

[9] C. C. CHEN - C. S. LIN, *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*. Comm. Pure Appl. Math. 55 (2002), 728–771.

[10] C. C. CHEN - C. S. LIN, *Topological degree for a mean field equation on Riemann surfaces*. Comm. Pure Appl. Math. 56 (2003), 1667–1727.

[11] W. CHEN - C. LI, *Prescribing Gaussian curvatures on surfaces with conical singularities*. J. Geom. Anal. 1 (1991), 359-372.

[12] W. DING - J. JOST - J. LI - G. WANG, *Existence results for mean field equations*. Ann. Inst. H. Poincaré Anal. Non Linéaire 16 (1999), 653–666.

[13] Z. DJADLI, *Existence result for the mean field problem on Riemann surfaces of all genus*. Comm. Contemp. Math., to appear.

[14] Z. DJADLI - A. MALCHIODI, *A fourth order uniformization theorem on some four manifolds with large total Q-curvature*. C. R. Math. Acad. Sci. Paris 340 (2005), 341–346.

[15] Z. DJADLI - A. MALCHIODI, *Existence of conformal metrics with constant Q-curvature*. Ann. of Math., to appear.

[16] G. DUNNE, *Self-dual Chern–Simons Theories*. Lecture Notes in Phys. 36, Springer, Berlin, 1995.

[17] J. JOST - C. S. LIN - G. WANG, *Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions*. Comm. Pure Appl. Math. 59 (2006), 526–558.

[18] J. JOST - G. WANG, *Analytic aspects of the Toda system. I. A Moser–Trudinger inequality*. Comm. Pure Appl. Math. 54 (2001), 1289–1319.

[19] J. JOST - G. WANG, *Classification of solutions of a Toda system in* $\mathbb{R}^2$. Int. Math. Res. Not. 2002, no. 6, 277–290.

[20] M. K. H. KIESSLING, *Statistical mechanics of classical particles with logarithmic interactions*. Comm. Pure Appl. Math. 46 (1993), 27–56.

[21] J. LI - Y. LI, *Solutions for Toda systems on Riemann surfaces*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4 (2005), 375–388.

[22] Y. Y. LI, *Harnack type inequality: The method of moving planes*. Comm. Math. Phys. 200 (1999), 421–444.

[23] M. LUCIA - M. NOLASCO, SU(3) *Chern–Simons vortex theory and Toda systems*, J. Differential Equations 184 (2002), 443–474.

[24] A. MALCHIODI, *Compactness of solutions to some geometric fourth-order equations*. J. Reine Angew. Math. 594 (2006), 137–174.

[25] M. NOLASCO - G. TARANTELLO, *On a sharp Sobolev-type inequality on two-dimensional compact manifolds*. Arch. Ration. Mech. Anal. 145 (1998), 161–195.

[26] M. NOLASCO - G. TARANTELLO, *Vortex condensates for the SU(3) Chern–Simons theory*. Comm. Math. Phys. 213 (2000), 599–639.

[27] I. SHAFRIR - G. WOLANSKY, *Moser–Trudinger and logarithmic HLS inequalities for systems*. J. Eur. Math. Soc. 7 (2005), 413–448.

[28] I. SHAFRIR - G. WOLANSKY, *Moser–Trudinger type inequalities for systems in two dimensions*. C. R. Acad. Sci. Paris Sér. I Math. 333 (2001), 439–443.

[29] M. STRUWE, *The existence of surfaces of constant mean curvature with free boundaries*. Acta Math. 160 (1988), 19–64.

[30] M. STRUWE - G. TARANTELLO, *On multivortex solutions in Chern–Simons gauge theory*. Boll. Un. Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1998), 109–121.

[31] Y. YANG, *Solitons in Field Theory and Nonlinear Analysis*. Springer, 2001.

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