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Mathematical physics. — *Spectral analysis of transfer operators associated to Farey fractions*, by CLAUDIO BONANNO, SANDRO GRAFFI and STEFANO ISOLA, communicated by S. Graffi.

ABSTRACT. — The spectrum of a one-parameter family of signed transfer operators associated to the Farey map is studied in detail. We show that when acting on a suitable Hilbert space of analytic functions they are self-adjoint and exhibit absolutely continuous spectrum and no non-zero point spectrum. Polynomial eigenfunctions when the parameter is a negative half-integer are also discussed.

KEY WORDS: Transfer operators; Farey fractions; spectral theory; period functions; self-reciprocal functions.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 37C30, 47A10, 11B57.

1. PRELIMINARIES AND STATEMENT OF THE MAIN RESULTS

Let $F : [0,1] \to [0,1]$ be the *Farey map* defined by

$$F(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } 0 \le x \le 1/2, \\ \dfrac{1-x}{x} & \text{if } 1/2 \le x \le 1. \end{cases} \tag{1.1}$$

Its name can be related to the following observation. If we expand $x \in [0,1]$ in a continued fraction, i.e.

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \equiv [a_1, a_2, a_3, \dots]$$

then

$$x = [a_1, a_2, a_3, \dots] \mapsto F(x) = [a_1 - 1, a_2, a_3, \dots] \tag{1.2}$$

with $[0, a_2, a_3, \dots] \equiv [a_2, a_3, \dots]$. In other words, let $\mathcal{F}_n$ be the ascending sequence of irreducible fractions between 0 and 1 constructed inductively in the following way: set first $\mathcal{F}_1 = \left(\frac{0}{1}, \frac{1}{1}\right)$; then $\mathcal{F}_n$ is obtained from $\mathcal{F}_{n-1}$ by inserting among each pair of neighbours $\frac{a'}{b'}$ and $\frac{a''}{b''}$ in $\mathcal{F}_{n-1}$ their Farey sum $\frac{a}{b} := \frac{a'+a''}{b'+b''}$. Thus

$$\mathcal{F}_2 = \left(\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}\right), \quad \mathcal{F}_3 = \left(\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}\right), \quad \mathcal{F}_4 = \left(\tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1}\right)$$

and so on. The elements of $\mathcal{F}_n$ are called *Farey fractions*. It is easy to verify that the set of pre-images $\bigcup_{k=0}^{n} F^{-k}\{0\}$ coincides with $\mathcal{F}_n$ for all $n \ge 1$. This implies that

$\bigcup_{k=0}^{\infty} F^{-k}\{0\} = \mathbb{Q} \cap [0,1]$. These two observations are related by the fact that a rational number $\frac{a}{b}$ belongs to $\mathcal{F}_n \setminus \mathcal{F}_{n-1}$ if and only if its continued fraction expansion $\frac{a}{b} = [a_1, \dots, a_k]$ with $a_k > 1$ is such that $\sum_{i=1}^{k} a_i = n$.

In this paper we shall study a family of *signed generalized transfer operators* $\mathcal{P}_q^{\pm}$ associated to the map F, whose action on a function $f : [0,1] \to \mathbb{C}$ is given by a weighted sum over the values of f on the set $F^{-1}(x)$, namely

$$f(x) \mapsto (\mathcal{P}_q^{\pm} f)(x) = \left(\frac{1}{x+1}\right)^{2q} \left[f\left(\frac{x}{x+1}\right) \pm f\left(\frac{1}{x+1}\right)\right] \tag{1.3}$$

where q is a real or complex parameter. The operator $\mathcal{P}_1^{+}$ is referred to as the *Perron–Frobenius* operator for the map F: its fixed function is the density of an absolutely continuous F-invariant measure. In this case one easily checks that the function $1/x$ has this property. However, since $1/x$ does not belong to $L^1([0,1], dx)$ the statistical properties of the map F have to be described in the framework of infinite ergodic theory [Aa]. We refer to [Bal] for a general review of transfer operator techniques in dynamical systems theory. Here, one motivation to study signed transfer operators arises from their appearing in dynamical zeta functions such as Selberg and Ruelle's (see [DEIK, Corollary 3.13], and also [BI]).

Using the Farey fractions, the iterates $\mathcal{P}_q^{\pm n} f$ of the above operators can be expressed as suitable sums over the *Stern–Brocot tree*, the binary tree with root node 1 and whose n-th level L_n is given by $L_n = (\mathcal{F}_n \setminus \mathcal{F}_{n-1}) \cup S(\mathcal{F}_n \setminus \mathcal{F}_{n-1})$, where S is the map $S : x \mapsto 1/x$ and the elements of $S(\mathcal{F}_n \setminus \mathcal{F}_{n-1})$ are in reverse order. An important feature of this tree is that each positive rational number appears as a vertex exactly once. The left part of the Stern–Brocot tree (starting from the node $\frac{1}{2}$) is called the *Farey tree*, with vertex set $\mathbb{Q} \cap (0,1)$.

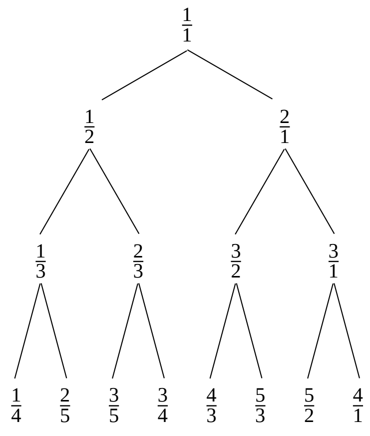

FIG. 1. First four levels of the Stern–Brocot tree.

An easy generalisation of Proposition 5.9 in [DEIK] yields, for all $x \in \mathbb{R}_+$ and $q \in \mathbb{C}$,

$$(\mathcal{P}_q^{\pm n} f)(x) = \sum_{\frac{a}{b} \in L_n} \frac{f\left(\frac{n_0(x,a/b)}{ax+b}\right) \pm f\left(\frac{n_1(x,a/b)}{ax+b}\right)}{(ax+b)^{2q}} \tag{1.4}$$

where $n_0(x, a/b) = \mu x + \nu$ and $n_1(x, a/b) = (a - \mu)x + b - \nu$, for some $0 \leq \mu \leq a$ and $0 \leq \nu \leq b$. In particular $n_0(x, a/b) + n_1(x, a/b) = ax + b$.

In Section 2 we prove

THEOREM 1.1. *For each $q \in (0, \infty)$ there is a Hilbert space $\mathcal{H}_q$ of analytic functions on which the operators $\mathcal{P}_q^{\pm}$ are bounded, self-adjoint and iso-spectral. Their common spectrum is $\{0\} \cup (0, 1]$, with $(0, 1]$ purely absolutely continuous.*

REMARK 1.2. From thermodynamic formalism it follows that $\mathcal{P}_q^+$ for $q \in (-\infty, 1)$, when acting on a suitable Banach space, has a leading eigenvalue $\lambda(q) \geq 1$ which is a differentiable and decreasing function of q with $\lim_{q \to 1_-} \lambda(q) = 1$ (and $\lambda(q) = 1$ for all $q \geq 1$, see [PS]). From the above theorem we see that the corresponding eigenfunction does not belong to $\mathcal{H}_q$ (for $q = 1$ it is just the invariant density $1/x$). Moreover,

$$\log \lambda(q) = \lim_{n \to \infty} \frac{1}{n} \log((\mathcal{P}_q^+)^n 1)(0) .$$

Note that by (1.4) we can write

$$((\mathcal{P}_q^+)^n 1)(0) = 2 \sum_{\frac{a}{b} \in \mathcal{F}_n \setminus \{\frac{0}{1}\}} b^{-2q}$$

and the above sum is equal to the *partition function* $Z_{n-1}(2q)$ at (inverse) temperature $2q$ of the number-theoretical spin chain introduced by Andreas Knauf in [Kn].

REMARK 1.3. Also the operators P_q^- have eigenfunctions which do not belong to the space H_q. Indeed, one easily checks that the function $f(x) = (1-x)/x$ is an eigenfunction of $\mathcal{P}_q^-$ for $q = 1/2$ and eigenvalue 1. But, again, this function does not belong to $\mathcal{H}_{1/2}$.

There are interesting functional symmetries related to the eigenvalue equation for $\mathcal{P}_q^{\pm}$, which can be rephrased in terms of Hankel transforms. The construction of Section 2 allows for a complete account of the corresponding self-reciprocal functions in $L^2(\mathbb{R}_+)$, discussed in Section 3. Finally, in Section 4 we characterise all polynomial eigenvectors of $\mathcal{P}_q^{\pm}$ when $q = -k/2, k \geq 0$.

2. THE SPECTRUM OF $\mathcal{P}_q^{\pm}$ FOR REAL POSITIVE q

In this section we give the proof of Theorem 1.1, hence we restrict ourselves to the case $q \in (0, \infty)$. The proof follows from the results of the following subsections.

2.1. *An invariant Hilbert space*

In this subsection we introduce a family of Hilbert spaces $\mathcal{H}_q$, where $q \in (0, \infty)$, and give a representation of the operators $\mathcal{P}_q^{\pm}$ on $\mathcal{H}_q$.

DEFINITION 2.1. *For $q \in (0,\infty)$ we denote by $\mathcal{H}_q$ the Hilbert space of all complex-valued functions f which can be represented as a generalised Borel transform*

$$f(x) = \mathcal{B}_q[\varphi](x) := \frac{1}{x^{2q}} \int_0^\infty e^{-t/x} e^t \varphi(t)\, m_q(dt), \qquad \varphi \in \mathrm{L}^2(m_q), \tag{2.1}$$

with inner product

$$(f_1, f_2) = \int_0^\infty \varphi_1(t)\overline{\varphi_2(t)}\, m_q(dt) \quad \text{if } f_i = \mathcal{B}_q[\varphi_i] \tag{2.2}$$

and measure $(p = 2q-1)$

$$m_q(dt) = t^p e^{-t}\, dt. \tag{2.3}$$

Function spaces related to that introduced above have been used in [Is], [GI] and [Pre]. In [Is] an explicit connection between the approach presented here and Mayer's work on the transfer operator for the Gauss map [Ma] is established by means of a suitable operator-valued power series.

REMARK 2.2. For $q \in \mathbb{C}$ with $\mathrm{Re}\, q > 0$, the space $\mathcal{H}_q$ can be regarded as a complex Hilbert space. If we set

$$\chi_p(x) := x^p \qquad (p = 2q-1), \tag{2.4}$$

an alternative representation for $f \in \mathcal{H}_q$ can be obtained by a simple change of variable when x is real and positive:

$$(\chi_p \cdot f)(x) = \int_0^\infty e^{-s} (\chi_p \cdot \varphi)(sx)\, ds. \tag{2.5}$$

Note that a function $f \in \mathcal{H}_q$ is analytic in the disk

$$D_1 = \{x \in \mathbb{C} : \mathrm{Re}\, 1/x > 1/2\} = \{x \in \mathbb{C} : |x-1| < 1\}. \tag{2.6}$$

In particular,

$$(\chi_p \cdot \varphi)(t) = \sum_{n=0}^\infty \frac{a_n}{n!} t^n \;\Rightarrow\; (\chi_p \cdot f)(x) = \sum_{n=0}^\infty a_n x^n \tag{2.7}$$

in the sense of formal power series. So the power series of $\chi_p \cdot \varphi$ is obtained by Borel transforming that of $\chi_p \cdot f$, in the usual sense. This justifies the name of the integral transform (2.1).

REMARK 2.3. The invariant density $1/x$ for the Farey map, that is, the fixed function of $\mathcal{P}_1^+$, is the generalised Borel transform (for $q = 1$) of the function $\varphi(t) = 1/t$, which, however, does not belong to $\mathrm{L}^2(m_1)$.

Let us now study the Hilbert space $\mathrm{L}^2(m_q)$. First of all we notice that the measure $m_q(dt)$ is finite: indeed,

$$\int_0^\infty m_q(dt) = \Gamma(2q). \tag{2.8}$$

Second, for the linearly independent family of functions $f_n(t) := t^n/n!$ $(n \geq 0)$ we have

$$(f_n, f_m) = \frac{\Gamma(n+m+2q)}{n!m!}. \tag{2.9}$$

This implies that the (generalised) Laguerre polynomials $L_n^p(t)$ ($n \geq 0$, $\mathrm{Re}\, p > -1$) given by

$$e_n(t) := L_n^p(t) = \sum_{m=0}^{n} \binom{n+p}{n-m} \frac{(-t)^m}{m!} \tag{2.10}$$

form a complete orthogonal basis in $\mathrm{L}^2(m_q)$, with

$$(e_n, e_m) = \frac{\Gamma(n+2q)}{n!} \delta_{n,m}. \tag{2.11}$$

Moreover, using [GR, p. 850] and (2.11) we get, for $m \leq n$,

$$\begin{aligned}(f_n, e_m) &= (-1)^m \frac{\Gamma(n+2q)}{m!(n-m)!} = (-1)^m \binom{n}{m} \|e_n\|^2 \\ &= (-1)^m \frac{\Gamma(n+2q)}{\Gamma(m+2q)(n-m)!} \|e_m\|^2 \\ &= (-1)^m \binom{n+p}{n-m} \|e_m\|^2.\end{aligned} \tag{2.12}$$

In particular $(f_n, e_n) = (-1)^n \|e_n\|^2$. Also note that $(f_n, e_m) = 0$ for $m > n$. Comparing to (2.10) we obtain the following result:

LEMMA 2.4. *For each $n \in \mathbb{N}_0$ the numbers*

$$a_{n,m} := \begin{cases} (-1)^m \binom{n+p}{n-m} & \text{if } m \leq n, \\ 0 & \text{if } m > n, \end{cases}$$

are the Fourier coefficients of f_n with respect to the basis (e_m), i.e.

$$a_{n,m} = \frac{(f_n, e_m)}{\|e_m\|^2}.$$

Moreover,

$$f_n = \sum_{m=0}^{n} a_{n,m} e_m, \qquad e_n = \sum_{m=0}^{n} a_{n,m} f_m.$$

REMARK 2.5. In particular, the $(n+1)\times(n+1)$ lower triangular matrix $A_n := (a_{i,j})_{0\le i,j\le n}$ satisfies $A_n^2 = I_{n+1}$. Therefore, the operator $\Pi_n : \mathrm{L}^2(m_q) \to \mathrm{L}^2(m_q)$ acting as

$$\Pi_n : \sum_{s=0}^{\infty} c_s e_s \mapsto \sum_{s=0}^{\infty} c_s \sum_{r=0}^{n} \frac{(f_r, e_s)}{\|e_s\|^2} f_r = \sum_{r=0}^{n} d_r f_r$$

with

$$d_r := \sum_{s=0}^{r} a_{r,s} c_s \quad \text{or} \quad \mathbf{d}^{(n)} = A_n \mathbf{c}^{(n)}$$

where we have set $\mathbf{c}^{(n)} = (c_0, c_1, \dots, c_n)^T$ and similarly for $\mathbf{d}^{(n)}$, is the orthogonal projection onto the linear subspace spanned by $(1, t, t^2/2!, \dots, t^n/n!)$.

Let us now consider the action of the transform $\mathcal{B}_q$ on the functions (e_n) and (f_m). We have

$$\mathcal{B}_q[e_n](x) = \sum_{m=0}^{n} \Gamma(2q+m) \binom{n+p}{n-m} \frac{(-x)^m}{m!} = (n+1)_p (1-x)^n \tag{2.13}$$

where $(a)_p := \Gamma(a+p)/\Gamma(a) = a(a+1)\cdots(a+p-1)$ is the shifted factorial, and

$$\mathcal{B}_q[f_n](x) = (n+1)_p x^n. \tag{2.14}$$

The next result describes the action of $\mathcal{P}_q^{\pm}$ on the Hilbert space $\mathcal{H}_q$.

PROPOSITION 2.6. *For $q \in (0,\infty)$ the space $\mathcal{H}_q$ is invariant for $\mathcal{P}_q^{\pm}$, and $\mathcal{P}_q^{\pm} : \mathcal{H}_q \to \mathcal{H}_q$ are positive operators, isomorphic to self-adjoint compact perturbations of the multiplication operator $M : \mathrm{L}^2(m_q) \to \mathrm{L}^2(m_q)$ given by*

$$(M\varphi)(t) = e^{-t}\varphi(t).$$

More specifically,

$$\mathcal{P}_q^{\pm}\mathcal{B}_q[\varphi] = \mathcal{B}_q[P^{\pm}\varphi]$$

where $P^{\pm} = M \pm N$ and $N : \mathrm{L}^2(m_q) \to \mathrm{L}^2(m_q)$ is the symmetric integral operator given by

$$(N\varphi)(t) = \int_0^{\infty} \frac{J_p\left(2\sqrt{st}\right)}{(st)^{p/2}} \varphi(s)\, m_q(ds)$$

where J_p denotes the Bessel function of order p.

PROOF. The representation of $\mathcal{P}_q^{\pm}$ on $\mathcal{H}_q$ follows from a direct computation (see [Is], [GI]). The positivity amounts to

$$((M \pm N)\varphi, \varphi) \ge 0 \quad \forall \varphi \in \mathrm{L}^2(m_q),\ \|\varphi\| = 1, \tag{2.15}$$

and can be checked by expanding φ in the basis of (normalised) Laguerre polynomials. Indeed, a calculation using [GR, pp. 849–850] yields

$$\frac{(Me_n, e_n)}{\|e_n\|^2} = 2^{-2n-2q} \binom{2n+p}{n}$$

and

$$\frac{(Ne_n, e_n)}{\|e_n\|^2} = 2^{-n-2q}\binom{n+p}{n}{}_2F_1(-n, n+2q; 2q; 1/2) = 2^{-n-2q}P_n^{(p,0)}(0)$$

where $P_n^{(a,b)}(x)$ denotes the Jacobi polynomial ([AAR, p. 99]). Since

$$P_n^{(p,0)}(0) = (-2)^{-n}\sum_{k=0}^{n}(-1)^k\binom{n+p}{k}\binom{n}{k}$$

and

$$\binom{2n+p}{n} = \sum_{k=0}^{n}\binom{n+p}{k}\binom{n}{k}$$

we get

$$\frac{((M \pm N)e_n, e_n)}{\|e_n\|^2} = \frac{1}{2^{2n+2q}}\sum_{k=0}^{n}(1 \pm (-1)^{n-k})\binom{n+p}{k}\binom{n}{k}.$$

An easy generalisation of this calculation shows that

$$((M \pm N)e_n, e_k) \geq 0 \quad \forall k, n$$

and thus (2.15). Finally, $N\varphi$ can be written as $\int_0^\infty k(s,t)\varphi(s)\, m_q(ds)$ with symmetric kernel

$$k(s,t) = \frac{J_p\left(2\sqrt{st}\right)}{(st)^{p/2}}. \tag{2.16}$$

From the estimates $J_p(t) \sim 2^{-p}t^p/\Gamma(p+1)$ as $t \to 0^+$ and $J_p(t) = O(t^{-1/2})$ as $t \to \infty$ ([E, Vol. II]), we see that the kernel $k(s,t)$ is bounded and continuous. □

We can now describe the action of $P^\pm$ on (e_n) and (f_n). Applying the integral representation (see [E, Vol. II, p. 190])

$$n!e^{-t}L_n^p(t) = \int_0^\infty \frac{J_p(2\sqrt{st})}{(st)^{p/2}}s^n\, m_q(ds)$$

we get

$$M^{-1}Nf_n = e_n, \qquad M^{-1}Ne_n = f_n. \tag{2.17}$$

2.2. *Functional symmetries*

Let us introduce an isometry which turns out to be useful for the characterisation of eigenfunctions of the operators $\mathcal{P}_q^\pm$. Let $\mathcal{J}_q$ be the involution defined by

$$(\mathcal{J}_q f)(x) := \frac{1}{x^{2q}} f\left(\frac{1}{x}\right) \tag{2.18}$$

and consider its action on the Hilbert space $\mathcal{H}_q$. We have the following

PROPOSITION 2.7. *For any $\varphi \in \mathrm{L}^2(m_q)$,*

$$\mathcal{J}_q \mathcal{B}_q[\varphi] = \mathcal{B}_q[J\varphi] \tag{2.19}$$

where $J := NM^{-1}$ is a bounded operator in $\mathrm{L}^2(m_q)$ with $\|J\| \le 2\pi$. If moreover $\mathcal{P}_q^{\pm} f = \lambda f$ for some $\lambda \neq 0$ then f satisfies the functional equation

$$\mathcal{J}_q f = \pm f. \tag{2.20}$$

PROOF. The representation of $\mathcal{J}_q$ in $\mathcal{H}_q$ is easily checked by first noting that for any $f \in \mathcal{H}_q$ the function $\mathcal{J}_q f$ can be written as an ordinary Laplace transform, i.e.

$$f(x) = \mathcal{B}_q[\varphi](x) \;\Rightarrow\; (\mathcal{J}_q f)(x) = \int_0^\infty e^{-tx} (\chi_p \cdot \varphi)(t)\, dt, \tag{2.21}$$

and then using Tricomi's theorem ([Sne, p. 165]). Let us prove the bound on $\|J\|$. Adapting formula (33) of [RS, Vol. IV] to our situation we get, for all $\varphi \in \mathrm{L}^2(m_q)$ and $\lambda \in [0,1]$,

$$\|N(M-\lambda)^{-1}\varphi\|^2 \le \int_0^1 \|N(M-\lambda)^{-1}\varphi\|^2\, d\lambda \le 2\pi \int_{-\infty}^{\infty} \|Ne^{i\tau M}\varphi\|^2\, d\tau. \tag{2.22}$$

On the other hand, we claim that

$$\int_{-\infty}^{\infty} \|Ne^{i\tau M}\varphi\|^2\, d\tau \le 2\pi \int_0^\infty e^{-t} \left(\int_0^\infty |J_p(2\sqrt{st})|^2 |\varphi(s)|^2 s^p e^{-s}\, ds \right) dt. \tag{2.23}$$

To prove (2.23) we write

$$(Ne^{i\tau M}\varphi)(t) = \int_0^\infty \frac{J_p(2\sqrt{st})}{(st)^{p/2}} e^{i\tau e^{-s}} \varphi(s) s^p e^{-s}\, ds$$

so that interchanging the order of integration yields

$$\|Ne^{i\tau M}\varphi\|^2 = \int_0^\infty |G(t,\tau)|^2 e^{-t}\, dt$$

where we have set

$$\begin{aligned} G(t,\tau) &= \int_0^\infty J_p(2\sqrt{st}) e^{i\tau e^{-s}} s^{p/2} \varphi(s) e^{-s}\, ds \\ &= -\int_0^1 J_p(2\sqrt{-t\ln u}) e^{i\tau u} (-\ln u)^{p/2} \varphi(-\ln u)\, du. \end{aligned}$$

The estimate (2.23) now follows by applying the Fourier–Plancherel theorem:

$$\begin{aligned} \int_{-\infty}^{\infty} |G(t,\tau)|^2\, d\tau &= 2\pi \int_0^1 |J_p(2\sqrt{-t\ln u})\varphi(-\ln u)|^2 (-\ln u)^p\, du \\ &= 2\pi \int_0^\infty |J_p(2\sqrt{st})|^2 |\varphi(s)|^2 s^p e^{-s}\, ds. \end{aligned}$$

Hence, putting together (2.22) and (2.23), we have

$$\|N(M-\lambda)^{-1}\varphi\|^2 \leq 4\pi^2 \int_0^\infty e^{-t} \left(\int_0^\infty |J_p(2\sqrt{st})|^2 |\varphi(s)|^2 s^p e^{-s}\, ds \right) dt.$$

The right hand side is bounded above by

$$4\pi^2 \|\varphi\|^2 \int_0^\infty e^{-t} \sup_{st \geq 0} |J_p(2\sqrt{st})|^2\, dt =: 4\pi^2 C \|\varphi\|^2.$$

Using $\sup_{x\geq 0} |J_p(2\sqrt{x})|^2 = 1$ we get $C = 1$. Therefore

$$\|N(M-\lambda)^{-1}\|^2 \leq 4\pi^2 \quad \forall \lambda \in [0,1].$$

Choosing $\lambda = 0$ we get $\|J\| \leq 2\pi$ as claimed.

To finish the proof, we note that if $\varphi \in \mathrm{L}^2(m_q)$ the functions $M\varphi$ and $N\varphi$ are bounded at infinity. Therefore, if $f \in \mathcal{H}_q$ satisfies $\mathcal{P}_q^{\pm} f = \lambda f$ with $\lambda \neq 0$, then f extends analytically from the disk D_1 to the half-plane $\{\mathrm{Re}\, x > 0\}$. In addition the expression $(\mathcal{P}_q^{\pm} f)(x)$ reproduces itself times ± 1 if transformed by substituting $1/x$ for x and dividing through x^{2q}. Hence (2.20) holds. □

REMARK 2.8. Note that (2.18) is only a necessary condition for f to be an eigenfunction (with $\lambda \neq 0$). For instance the function $f(x) = x^{-q}$ (which does not belong to $\mathcal{H}_q$), although plainly satisfying (2.18) for all $q \in (0, \infty)$, is an eigenfunction of $\mathcal{P}_q^{+}$ only for $q = 1$ (with $\lambda = 1$).

REMARK 2.9. By applying Proposition 2.7, the eigenvalue equations $\mathcal{P}_q^{\pm} f = \lambda f$, with $\lambda \neq 0$, can be rewritten as the three-term functional equations

$$\lambda f(x) - f(x+1) = \pm \frac{1}{x^{2q}} f\left(1 + \frac{1}{x}\right), \tag{2.24}$$

which for $\lambda = 1$ are studied in [Le] and [LeZa].

2.3. *The spectrum of $P^{\pm}$ in* $\mathrm{L}^2(m_q)$

It now remains to study the spectrum of the operators $P^{\pm}$ in $\mathrm{L}^2(m_q)$. Let us start by studying the operators

$$Q^{\pm} = M^{-1} P^{\pm} = I \pm M^{-1} N. \tag{2.25}$$

We first show that they are bounded in $\mathrm{L}^2(m_q)$.

LEMMA 2.10. *We have* $\|Q^{\pm}\| \leq 1 + 2\pi$.

PROOF. The adjoint of the operator $J = NM^{-1}$ dealt with in the previous subsection exists and equals $J^* = M^{-1}N$. A priori it is not defined on the whole space $L^2(m_q)$. Recall, however, that J^* is continuous if and only if J is, and $\|J^*\| = \|J\|$. The assertion now follows from Proposition 2.7. □

Recall now the orthogonal basis of $\mathrm{L}^2(m_q)$ given by $e_n(t)$ (see (2.10)) and the independent family of functions $f_n(t) = t^n/n!$. We introduce the families of functions

$$\ell_n^{\pm}(t) := e_n(t) \pm f_n(t), \qquad h_n^{\pm}(t) := e^{-t}(e_n(t) \pm f_n(t)) \tag{2.26}$$

and consider the linear manifolds spanned by them.

PROPOSITION 2.11. *The linear manifolds $\mathcal{E}^{\pm} \subset \mathrm{L}^2(m_q)$ defined by*

$$\mathcal{E}^{\pm} := \left\{ \sum_{n=0}^{m} c_n h_n^{\pm} : c_n \in \mathbb{C},\ 0 \le n \le m,\ m \ge 0 \right\} \tag{2.27}$$

have the following properties:

(i) *they are fixed under the operators $\pm J$, i.e. $\pm J\varphi = \varphi$ for all $\varphi \in \mathcal{E}^{\pm}$;*
(ii) *their intersection is the trivial subspace, i.e. $\mathcal{E}^{+} \cap \mathcal{E}^{-} = \{0\}$;*
(iii) *they are dense, i.e. $\overline{\mathcal{E}^{\pm}} \equiv \mathrm{Span}\left\{h_n^{\pm}\right\}_{n\ge 0} = \mathrm{L}^2(m_q)$.*

PROOF. We first use (2.13) and (2.14) to get

$$\mathcal{B}_q[h_n^{\pm}](x) = (n+1)_p \frac{1 \pm x^n}{(1+x)^{n+2q}}, \tag{2.28}$$

hence $\mathcal{J}_q \mathcal{B}_q[h_n^{\pm}](x) = \pm \mathcal{B}_q[h_n^{\pm}](x)$. Now (i) follows upon application of Proposition 2.7.

(ii) follows at once from the fact that J is an involution.

Finally, from the proof of Proposition 2.6 and (2.17) one readily sees that $(h_n^{\pm}, e_n) > 0$ for all $n \ge 0$. This yields the density of $\mathcal{E}^{\pm}$ in $\mathrm{L}^2(m_q)$. □

Let us now consider the functions $(\ell_n^{\pm})$. From the definition it follows that $\ell_n^{+}(t)$ is a polynomial of degree $2k$ for $n = 2k$ and $n = 2k+1$ ($k \ge 0$), whereas $\ell_n^{-}(t)$ has degree $2k+1$ for $n = 2k+1$ and $n = 2k+2$ ($k \ge 0$). Moreover, $(\ell_n^{\pm}, e_n) = (1 \pm (-1)^n)\|e_n\|^2$ so that

$$\begin{gathered} (\ell_{2k+1}^{+}, e_{2k+1}) = (\ell_{2k+2}^{-}, e_{2k+2}) = 0, \\ (\ell_{2k}^{+}, e_{2k}) = 2\|e_{2k}\|^2, \\ (\ell_{2k+1}^{-}, e_{2k+1}) = 2\|e_{2k+1}\|^2. \end{gathered} \tag{2.29}$$

PROPOSITION 2.12. *Let $H^{\pm} := \mathrm{Span}\left\{\ell_n^{\pm}\right\}_{n\ge 0}$. Then*

(i) $\mathrm{L}^2(m_q) = H^{+} \oplus H^{-}$;
(ii) $Q^{\pm}|_{H^{\pm}} = 2I$ *and* $Q^{\pm}|_{H^{\mp}} = 0$.

PROOF. (i) By (2.29), H^{+} and H^{-} do not have common non-zero vectors, thus $H^{+} \cap H^{-} = \{0\}$. Moreover, let $\varphi \in \mathrm{L}^2(m_q)$ be such that $\varphi \perp H^{+} \oplus H^{-}$. Since $(\ell_n^{\pm}, e_n) = (1 \pm (-1)^n)\|e_n\|^2$ we get $\varphi = 0$.

(ii) We recall (2.17):

$$M^{-1}Nf_n = e_n, \qquad M^{-1}Ne_n = f_n.$$

From this we get

$$Q^{\pm}\ell_n^{\pm} = 2\ell_n^{\pm} \quad \text{and} \quad Q^{\pm}\ell_n^{\mp} = 0.$$

For $\varphi = \sum_{n=0}^{m} c_n \ell_n^{\pm}$ we have by linearity $Q^{\pm}\varphi = 2\varphi$ so that $\|Q^{\pm}\varphi\| = 2\|\varphi\|$, independently of m. This implies $Q^{\pm}\varphi = 2\varphi$ for all $\varphi \in H^{\pm}$. Hence $Q^{\pm}H^{\pm} \subseteq H^{\pm}$ and $Q^{\pm}|_{H^{\pm}} = 2I$. In the same way one proves that $Q^{\pm}|_{H^{\mp}} = 0$. □

REMARK 2.13. From the above it follows that the operators $Q^{\pm}$ are bounded in $\mathrm{L}^2(m_q)$ with $\|Q^{\pm}\| = 2$.

The operators $P^{\pm}$ are self-adjoint and positive on $\mathrm{L}^2(m_q)$, hence their spectrum is real and positive. Moreover, $\|P^{\pm}\| \le \|Q\|\,\|M\| = 2$. Hence $\sigma(P^{\pm}) \subseteq [0, 2]$. From the previous results we have information on the point spectrum $\sigma_p(P^{\pm})$.

COROLLARY 2.14. *In* $\mathrm{L}^2(m_q)$ *we have* $\operatorname{Ker} P^{\pm} = H^{\mp}$ *and* $\sigma_p(P^{\pm}) = \{0\}$ *with infinite multiplicity.*

PROOF. We first observe that since $\operatorname{Ker} M = \{0\}$, Proposition 2.12 yields

$$\operatorname{Ker} P^{\pm} = \operatorname{Ker}(MQ^{\pm}) = \operatorname{Ker} Q^{\pm} = H^{\mp}.$$

Now suppose that $P^{\pm}\varphi = \lambda\varphi$ for some $0 < \lambda \le 2$ and $\varphi \not\equiv 0$. Then, since $P^{\pm}$ are self-adjoint, we can assume that $\varphi \in H^{\pm}$ and hence $P^{\pm}\varphi = MQ^{\pm}\varphi = 2M\varphi$. Therefore $(2M - \lambda)\varphi = 0$, which implies $\varphi \equiv 0$. □

To discuss the rest of the spectrum, we first characterise in more detail the nature of the perturbation operator N.

PROPOSITION 2.15. *For* $\operatorname{Re} q > 0$ *the operator* $N : \mathrm{L}^2(m_q) \to \mathrm{L}^2(m_q)$ *is nuclear (and hence of trace class). Its spectrum is given by*

$$\sigma(N) = \{0\} \cup \{(-1)^k \alpha^{2(q+k)}\}_{k\ge 0} \tag{2.30}$$

where $\alpha = (\sqrt{5} - 1)/2$ *is the golden ratio. Each eigenvalue* $\lambda_k \in \sigma(N)$ *is simple and the corresponding (normalised) eigenfunction* ψ_k *is given by*

$$\psi_k(t) = \sqrt{\frac{5^q k!}{\Gamma(k + 2q)}}\; L_k^p(\sqrt{5}\,t)\exp(-\alpha t). \tag{2.31}$$

COROLLARY 2.16. *For* $\operatorname{Re} q > 0$,

$$\operatorname{tr}(N) = \frac{1}{\sqrt{5}}\alpha^p \quad \textit{and} \quad \|N\| = \alpha^{2\operatorname{Re} q} < 1.$$

PROOF OF PROPOSITION 2.15. Expanding the kernel of N (see (2.16)) in the basis $(e_n)_{n\ge 0}$, we get (see [Sze, p. 102])

$$\frac{J_p\left(2\sqrt{st}\right)}{(st)^{p/2}} = \sum_{n=0}^{\infty} e_n(s)\frac{e^{-t}t^n}{\Gamma(n + 2q)}.$$

This yields

$$N\varphi = \sum_{n\ge 0} (\varphi, e_n) g_n$$

where $g_n(t) = Ne_n(t) = e^{-t}t^n/n!$. Since

$$\|e_n\| = \sqrt{\frac{\Gamma(n+2q)}{n!}}, \qquad \|g_n\| = \frac{\sqrt{\Gamma(2n+2q)}}{n!3^{n+q}},$$

we have

$$\sum_n \|e_n\|\,\|g_n\| < \infty,$$

and therefore N is nuclear. To compute the spectrum of N we use the following Hankel transform (see [E, Vol. II]):

$$\int_0^\infty x^{p+1/2} e^{-bx^2} L_k^p(ax^2) J_p(xy)\sqrt{xy}\,dx = \frac{(b-a)^k y^{p+1/2}}{2^{p+1}b^{p+k+1}}\, e^{-y^2/4b} L_k^p\left(\frac{ay^2}{4b(a-b)}\right),$$

which can be recast in terms of the operator N as

$$N[L_k^p(2at)e^{-(2b-1)t}] = \frac{(b-a)^k}{2^{2q}b^{2q+k}} e^{-t/2b} L_k^p\left(\frac{at}{2b(a-b)}\right).$$

This becomes an eigenvalue equation in $\mathrm{L}^2(m_q)$ provided $2b = \alpha^{-1}$ and $2a = \sqrt{5}$. The normalisation constant results from (2.11) upon noting that

$$\|L_k^p(\sqrt{5}\,t)\exp(-\alpha t)\| = \frac{1}{5^{q/2}}\|L_k^p(t)\|.$$

This gives the eigenfunctions ψ_k, and the proof is complete. $\square$

We now put together the previous results. We have seen that for all $q \in (0,\infty)$ the operators $P^\pm = M \pm N$ when acting on $\mathrm{L}^2(m_q)$ are self-adjoint and positive with $\|M\| = 1$ and $\|N\| = \alpha^{2q}$.

The operator M is spectrally absolutely continuous ([Ka, p. 520]). Its spectrum, being the essential range of the multiplying function, coincides with $[0,1]$. This means that in the orthogonal decomposition $\mathrm{L}^2(m_q) = \mathrm{H}_{\mathrm{ac}}(M) \oplus \mathrm{H}_{\mathrm{s}}(M)$ of the Hilbert space into the subspace of absolute continuity $\mathrm{H}_{\mathrm{ac}}(M) = \Pi_{\mathrm{ac}}(M)\mathrm{L}^2(m_q)$ and that of singularity $\mathrm{H}_{\mathrm{s}}(M) = \Pi_{\mathrm{s}}(M)\mathrm{L}^2(m_q)$, we have $\mathrm{H}_{\mathrm{s}}(M) = 0$ (and thus $\Pi_{\mathrm{ac}}(M) = I$).

On the other hand, N_q is of trace class. Therefore, applying the Kato–Rosenblum theorem (see [Ka, p. 542] or [RS, Vol. III, p. 26]), we obtain

PROPOSITION 2.17. *The operator M is unitarily equivalent to the spectrally absolutely continuous part of $P^\pm$. Hence on $\mathrm{L}^2(m_q)$ we have $\sigma_{\mathrm{ac}}(P^\pm) = (0,1]$.*

REMARK 2.18. The equivalence is realised by means of the one-parameter family of unitary operators

$$W(\tau) = e^{i\tau P}e^{-i\tau M}, \qquad -\infty < \tau < \infty.$$

The (strong) limits $W_\pm$ of $W(\tau)$ as $\tau \to \pm\infty$ are called the *wave operators* and $S = W_+^* W_-$ is the *scattering operator*, which is unitary from $L^2(m_q)$ to itself and commutes with M. The Kato–Rosenblum theorem says that in this case the wave operators $W_\pm$ exist and are complete, meaning that they are partial isometries with initial domain $L^2(m_q)$ and range $H_{ac}(P) = \Pi_{ac}(P)L^2(m_q)$. Therefore we have $W_\pm^* W_\pm = I$, $W_\pm W_\pm^* = \Pi_{ac}(P)$ and $PW_\pm = W_\pm M$ (see [RS, Vol. III, pp. 17–19]).

Putting together Proposition 2.6, Corollary 2.14 and Proposition 2.17, we get Theorem 1.1.

3. DIGRESSION: SELF-RECIPROCAL FUNCTIONS IN $L^2(\mathbb{R}_+)$

Given a continuous function ϕ on $\mathbb{R}_+$ and $q \in \mathbb{C}$ with $\operatorname{Re} q > 0$ (or $\operatorname{Re} p > -1$), the function $J\phi = NM^{-1}\phi$ considered in Section 2.2 can be viewed as a version of the *Hankel transform* of ϕ, i.e.

$$J\phi(t) := \int_0^\infty J_p(2\sqrt{st})\left(\frac{s}{t}\right)^{p/2} \phi(s)\, ds. \tag{3.1}$$

We can also define the conjugate transform $\tilde{J}$ as

$$\tilde{J} := \chi_q J \chi_p^{-1} \tag{3.2}$$

or else

$$\tilde{J}\phi(t) = \int_0^\infty J_p(2\sqrt{st})\left(\frac{t}{s}\right)^{p/2} \phi(s)\, ds. \tag{3.3}$$

From the asymptotic estimates on $J_p(t)$ we see that the conditions on ϕ sufficient for the absolute convergence of the integral (3.1) are $\phi(t) = O(t^{-a})$ as $t \to 0^+$ and $\phi(t) = O(t^{-b})$ as $t \to \infty$ with $a < 2 \operatorname{Re} q$ and $b > \operatorname{Re} q + 1/4$. For the integral (3.3) we have the same conditions with $b > 5/4 - q$ and $a < 1$.

Accordingly, the identity $\mathcal{J}_q f = \pm f$ for $f = \mathcal{B}_q[\varphi]$ can be rephrased as a *self-reciprocity* property for the functions φ and $\psi := \chi_p \cdot \varphi$, that is,

$$\mathcal{J}_q f = \pm f \;\Rightarrow\; J\varphi = \pm\varphi \text{ and } \tilde{J}\psi = \pm\psi. \tag{3.4}$$

LEMMA 3.1. *If* $\varphi \in L^2(\mathbb{R}_+)$ *then* $\varphi \in L^2(m_q) \cap L^2(\mathbb{R}_+)$ *provided* $\operatorname{Re} p \geq 0$. *Conversely, if* $\varphi \in L^2(m_q)$ and $J\varphi = \pm\varphi$ *then* $\varphi \in L^2(\mathbb{R}_+)$.

PROOF. The first implication is immediate. The second follows from the asymptotic estimates on $J_p(t)$. □

Therefore, we shall study self-reciprocal functions in $L^2(\mathbb{R}_+)$. Moreover, by a change of variables the conditions (3.4) can be recast in the form that the function

$$\phi(t) = 2^{-q+1/2} t^{p+1/2} \varphi\left(\frac{t^2}{2}\right) = 2^{q-1/2} t^{-p+1/2} \psi\left(\frac{t^2}{2}\right) \tag{3.5}$$

satisfies $K\phi = \pm\phi$ where K is the symmetric version of the Hankel transform given by

$$K\phi(t) := \int_0^\infty J_p(st)\sqrt{st}\,\phi(s)\,ds. \tag{3.6}$$

For Re $p > -1$ the simplest solution of $K\phi = \phi$ is $\phi(t) = \sqrt{2}\,t^{-1/2}$, which corresponds to $\varphi(t) = t^{-q}$ and $\psi(t) = t^{q-1}$. This solution has already been considered above and does not belong to $L^2(\mathbb{R}_+)$. We refer to [Tit, Chap. 9] for an analysis of the equation $K\phi = \phi$ in $L^2(\mathbb{R}_+)$.

For $a > 0$, let $S_a : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ be given by $(S_a\varphi)(t) := a^q\varphi(at)$. Then $JS_a = S_{1/a}J$. In particular, since $Je^{-t} = e^{-t}$ we see that $a^q e^{-at}$ and $a^{-q}e^{-t/a}$ is a Hankel transform pair for all $a > 0$. Now, the operator $\tilde{J}$ is adjoint to J in the sense that $\langle\psi, J\varphi\rangle = \langle\tilde{J}\psi, \varphi\rangle$ with $\langle\phi_1, \phi_2\rangle := \int_0^\infty \phi_1(t)\overline{\phi}_2(t)\,dt$. Hence, the identity

$$\int_0^\infty a^{-q}e^{-t/a}\psi_1(t)\,dt = \int_0^\infty a^q e^{-at}\psi_2(t)\,dt, \qquad a > 0, \tag{3.7}$$

must hold whenever ψ_1, ψ_2 is a pair with respect to the Hankel transform $\tilde{J}$. If moreover $\tilde{\psi}_2$ is another Hankel transform of ψ_1 then $\int_0^\infty e^{-at}(\psi_2 - \tilde{\psi}_2)\,dt = 0$ for all $a > 0$ so that $\psi_2 = \tilde{\psi}_2$ almost everywhere. Therefore the identity (3.7) is a necessary and sufficient condition for ψ_1, ψ_2 to be a pair with respect to the Hankel transform $\tilde{J}$. Let moreover

$$\psi^*(s) := \int_0^\infty \psi(t)t^{s-1}\,dt \tag{3.8}$$

be the *Mellin transform* of ψ. If there are two constants $a < b$ such that $\psi(t) = O(t^{-a})$ as $t \to 0^+$ and $\psi(t) = O(t^{-b})$ as $t \to \infty$ then the integral (3.3) converges for s in the strip $a < \mathrm{Re}\, s < b$ and $\psi^*(s)$ is a holomorphic function in this strip.

REMARK 3.2. If $\mathcal{P}_q^+ f = \lambda f$ then one easily checks that

$$\lambda = 1 + \frac{f(1)}{f(0)}, \qquad \frac{\lambda}{2}(\lambda - 1) = \frac{f(2)}{f(0)}.$$

Thus, if $\lambda \neq 1$ we have $f(0) \neq 0$ and

$$f(x) \sim f(0)x^{-2q}, \qquad x \to \infty.$$

Therefore, if $\mathrm{Re}\, q > 0$ then the Mellin transform f^* is analytic in the strip $0 < \mathrm{Re}\, s < 2\,\mathrm{Re}\, q$, and in this region we have

$$(\mathcal{J}_q f)(x) = f(x) \;\Rightarrow\; f^*(s) = f^*(2q - s).$$

Now, taking the Mellin transform of both sides in (3.7) we obtain

$$\Gamma(1 - s)\psi_1^*(s) = \Gamma(s + p)\psi_2^*(1 - p - s).$$

Note that if $\psi = \chi_p \cdot \varphi$, then $\psi^*(s) = \varphi^*(s + p)$. Moreover, Mellin transforming (3.5) gives

$$\phi^*(s) = 2^{s/2-3/4}\varphi^*(s/2 + p/2 + 1/4) = 2^{s/2-3/4}\psi^*(s/2 - p/2 + 1/4).$$

Therefore, if we define the weighted transforms $\bar{\varphi}^*$, $\tilde{\psi}^*$ and $\hat{\phi}^*$ as

$$\bar{\varphi}^*(s) := \frac{\varphi^*(s)}{\Gamma(s)}, \quad \tilde{\psi}^*(s) := \frac{\psi^*(s)}{\Gamma(s+p)}, \quad \hat{\phi}^*(s) := 2^{p/2+1} \frac{\phi^*(s)}{\Gamma(s/2+p/2+1/4)},$$

and take into account that $1 + p - (s/2 + p/2 + 1/4) = (1 - s)/2 + p/2 + 1/4$, we have the following result:

PROPOSITION 3.3. *The functions $\varphi, \psi, \phi \in \mathrm{L}^2(\mathbb{R}_+)$, related to each other by* (3.5)*, are jointly self-reciprocal, i.e. $J\varphi = \pm\varphi$, $\tilde{J}\psi = \pm\psi$ and $K\phi = \pm\phi$, if and only if*

$$\bar{\varphi}^*(s) = \pm\bar{\varphi}^*(1 + p - s), \quad \tilde{\psi}^*(s) = \pm\tilde{\psi}^*(1 - p - s), \quad \hat{\phi}^*(s) = \pm\hat{\phi}^*(1 - s).$$

The sequences $h_n^\pm$ introduced in (2.26) were our first example of self-reciprocal functions in $\mathrm{L}^2(\mathbb{R}_+)$, in the sense that $Jh_n^\pm = \pm h_n^\pm$ for all $n \geq 0$. Even more interesting self-reciprocal functions are provided by the conjugate sequences $\varphi_n, \psi_n \in \mathrm{L}^2(\mathbb{R}_+)$, $n \geq 0$, defined for Re $p > -1$ by

$$\varphi_n(t) := \sqrt{\frac{2^{p+1} n!}{\Gamma(n+p+1)}} e^{-t} L_n^p(2t), \quad \psi_n(t) := (\chi_p \cdot \varphi_n)(t), \tag{3.9}$$

and satisfying the condition $\langle \varphi_n, \psi_m \rangle = \delta_{n,m}$. They are related to the sequences $h_n^\pm$ by (see [E, Vol. II, p. 192])

$$\varphi_n = (-1)^n \sqrt{\frac{2^{p+1} n!}{\Gamma(n+p+1)}} \sum_{m=0}^n \binom{n+p}{n-m} (-2)^m \left(\frac{h_m^+ + h_m^-}{2} \right).$$

Thus

$$J\varphi_n = (-1)^n \sqrt{\frac{2^{p+1} n!}{\Gamma(n+p+1)}} \sum_{m=0}^n \binom{n+p}{n-m} (-2)^m \left(\frac{h_m^+ - h_m^-}{2} \right),$$

which, compared to (2.10), yields

$$J\varphi_n = (-1)^n \varphi_n, \quad \tilde{J}\psi_n = (-1)^n \psi_n. \tag{3.10}$$

Note that

$$\mathcal{B}_q[\varphi_n](x) = (n+1)_p \frac{(1-x)^n}{(1+x)^{n+2q}} \tag{3.11}$$

so that $\mathcal{J}_q \mathcal{B}_q[\varphi_n] = (-1)^n \mathcal{B}_q[\varphi_n]$, as expected (cf. (2.28)).

Moreover,

$$\bar{\varphi}_n^*(s) = \frac{(p+1)_n}{n!} \, {}_2F_1(-n, s; p+1; 2), \tag{3.12}$$

which satisfies the functional equation of Proposition 3.3 because of Pfaff's identity ([AAR, Theorem 2.2.5]) which implies

$${}_2F_1(-n, b; c; 2) = (-1)^n \, {}_2F_1(-n, c - b; c; 2).$$

Finally, the orthonormal family $\{\phi_n\}$ of $\mathrm{L}^2(\mathbb{R}_+)$ given by

$$\phi_n(t) := \sqrt{\frac{2n!}{\Gamma(n+p+1)}} e^{-t^2/2} t^{p+1/2} L_n^p(t^2) \tag{3.13}$$

satisfies

$$K\phi_n = (-1)^n \phi_n, \quad n \geq 0. \tag{3.14}$$

Thus, the families $\varphi_n, \psi_n, \phi_n$ furnish *a complete characterization of self-reciprocal functions in* $\mathrm{L}^2(\mathbb{R}_+)$ *for the Hankel transforms* J, $\tilde{J}$, K.

REMARK 3.4. The functions ϕ_n are also solutions of the differential equation

$$\phi_n'' - \left(\frac{p^2 - 1/4}{t^2} + t^2 - 4n - 2p - 2\right)\phi_n = 0 \tag{3.15}$$

as one can check using, e.g., [E, Vol. II, p. 188]. More specifically, the second order differential operator H given by[1]

$$H := \frac{1}{2}\left(-\frac{d^2}{dt^2} + \frac{p^2 - 1/4}{t^2} + t^2\right) \tag{3.16}$$

has for real $p \geq 1$ a unique self-adjoint extension on $\mathrm{C}_0^\infty(\mathbb{R}_+)$ which has an integer-spaced spectrum so that

$$H\phi_n = (2n + p + 1)\phi_n, \quad n \geq 0. \tag{3.17}$$

For $-1 < p < 1$ there is more than one self-adjoint extension, one of which, however, still satisfies (3.17). Comparing (3.14) and (3.17) one may regard the unitary mapping K from $\mathrm{L}^2(\mathbb{R}_+)$ onto itself as a hyperdifferential operator of the form ($2q = p + 1$)

$$K = e^{i\pi q} \exp\left(-\frac{i\pi}{2} H\right) \tag{3.18}$$

and acting on a suitable class of analytic functions (see [Bar] and [Wo] for a discussion on this and related correspondences).

4. POLYNOMIAL EIGENFUNCTIONS OF $\mathcal{P}_q^{\pm}$ FOR $q = -k/2$

Although the eigenfunction $f^{(q)}(x)$ corresponding to the leading eigenvalue $\lambda(q)$ does not belong to the space $\mathcal{H}_q$ (see Remark 1.2), we shall see that explicit expressions for $\lambda(q)$ and $f^{(q)}(x)$ can be obtained when $q = -k/2$ with k a non-negative integer. Note that these values correspond exactly to the simple poles of $\Gamma(2q)$ and thus, by (2.8), to the q-values

[1] In quantum mechanics this corresponds to the Schrödinger operator for a two-dimensional isotropic harmonic potential (see [RS, Vol. II, p. 161]).

where the measure m_q has an infinite mass. On the other hand, for $q = -k/2$ the operators $\mathcal{P}_q^\pm$ take the form

$$\mathcal{P}_{-k/2}^\pm f(x) = (x+1)^k \left[f\left(\frac{x}{x+1}\right) \pm f\left(\frac{1}{x+1}\right)\right]$$

so that they leave invariant the vector space $\bigoplus_{n=0}^k \mathbb{C}x^n$ of polynomials of degree $\le k$. In particular, we expect $f^{(-k/2)}(x)$ to be a polynomial of degree k with real coefficients.

To warm up, a direct calculation yields

$$\begin{aligned}
f^{(0)}(x) &= 1, & \lambda(0) &= 2,\\
f^{(-1/2)}(x) &= x+1, & \lambda(-1/2) &= 3,\\
f^{(-1)}(x) &= x^2 + \frac{\sqrt{17}-1}{2}x + 1, & \lambda(-1) &= \frac{5+\sqrt{17}}{2},\\
f^{(-3/2)}(x) &= x^3 + 2x^2 + 2x + 1, & \lambda(-3/2) &= 7,\\
f^{(-2)}(x) &= x^4 + \frac{\sqrt{113}-1}{4}x^3 + 3x^2 + \frac{\sqrt{113}-1}{4}x + 1, & \lambda(-2) &= \frac{11+\sqrt{113}}{2}.
\end{aligned}$$

To say more we first need the following result.

LEMMA 4.1. *The $(k+1)\times(k+1)$ real positive matrix M_k defined as*

$$M_k(i,j) := \begin{cases} \binom{k-i}{j-i} & \text{if } i<j, \\ 2 & \text{if } i=j, \\ \binom{i}{j} & \text{if } i>j, \end{cases} \qquad (0 \le i, j \le k)$$

has the following properties:

(i) *the symmetry $M_k(i,j) = M_k(k-i, k-j)$ holds for all $0 \le i, j \le k$;*
(ii) *the sum S_i of the entries in row i equals $2^i + 2^{k-i}$;*
(iii) *the sum R_j of the entries in column j equals $\binom{k+2}{j+1}$;*
(iv) *if $M_k\Phi = \lambda\Phi$ with $\mathbb{C}^{k+1} \ni \Phi := (b_0, b_1, \dots, b_k)^T$ and $\lambda \neq 0$ then Φ is either a palindrome or a skew-palindrome, i.e. $b_i = \pm b_{k-i}$ for $0 \le i \le k$;*
(v) *$\sigma(M_k) \subset \mathbb{R}$ for all $k \in \mathbb{N}\cup\{0\}$;*
(vi) *$1 \in \sigma(M_k)$ for all $k \in \mathbb{N}$.*

PROOF. (i)–(iii) follow by direct computation. To prove (iv) we write the eigenvalue equation componentwise:

$$\lambda b_i = \sum_{j=0}^k M(i,j) b_j \qquad (0 \le i \le k),$$

which yields, using the symmetry (i),

$$\lambda b_{k-i} = \sum_{j=0}^k M(k-i,j)b_j = \sum_{j=0}^k M(k-i,k-j)b_{k-j} = \sum_{j=0}^k M(i,j)b_{k-j}$$

so that $M_k\Phi = \lambda\Phi$ if and only if $M_k\Phi' = \lambda\Phi'$ with $\Phi' := (b_k, b_{k-1}, \dots, b_0)^T$. If λ is (geometrically) simple then $\Phi = \pm\Phi'$ because $\|\Phi\| = \|\Phi'\|$ where $\| \ \|$ is the euclidean norm in $\mathbb{C}^{k+1}$. In other words, $\Phi = \pm\Phi'$ is a necessary and sufficient condition for Φ and Φ' to be linearly dependent. Assume now that $M_k\Phi = \lambda\Phi$ with λ of geometric (and thus algebraic) multiplicity greater than 1. If Φ and Φ' are linearly dependent then we are done. Suppose they are not. Then the vectors $\Psi_\pm := \Phi \pm \Phi'$ should also be two linearly independent eigenvectors. But this is impossible because $\Psi'_\pm = \pm\Psi_\pm$. This concludes the proof of (iv).

As for (v), we observe that M_1 is symmetric and for each $k \geq 2$ it is not difficult to realise that one can construct recursively a positive symmetric $(k+1) \times (k+1)$ matrix N_k such that the product $M_k N_k$ is symmetric as well. For instance for $k = 4$ one gets

$$M_4 = \begin{pmatrix} 2 & 4 & 6 & 4 & 1 \\ 1 & 2 & 3 & 3 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 3 & 3 & 2 & 1 \\ 1 & 4 & 6 & 4 & 2 \end{pmatrix}, \quad N_4 = \begin{pmatrix} 1 & 13 & 1 & 7 & 18 \\ 13 & 1 & 3 & 2 & 13 \\ 1 & 3 & 3 & 7 & 1 \\ 7 & 2 & 7 & 1 & 7 \\ 18 & 13 & 1 & 7 & 1 \end{pmatrix}.$$

Then apply Theorem 1 in [DH]. Finally, the vector $\Phi = (1, 0, \dots, 0, -1)^T$ always satisfies $M_k\Phi = \Phi$, which yields (vi). □

REMARK 4.2. If one defines a pseudo-scalar product of $\Phi = (b_0, b_1, \dots, b_k)^T$ and $\Psi = (c_0, c_1, \dots, c_k)^T$ as $\langle\Phi, \Psi\rangle := \sum_{i=0}^k b_i \bar{c}_{k-i}$ then the symmetry stated in (i) amounts to $\langle M_k\Phi, \Psi\rangle = \langle\Phi, M_k^T\Psi\rangle$. Moreover, (iv) implies that if $M_k\Phi = \lambda\Phi$ with $\lambda \neq 0$ then $\langle\Phi, \Phi\rangle = \pm\|\Phi\|^2$.

THEOREM 4.3. *Let $q = -k/2$, $k \geq 1$. The polynomial*

$$f(x) = \sum_{i=0}^{k} \binom{k}{i} b_i x^i \tag{4.1}$$

satisfies $\mathcal{P}_q^\pm f = \lambda f$ with $\lambda \neq 0$ if and only if the vector $\Phi = (b_0, b_1, \dots, b_k)^T$ satisfies $M_k\Phi = \lambda\Phi$ and is either a palindrome (if $\mathcal{P}_q^+ f = \lambda f$) or a skew-palindrome (if $\mathcal{P}_q^- f = \lambda f$).

COROLLARY 4.4. *The eigenvector corresponding to the simple positive maximal eigenvalue $\lambda(-k/2)$ of M_k is always palindromic and we have the bounds*

$$s - 1 + h \leq \lambda \leq S - 1 + 1/g$$

where

$$S := \max_i S_i = 2^k + 1, \quad s := \min_i S_i = \begin{cases} 2^{k/2+1} + 2^{k/2-1}, & k \text{ even}, \\ 2^{k+1/2} + 2^{k-1/2}, & k \text{ odd}, \end{cases}$$

and

$$h = \frac{-s + 2 + \sqrt{s^2 + 4(S-s)}}{2}, \quad g = \frac{S - 2 + \sqrt{S^2 - 4(S-s)}}{2(s-1)}.$$

PROOF. Put together the above and [MM, p. 155, eq. (9)]. □

PROOF OF THEOREM 4.3. Set

$$f(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0. \tag{4.2}$$

The condition $\mathcal{J}_q f = \pm f$ implies that the sequence of coefficients a_i is either a palindrome or a skew-palindrome, i.e. $a_i = \pm a_{k-i}$ $(0 \le i \le k)$. Inserting the function $f(x)$ written above into (2.24) with $q = -k/2$ and $k \ge 1$ we get

$$\begin{aligned} \lambda \sum_{i=0}^{k} a_i x^i &= \sum_{i=0}^{k} a_i \sum_{j=0}^{i} \binom{i}{j} (x^j \pm x^{k-j}) \\ &= \sum_{j=0}^{k} \left[\sum_{l=j}^{k} \binom{l}{j} a_l \right] (x^j \pm x^{k-j}) \\ &= \sum_{i=0}^{k} \left[\sum_{l=i}^{k} \binom{l}{i} a_l \pm \sum_{l=k-i}^{k} \binom{l}{k-i} a_l \right] x^i, \end{aligned} \tag{4.3}$$

which in both cases yields

$$\lambda a_i = \sum_{l=0}^{i} \binom{k-l}{k-i} a_l + \sum_{l=i}^{k} \binom{l}{i} a_l \qquad (0 \le i \le k). \tag{4.4}$$

Defining new coefficients b_i so that

$$a_i = \binom{k}{i} b_i \qquad (0 \le i \le k) \tag{4.5}$$

and using the identities

$$\binom{k-l}{k-i}\binom{k}{l} = \binom{k}{i}\binom{i}{l} \quad \text{and} \quad \binom{l}{i}\binom{k}{l} = \binom{k}{i}\binom{k-i}{l-i}$$

we see that the above recursion becomes

$$\lambda b_i = \sum_{l=0}^{i} \binom{i}{l} b_l + \sum_{l=i}^{k} \binom{k-i}{l-i} b_l \qquad (0 \le i \le k), \tag{4.6}$$

and the proof is complete. □

EXAMPLE 4.5. For $k = 4$ we find

$$\mathrm{sp}(M_4) = \left\{ \frac{11 + \sqrt{113}}{2}, 1, \frac{11 - \sqrt{113}}{2}, -1, -1 \right\}$$

and the corresponding eigenvectors are

$$\Phi_1 = \begin{pmatrix} 1 \\ \frac{\sqrt{113}-1}{16} \\ 1/2 \\ \frac{\sqrt{113}-1}{16} \\ 1 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} 1 \\ \frac{\sqrt{113}+1}{16} \\ 1/2 \\ \frac{\sqrt{113}+1}{16} \\ 1 \end{pmatrix}$$

$$\Phi_4 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 3 \end{pmatrix}, \quad \Phi_5 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Therefore the spectrum of M_4 yields three eigenfunctions for $\mathcal{P}^+_{-2}$:

$$h_1(x) = x^4 + \frac{\sqrt{113}-1}{4}x^3 + 3x^2 + \frac{\sqrt{113}-1}{4}x + 1,$$
$$h_3(x) = x^4 + \frac{\sqrt{113}+1}{4}x^3 + 3x^2 + \frac{\sqrt{113}+1}{4}x + 1,$$
$$h_4(x) = -3x^4 + 12x^2 - 3$$

and two eigenfunctions for $\mathcal{P}^-_{-2}$:

$$h_2(x) = x^4 - 1, \qquad h_5(x) = 4x(1-x^2).$$

REMARK 4.6. Eigenvectors of M_k corresponding to the eigenvalue 1 are related to the period functions for the modular group (see [CM]). In particular, for $k \in \mathbb{N}$ the eigenvectors $(1, 0, \dots, 0, -1)^T$ correspond to the fixed functions $x^k - 1$ of $\mathcal{P}^-_{-k/2}$ which yield the even part of the period functions corresponding to holomorphic Eisenstein forms of weight $k+2$. The odd parts are computed below in Proposition 4.7. Other linearly independent (skew-palindromic and palindromic) eigenvectors with eigenvalue 1 are expected for $k \geq 10$, as they are related to (the even and odd parts of) holomorphic cusp forms [A].

For the sake of completeness we end with the following result, a version of which is contained in [CM].

PROPOSITION 4.7. *Let B_m denote the m-th Bernoulli number. For $k \in \mathbb{N} \cup \{0\}$ the function $f_k(x) \in \bigoplus_{n=-1}^{k+1} \mathbb{C}x^n$ given by*

$$f_k(x) := \frac{\zeta(-k)}{2}(1+x^k) + (-1)^k k! \sum_{-1 \leq n \leq k+1} \frac{B_{n+1}B_{k+1-n}}{(n+1)!(k+1-n)!} x^n$$

satisfies $\mathcal{P}^+_{-k/2} f_k = f_k$ for k even and $f_k \equiv 0$ for k odd.

Two examples are

$$f_0(x) = \frac{1}{12}\left[x + \frac{1}{x} - 3\right], \qquad f_2(x) = \frac{1}{360}\left[5x - \left(x^3 + \frac{1}{x}\right)\right].$$

Note that for $k \geq 1$, the odd parts of the period functions mentioned in Remark 4.6 can be expressed as $\frac{(-1)^k}{k!}(f_k(x) - \frac{\zeta(-k)}{2}(1 + x^k))$ [Za1].

PROOF OF PROPOSITION 4.7. Consider the function $\psi_q(x)$ defined for $\operatorname{Re} q > 1$ by

$$\psi_q(x) = \frac{\zeta(2q)}{2}(1 + x^{-2q}) + \sum_{n,m\geq 1}(nx + m)^{-2q}.$$

It is shown in [Za2] that the function $\psi_q(x)$ has an analytic extension into the complex q-plane with a simple pole at $q = 1$, and the analytic continuation satisfies (2.24) with the $+$ sign and $\lambda = 1$ for all $q \in \mathbb{C} \setminus \{1\}$. Note that if $\operatorname{Re} q > 1$ then $\psi_q(\infty) = \frac{1}{2}\zeta(2q)$.

The proof then amounts to showing that for $q = -k/2$ the analytic extension of the function ψ_q is precisely f_k. This is achieved using standard Mellin transform techniques: start from the identity

$$\begin{aligned}\sum_{n,m\geq 1}(nx + m)^{-2q} &= \frac{1}{\Gamma(2q)}\int_0^\infty \sum_{n,m\geq 1} e^{-t(nx+m)}t^{2q-1}\,dt \\ &= \frac{1}{\Gamma(2q)}\int_0^\infty \frac{t^{2q-1}}{(e^t - 1)(e^{tx} - 1)}\,dt.\end{aligned}$$

Recalling that

$$\frac{1}{e^t - 1} = \sum_{r=-1}^{\infty}\frac{B_{r+1}}{(r+1)!}t^r = \frac{1}{t} - \frac{1}{2} + \sum_{l=1}^{\infty}\frac{B_{2l}}{(2l)!}t^{2l-1}$$

we get

$$\sum_{n,m\geq 1}(nx + m)^{-2q} = \frac{1}{\Gamma(2q)}\sum_{k=-2}^{\infty}\int_0^\infty c_k(x)t^{k+2q-1}\,dt$$

with $c_{-2} = 1/x$, $c_{-1} = -\frac{1}{2}(1 + 1/x)$, and for $k \geq 0$,

$$c_k = \sum_{-1\leq n\leq k+1}\frac{B_{n+1}B_{k+1-n}}{(n+1)!(k+1-n)!}x^n.$$

Now $\Gamma(2q)$ has simple poles at $2q = -k$ with $k = 0, 1, 2, \ldots$, with residues $(-1)^k/k!$. On the other hand, the above integral has simple poles at $2q = -k$ with $k = -2, -1, 0, 1, \ldots$, the residues being c_k. Therefore the analytic continuation of $\sum_{n,m\geq 1}(nx + m)^{-2q}$ has only two simple poles at $q = 1$ and $q = 1/2$ and the claimed expression for f_k follows by taking the limit $2q \to -k$ with $k \in \mathbb{N} \cup \{0\}$. The last assertion is a consequence of the identity $\zeta(-k) = -B_{k+1}/(k + 1)$ for k odd. □

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