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Ordinary differential equations. — *Homoclinic solutions to invariant tori in a center manifold*, by VITTORIO COTI ZELATI and MARTA MACRÌ, communicated on 14 December 2007.

ABSTRACT. — We consider the Lagrangian

$$L(y_1, \dot{y}_1, y_2, \dot{y}_2, q, \dot{q}) = \frac{1}{2}(\dot{y}_1^2 - \omega_1^2 y_1^2) + \frac{1}{2}(\dot{y}_2^2 - \omega_2^2 y_2^2) + \frac{1}{2}\dot{q}^2 + (1 + \delta(y_1, y_2))V(q),$$

where V is non-negative, periodic in q and such that $V(0) = V'(0) = 0$. We prove, using critical point theory, the existence of infinitely many solutions of the corresponding Euler–Lagrange equations which are asymptotic, as $t \to \pm\infty$, to invariant tori in the center manifold of the origin, that is, to solutions of the form $q(t) = 0$, $y_1(t) = R\cos(\omega_1 t + \varphi_1)$, $y_2(t) = R\cos(\omega_2 t + \varphi_2)$.

KEY WORDS: Heteroclinic orbits; critical point theory; invariant tori; center manifold.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35B40, 35J10, 35J60, 58C15.

1. INTRODUCTION

The study of solutions asymptotic to invariant manifolds is important in order to understand the global dynamics of Hamiltonian systems. It is well known that they can indicate the existence of a complicated—even chaotic—behavior for the system under consideration.

Global variational methods have been employed by many authors to prove existence of solutions homoclinic or heteroclinic to hyperbolic stationary points (see [7, 9, 18, 21]), and to prove chaotic behavior for time-dependent systems (always having a hyperbolic stationary point); see [20] and [12]. The same kind of techniques have also been employed to study existence of solutions asymptotic to periodic orbits and to more general invariant manifolds having some kind of minimizing property (see [19, 8]).

More recently Patrick Bernard [1, 2] has considered a class of Hamiltonian systems having a saddle-center stationary point and has proved existence of solutions homoclinic to periodic orbits in the (global) center manifold. Motivated by such papers we have further investigated the situation in [10, 11]. All these papers consider autonomous Lagrangian systems of the form

$$L(x, \dot{x}, q, \dot{q}) = \frac{1}{2}(\dot{x}^2 - \omega^2 x^2) + \frac{1}{2}\dot{q}^2 + V(x, q),$$

where $x \in \mathbb{R}$, $q \in \mathbb{T}$, $V(x, q) \geq V(x, 0) = 0$ for all x, q (actually, Bernard considers more general Hamiltonian systems, and his results apply also to $q \in \mathbb{T}^k$). In this case the center manifold of the stationary point is 2-dimensional and is foliated by periodic solutions.

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In this paper we extend some of the above results to cover a class of Lagrangian systems having a 4-dimensional center manifold by considering Lagrangians of the form

$$L(y_1, \dot{y}_1, y_2, \dot{y}_2, q, \dot{q}) = \frac{1}{2}(\dot{y}_1^2 - \omega_1^2 y_1^2) + \frac{1}{2}(\dot{y}_2^2 - \omega_2^2 y_2^2) + \frac{1}{2}\dot{q}^2 + (1 + \delta(y_1, y_2))V(q)$$

with $y_1, y_2 \in \mathbb{R}$, $q \in \mathbb{T} = \mathbb{R}/[0, 2\pi]$. We look for solutions of the corresponding Euler–Lagrange equations

$$\begin{cases} \ddot{q} = (1 + \delta(y_1, y_2))V'(q), \\ \ddot{y}_1 + \omega_1^2 y_1 = \dfrac{\partial}{\partial y_1}\delta(y_1, y_2)V(q), \\ \ddot{y}_2 + \omega_2^2 y_2 = \dfrac{\partial}{\partial y_2}\delta(y_1, y_2)V(q). \end{cases} \tag{1.1}$$

Note that the Lagrangian is autonomous and hence the total energy

$$\frac{1}{2}(\dot{y}_1^2 + \omega_1^2 y_1^2) + \frac{1}{2}(\dot{y}_2^2 + \omega_2^2 y_2^2) + \frac{1}{2}\dot{q}^2 - (1 + \delta(y_1, y_2))V(q) = E$$

is conserved along any solution.

Since we are assuming that V has a strict global minimum at $q = 0$ and it is periodic in $q \in \mathbb{R}$, such a system admits, for all $R_1, R_2 \geq 0$ and $\varphi_1, \varphi_2 \in \mathbb{R}$, the solutions

$$\begin{cases} q(t) \equiv 0, \\ y_1(t) = R_1 \cos(\omega_1 t + \varphi_1), \\ y_2(t) = R_2 \cos(\omega_2 t + \varphi_2), \end{cases}$$

having total energy $E = (\omega_1^2 R_1^2 + \omega_2^2 R_2^2)/2$. The set

$$\begin{aligned} \mathcal{T}_{R_1,R_2} = \{(&y_1 = R_1 \cos(\varphi_1),\ \dot{y}_1 = -\omega_1 R_1 \sin(\varphi_1), \\ &y_2 = R_2 \cos(\varphi_2),\ \dot{y}_2 = -\omega_2 R_2 \sin(\varphi_2),\ q = 0,\ \dot{q} = 0) \mid (\varphi_1, \varphi_2) \in \mathbb{T}^2\} \end{aligned}$$

is an invariant torus for the system (1.1).

We look for solutions asymptotic to such tori, that is, solutions $y_1(t), y_2(t), q(t)$ such that

$$\lim_{t \to +\infty} \text{dist}((y(t), \dot{y}(t), q(t), \dot{q}(t)), \mathcal{T}_{R_{1+},R_{2+}}) = 0,$$
$$\lim_{t \to -\infty} \text{dist}((y(t), \dot{y}(t), q(t), \dot{q}(t)), \mathcal{T}_{R_{1-},R_{2-}}) = 0.$$

By energy conservation,

$$E = \omega_1^2 R_{1+}^2 + \omega_2^2 R_{2+}^2 = \omega_1^2 R_{1-}^2 + \omega_2^2 R_{2-}^2. \tag{1.2}$$

This kind of problem has been studied using perturbative methods by many authors mainly in the case of a 2-dimensional center manifold (see [16, 17, 15, 13], and the more recent [4, 5, 23, 22, 14, 6]). In these papers it is proved that there exist many solutions homoclinic to invariant tori of energy $E > 0$ and "not too small" (one does not expect to find solutions homoclinic to the stationary point $P_0 = (y_1 = 0, \dot{y}_1 = 0, y_2 = 0, \dot{y}_2 = 0, q = 0, \dot{q} = 0)$, which has one-dimensional stable and unstable manifold).

In this paper we obtain existence of infinitely many solutions, homoclinic to the invariant tori, under different kinds of assumptions on δ and ω_i.

We remark that the condition that a solution is asymptotic to $\mathcal{T}_{R_{1\pm},R_{2\pm}}$ as $t \to \pm\infty$ can also be formulated by saying that $(y_1(t), y_2(t), q(t))$ is a solution of (1.1) which satisfies, for some $f_{1\pm}, f_{2\pm} \in [0, 2\pi)$ and $k \in \mathbb{Z} \setminus \{0\}$,

$$\begin{aligned} &\lim_{t\to\pm\infty} |y_1(t) - R_{1\pm}\cos(\omega_1 t + f_{1\pm})| = 0, \\ &\lim_{t\to\pm\infty} |y_2(t) - R_{2\pm}\cos(\omega_2 t + f_{2\pm})| = 0, \\ &\lim_{t\to-\infty} q(t) = 0, \qquad \lim_{t\to+\infty} q(t) = 2k\pi. \end{aligned} \tag{1.3}$$

Throughout this paper, we will assume that V and $\delta \in C^2(\mathbb{R})$ are such that

(V1) $V(q + 2\pi) = V(q)$ for all $q \in \mathbb{R}$;
(V2) $0 = V(0) < V(q)$ for all $q \in \mathbb{R} \setminus 2\pi\mathbb{Z}$;
(V3) $V''(0) = \mu > 0$;
(V4) $V'(q)q > 0$ for all $q \in [-\bar{\eta}, \bar{\eta}]$, $q \neq 0$;

(δ1) $-1 < \underline{\delta} \le \delta(y_1, y_2) \le \bar{\delta}$ for all $(y_1, y_2) \in \mathbb{R}^2$;
(δ2) $\|\nabla\delta(y_1, y_2)\| \le C$ for all $(y_1, y_2) \in \mathbb{R}^2$ and for some positive constant C;
(δ3) $|\langle\nabla\delta(y_1, y_2), (y_1, y_2)\rangle| \le 2\alpha$ for all $(y_1, y_2) \in \mathbb{R}^2$ where $1 + \underline{\delta} - \alpha > 0$.

We will often use the notation

$$V_\eta = \min\{V(s) : s \in [\eta, 2\pi - \eta]\} > 0, \qquad \eta > 0. \tag{1.4}$$

REMARK 1.5. Let us point out, for future reference, that (V3) implies that there is an $\eta_0 \in (0, \bar{\eta}/2)$ such that

$$\mu/2 \le V''(q) \le 2\mu \quad \text{for all } |q| \le \eta_0. \tag{1.6}$$

REMARK 1.7. The above assumptions on V and δ are satisfied, for example, if

$$V(q) = 1 - \cos q, \qquad \delta(y_1, y_2) = \delta_\infty \arctan \lambda^2(y_1 + y_2) \tag{1.8}$$

provided $0 < \delta_\infty < 4/(2\pi + \lambda^2)$.

REMARK 1.9. If $\delta(y_1, y_2) \equiv \delta_0$ is a constant, then under assumptions (V1)–(V2) there is a solution $q_0(t)$ of $\ddot{q} = (1+\delta_0)V'(q)$ homoclinic to 0 (see, for example, [7, 18]) and hence $(R_1\cos(\omega_1 t+f_1), R_2\cos(\omega_2 t+f_2), q_0(t))$ is a solution of our problem for all $R_1, R_2 \ge 0$, $f_1, f_2 \in [0, 2\pi)$.

A first result is the existence of at least one solution asymptotic to an invariant torus $\mathcal{T}_{R_{1-},R_{2-}}$ as $t \to -\infty$ and to $\mathcal{T}_{R_{1+},R_{2+}}$ as $t \to +\infty$.

THEOREM 1.10. *Assume V and δ satisfy* (V1)–(V4)*,* (δ1)–(δ3) *and*

(δ4) $\bar{\delta} + 2(\alpha - \underline{\delta}) < 1$.

Then there is a solution of the system (1.1) *satisfying* (1.3) *with $k = 1$ for some $R_{1\pm}, R_{2\pm} \in [0, +\infty)$ satisfying* (1.2)*, and for some $f_{1\pm}, f_{2\pm} \in [0, 2\pi]$.*

Note that we cannot prescribe, in the above theorem, the tori to which the solution we find is asymptotic, and not even its energy. We also observe that the above system should have a lot of solutions like the one we find.

In the next two theorems, we give, under different additional assumptions, a more precise result: the existence of infinitely many homoclinic solutions, that is, of solutions asymptotic as time goes to $\pm\infty$ to the same torus $\mathcal{T}_{R_1,R_2}$ (different solutions will in general be homoclinic to different tori). The different solutions are characterized, as in [10], by different "phase shifts" between $t = -\infty$ and $t = +\infty$ (φ_i in the statement of the theorem).

THEOREM 1.11. *Assume V and δ satisfy* (V1)–(V4)*,* (δ1)–(δ3) *and*

$$(\delta 4') \qquad \bar{\delta} - \underline{\delta} + \alpha \le \frac{\bar{\eta}(1 + \underline{\delta} - \alpha)^{3/2}}{2\pi^2 + (1 + \bar{\delta})\|V\|_\infty} \sqrt{\frac{V_{\bar{\eta}/2}}{2}}$$

Assume moreover that

(E) *system* (1.1) *has no zero energy solutions satisfying* $\lim_{t\to-\infty} q(t) = 0$ *and* $\lim_{t\to+\infty} q(t) = 2\pi$.

Then for any $\varphi_1 \in (0, 2\pi)$ and

(ω1) *any $\varphi_2 \in (0, 2\pi)$ if $\omega_2/\omega_1 \notin \mathbb{Q}$,*
(ω2) *$\varphi_2 = j\varphi_1$ if $\omega_2/\omega_1 = j \in \mathbb{Q}$,*

there exist R_1 and R_2 and a solution $(y_1(t), y_2(t), q(t))$ of (1.1)*, satisfying* (1.3) *with $k = 1$, $R_{1\pm} = R_1$, $R_{2\pm} = R_2$, $\omega_1^2 R_1^2 + \omega_2^2 R_2^2 > 0$, $\varphi_i \equiv f_{i-} - f_{i+} \mod 2\pi$. Furthermore, $q(t) \in [0, 2\pi]$ for all t.*

REMARK 1.12. Assumption (δ4′) has already been used in [10] and it is satisfied if V and δ are of the form (1.8) with $\delta_\infty < 0.02$ and $\lambda = 1$. We also remark that (δ4′) implies (δ4).

Assumption (E) is used in order to prove the existence of infinitely many solutions. If it is violated, we obtain a solution homoclinic to the stationary point $P_0 = (y_1 = 0, \dot{y}_1 = 0, y_2 = 0, \dot{y}_2 = 0, q = 0, \dot{q} = 0)$. As already remarked above, this should not be the case in general (see [3]).

The next theorem shows the same result under a different set of assumptions. In order to state this result, let us introduce the notation

$$(1.13) \qquad \begin{aligned} &\bar{c} = 2\pi^2 + (1 + \bar{\delta})\|V\|_\infty, \quad K = 1 + \underline{\delta} - \alpha, \quad \bar{K} = \max\{\bar{c}/K, \bar{c}\} > 1, \\ &\mathcal{V} = \frac{\bar{\eta}}{12}\sqrt{\frac{1 + \delta(0,0)}{2}V_{\bar{\eta}/2}}, \quad C_{\varphi_i} = 2 + \left|\cot\frac{\varphi_i}{2}\right| \end{aligned}$$

THEOREM 1.14. *Assume V and δ satisfy* (V1)–(V4), (δ1)–(δ3), (E). *Fix any $\varphi_1 \in (0, 2\pi)$ and φ_2 as in* (ω1)–(ω2) *and assume that*

$$C_{\varphi_i} \frac{\|\nabla\delta\|_\infty \bar{K}}{\omega_i} \max\{1, \|\nabla\delta\|_\infty \bar{K}\} < \frac{\mathcal{V}}{2\bar{K}} \quad \forall i = 1, 2. \tag{1.15}$$

Then there exist R_1 and R_2 and a solution $(y_1(t), y_2(t), q(t))$ of (1.1) *satisfying* (1.3) *with $k = 1$, $R_{1\pm} = R_1$, $R_{2\pm} = R_2$, $\omega_1^2 R_1^2 + \omega_2^2 R_2^2 > 0$, $\varphi_i \equiv f_{i-} - f_{i+} \mod 2\pi$. Furthermore, $q(t) \in [0, 2\pi]$ for all t.*

REMARK 1.16. The above theorem states that one can find homoclinic solutions for all "phase shifts" uniformly far from 0 and 2π (so that C_{φ_i} is uniformly bounded) provided ω_i are large enough or δ is small enough.

Solutions of our problem will be found using variational methods as limits as $T \to +\infty$ of solutions of the following boundary value problem:

$$\text{(PT)} \qquad \begin{cases} \ddot{q} = (1 + \delta(y_1, y_2))V'(q), \\ \ddot{y}_1 + \omega_1^2 y_1 = \dfrac{\partial}{\partial y_1}\delta(y_1, y_2)V(q), \\ \ddot{y}_2 + \omega_2^2 y_2 = \dfrac{\partial}{\partial y_2}\delta(y_1, y_2)V(q), \\ q(0) = 0, \quad q(T) = 2\pi, \\ y_1(0) = y_1(T), \quad \dot{y}_1(0) = \dot{y}_1(T), \\ y_2(0) = y_2(T), \quad \dot{y}_2(0) = \dot{y}_2(T). \end{cases}$$

Existence of a solution for this boundary value problem will be proved using critical point theory, in particular a min-max procedure similar to the one introduced by Bernard in [2] and then used in [10, 11]. Some of the proofs in Sections 2 and 3 are closely related to those of [10, 11].

2. PRELIMINARIES

We first remark that the limit conditions

$$\begin{aligned} &\lim_{t\to-\infty} |y_i(t) - R_{i-}\cos(\omega_i t + f_{i-})| = 0, \\ &\lim_{t\to+\infty} |y_i(t) - R_{i+}\cos(\omega_i t + f_{i+})| = 0 \end{aligned} \tag{2.1}$$

for $i = 1, 2$ are satisfied by all solutions of

$$\ddot{y}_i + \omega_i^2 y_i = \frac{\partial\delta}{\partial y_i}(y_1, y_2)V(q) \tag{2.2}$$

such that $\int_{\mathbb{R}} V(q(t))\,dt < +\infty$. Indeed, all solutions of (2.2) can be expressed, for suitable $R_{i\pm}$, $f_{i\pm}$, $i = 1, 2$, as

$$y_i(t) = \frac{1}{\omega_i}\int_{-\infty}^{t} \frac{\partial\delta}{\partial y_i}(y_1, y_2)V(q)\sin\omega_i(t - s)\,ds + R_{i-}\cos(\omega_i t + f_{i-}) \tag{2.3}$$

or

$$y_i(t) = -\frac{1}{\omega_i}\int_t^{\infty} \frac{\partial\delta}{\partial y_i}(y_1, y_2)V(q)\sin\omega_i(t-s)\,ds + R_{i+}\cos(\omega_i t + f_{i+}), \tag{2.4}$$

and passing to the limit as $t \to \pm\infty$ in these expressions we obtain (2.1).

Now we give some preliminary estimates on the solutions of the boundary value problem (PT). In the following we will often use the notation

$$Q_i(y) = \int_0^T (\dot{y}^2 - \omega_i^2 y^2)\,ds. \tag{2.5}$$

LEMMA 2.6. *Assume that $T \neq 2\pi N/\omega_i$ for all N and $i = 1, 2$ and that y_i is a solution of*

$$\begin{cases} \ddot{y}_i + \omega_i^2 y_i = \dfrac{\partial}{\partial y_i}\delta(y_1, y_2)V(q), \\ y_i(0) - y_i(T) = \dot{y}_i(0) - \dot{y}_i(T) = 0. \end{cases} \tag{2.7}$$

Then for $i = 1, 2$,

$$\begin{aligned}
\|y_i\|_\infty &\le \frac{1}{\omega_i}\left(2 + \left|\cot\frac{\omega_i T}{2}\right|\right)\|\nabla\delta\|_\infty \int_0^T V(q(t))\,dt, \\
\|\dot{y}_i\|_\infty &\le \left(2 + \left|\cot\frac{\omega_i T}{2}\right|\right)\|\nabla\delta\|_\infty \int_0^T V(q(t))\,dt, \\
|Q_i(y_i)| &\le \frac{1}{\omega_i}\left(2 + \left|\cot\frac{\omega_i T}{2}\right|\right)\|\nabla\delta\|_\infty^2 \left(\int_0^T V(q(t))\,dt\right)^2, \\
\left|\frac{1}{2}Q_1(y_1) + \frac{1}{2}Q_2(y_2)\right| &\le \alpha \int_0^T V(q(t))\,dt.
\end{aligned}$$

PROOF. An easy calculation shows that for all $T \neq 2\pi N/\omega_i$ the solution of (2.7) is given by

$$\begin{aligned}
y_i(t) = &\frac{1}{\omega_i}\int_0^t \frac{\partial}{\partial y_i}\delta(y_1, y_2)V(q)\sin\omega_i(t-s)\,ds \\
&+ \left(\frac{H_i}{2} + \frac{L_i}{2}\cot\frac{\omega_i T}{2}\right)\cos\omega_i t + \left(\frac{L_i}{2} - \frac{H_i}{2}\cot\frac{\omega_i T}{2}\right)\sin\omega_i t
\end{aligned}$$

where

$$\begin{aligned}
H_i &= \frac{1}{\omega_i}\int_0^T \frac{\partial}{\partial y_i}\delta(y_1, y_2)V(q)\sin\omega_i(T-s)\,ds, \\
L_i &= \frac{1}{\omega_i}\int_0^T \frac{\partial}{\partial y_i}\delta(y_1, y_2)V(q)\cos\omega_i(T-s)\,ds.
\end{aligned}$$

We first observe that

$$|H_i| \leq \frac{1}{\omega_i}\|\nabla\delta\|_\infty \int_0^T V(q), \qquad |L_i| \leq \frac{1}{\omega_i}\|\nabla\delta\|_\infty \int_0^T V(q).$$

Then we have the following estimates:

$$\begin{aligned}\|y_i\|_\infty &\leq \frac{1}{\omega_i}\|\nabla\delta\|_\infty \int_0^t V(q) + \left|\frac{H_i}{2}\right| + \left|\frac{L_i}{2}\right| + \left(\left|\frac{L_i}{2}\right| + \left|\frac{H_i}{2}\right|\right)\left|\cot\frac{\omega_i T}{2}\right| \\ &\leq \frac{1}{\omega_i}\|\nabla\delta\|_\infty \int_0^T V(q) + \left(1 + \left|\cot\frac{\omega_i T}{2}\right|\right)\frac{1}{\omega_i}\|\nabla\delta\|_\infty \int_0^T V(q) \\ &= \left(2 + \left|\cot\frac{\omega_i T}{2}\right|\right)\frac{1}{\omega_i}\|\nabla\delta\|_\infty \int_0^T V(q).\end{aligned}$$

Similarly we can estimate $\|\dot{y}_i\|_\infty$. Finally, an integration by parts shows that

$$Q_i(y_i) = \int_0^T -\frac{\partial}{\partial y_i}\delta(y_1, y_2) y_i(t) V(q(t))\, dt \qquad \text{for all } i = 1, 2,$$

so that

$$\begin{aligned}\left|\int_0^T (\dot{y}_i^2 - \omega_i^2 y_i^2)\, dt\right| &\leq \|y_i\|_\infty \int_0^T \left|\frac{\partial}{\partial y_i}\delta(y_1, y_2)\right| V(q(t))\, dt \\ &\leq \frac{1}{\omega_i}\left(2 + \left|\cot\frac{\omega_i T}{2}\right|\right)\|\nabla\delta\|_\infty^2 \left(\int_0^T V(q(t))\, dt\right)^2,\end{aligned}$$

and

$$\begin{aligned}\left|\frac{1}{2}Q_1(y_1) + \frac{1}{2}Q_2(y_2)\right| &= \left|\int_0^T -\frac{1}{2}\langle\nabla\delta(y_1, y_2), (y_1, y_2)\rangle V(q(t))\, dt\right| \\ &\leq \alpha \int_0^T V(q(t))\, dt. \qquad \square\end{aligned}$$

Let us remark that Lemma 2.6 requires the condition $T \neq 2\pi N/\omega_i$ for $i = 1, 2$. The following lemma shows that there exists a sequence $T_N \to +\infty$ such that $\omega_i T_N \notin 2\pi\mathbb{N}$ for $i = 1, 2$.

LEMMA 2.8. *Fix $\varphi_1, \varphi_2 \in (0, 2\pi)$ and assume $\omega_2/\omega_1 \leq 1$ (otherwise exchange ω_1 with ω_2). Then, choosing $T_N = (2\pi N + \varphi_1)/\omega_1$, we have $\omega_1 T_N \bmod 2\pi = \varphi_1$ for all $N \in \mathbb{N}$. Moreover, we can extract a subsequence T_{N_k} such that*

$$\begin{aligned}&\omega_2 T_{N_k} \bmod 2\pi \to \varphi_2 && \text{as } k \to +\infty \text{ if } \omega_2/\omega_1 \notin \mathbb{Q}, \\ &\omega_2 T_{N_k} \bmod 2\pi = (\omega_2/\omega_1)\varphi_1 && \text{for all } k \text{ if } \omega_2/\omega_1 \in \mathbb{Q}.\end{aligned}$$

PROOF. If $\omega_2/\omega_1 \notin \mathbb{Q}$, then the set $\{\omega_2 T_N \bmod 2\pi\}_{N\in\mathbb{N}}$ is dense in $(0, 2\pi)$. Therefore, for any fixed $\varphi_2 \in (0, 2\pi)$, there is a sequence $\{N_k\}_{k\in\mathbb{N}}$ such that $\omega_2 T_{N_k} = 2\pi l_k + \varphi_{2k}$ with $l_k \in \mathbb{N}$ and $\varphi_{2k} \in (\varphi_2 - 1/k, \varphi_2 + 1/k)$, so that $\omega_2 T_{N_k} \bmod 2\pi \to \varphi_2$ as $k \to +\infty$.

Let $\omega_2/\omega_1 \in \mathbb{Q}$. If $\omega_2/\omega_1 = 1$ the statement is trivial. Otherwise we can assume $\omega_2 = (n/m)\omega_1$ with $n, m \in \mathbb{N}$ coprime and $m > n$. Then, setting $N_k = km$ for all k, we have $\omega_2 T_{N_k} = (n/m)(2\pi km + \varphi_1) = 2\pi nk + (n/m)\varphi_1$, that is, $\omega_2 T_{N_k} \bmod 2\pi = (n/m)\varphi_1 = (\omega_2/\omega_1)\varphi_1$ for all k. □

From now on we will denote by T_N the subsequence as in Lemma 2.8 so that, for φ_1, φ_2 chosen according to $(\omega 1)$–$(\omega 2)$, we have

$$\omega_1 T_N = 2\pi N + \varphi_1, \quad \omega_2 T_N = 2\pi l_N + \varphi_{2N} \quad \text{with } \varphi_{2N} \to \varphi_2, \tag{2.9}$$

and we will use the notation

$$C_i^N = 2 + \left|\cot \frac{\omega_i T_N}{2}\right| \quad \text{for } i = 1, 2; \tag{2.10}$$

note that, for T_N as in Lemma 2.8, we have $C_1^N = C_{\varphi_1}$ (see (1.13)) and $C_2^N = C_{\varphi_{2N}} \to C_{\varphi_2}$ as $N \to +\infty$.

The following lemma is a direct consequence of Lemma 2.6.

LEMMA 2.11. *Let T_N be as in Lemma 2.8 and let y_i^N be a solution of (2.7) for $i = 1, 2$. Then for $i = 1, 2$,*

$$\|y_i^N\|_\infty \le \frac{C_i^N}{\omega_i} \|\nabla\delta\|_\infty \int_0^{T_N} V(q(t))\,dt,$$

$$\|\dot{y}_i^N\|_\infty \le C_i^N \|\nabla\delta\|_\infty \int_0^{T_N} V(q(t))\,dt,$$

$$|Q_i(y_i^N)| \le \frac{C_i^N}{\omega_i} \|\nabla\delta\|_\infty^2 \left(\int_0^{T_N} V(q(t))\,dt\right)^2,$$

where $C_1^N = C_{\varphi_1}$ and $C_2^N = C_{\varphi_{2N}} \to C_{\varphi_2}$ as $N \to +\infty$.

REMARK 2.12. If (y_{1N}, y_{2N}, q_N) is a solution of (PT) in the interval $[0, T_N]$, then $y_{iN}(t)$, $i = 1, 2$, can be expressed, for suitable constants A_{iN} and μ_{iN} (such that the periodic boundary conditions are satisfied), as

$$y_{iN}(t) = \frac{1}{\omega_i} \int_0^t \frac{\partial\delta}{\partial y_i}(y_{1N}, y_{2N}) V(q_N) \sin\omega_i(t-s)\,ds + A_{iN}\cos(\omega_i t + \mu_{iN}), \tag{2.13}$$

or

$$\begin{aligned} y_{iN}(t) = -\frac{1}{\omega_i} \int_t^{T_N} &\frac{\partial\delta}{\partial y_i}(y_{1N}, y_{2N}) V(q_N) \sin\omega_i(t-s)\,ds \\ &+ A_{iN}\cos(\omega_i t + \mu_{iN} - \varphi_{iN}), \end{aligned} \tag{2.14}$$

with $\varphi_{iN} \equiv \omega_i T_N \bmod 2\pi$.

3. Variational setting and min-max procedure

From now on we assume that T_N satisfies (2.9). Let

$$E_N = \{x \in H^1_{\mathrm{loc}}(\mathbb{R}) \mid x \text{ is } T_N\text{-periodic}\}$$

with scalar product $(u, v) = \int_0^{T_N} (\dot{u}\dot{v} + uv)$ and

$$\Gamma_N = \{q \in H^1(0, T_N) \mid q(0) = 0,\ q(T_N) = 2\pi\},$$

and let, for all $(y_1, y_2, q) \in E_N \times E_N \times \Gamma_N$,

$$f_N(y_1, y_2, q) = \int_0^{T_N} \left[\frac{\dot{y}_1^2 - \omega_1^2 y_1^2}{2} + \frac{\dot{y}_2^2 - \omega_2^2 y_2^2}{2} + \frac{\dot{q}^2}{2} + (1 + \delta(y_1, y_2))V(q) \right].$$

In the following we will omit the subscript N when it is possible. It is straightforward to show that

LEMMA 3.1. *$f \in C^1(E \times E \times \Gamma; \mathbb{R})$. If $\nabla f(y_1, y_2, q) = 0$ then (y_1, y_2, q) solves* (PT).

We recall (see for example [10]), that, for fixed $N \in \mathbb{N}$, to the quadratic form $Q_1(y)$ on E (see (2.5)), there is associated a splitting $E = E_1^- \oplus E_1^+$, and to $Q_2(y)$ a splitting $E = E_2^- \oplus E_2^+$. More precisely, for $i = 1, 2$ let

$$E_i^- = \left\{ y(s) = a_0^i + \sum_{\{k \mid 2\pi k < \omega_i T_N\}} \left(a_k^i \cos \frac{2\pi k}{T_N} s + b_k^i \sin \frac{2\pi k}{T_N} s \right) \right\},$$

$$E_i^+ = \left\{ y(s) = \sum_{\{k \mid 2\pi k > \omega_i T_N\}} \left(a_k^i \cos \frac{2\pi k}{T_N} s + b_k^i \sin \frac{2\pi k}{T_N} s \right) \right\}.$$

Then, for all $y \in E$ and $i = 1, 2$, $y = y_i^+ + y_i^-$, $y_i^+ \in E_i^+$, $y_i^- \in E_i^-$ and $\int_0^T y_i^+ y_i^- = 0$, $\int_0^T \dot{y}_i^+ \dot{y}_i^- = 0$.

Note also that for suitable positive constants $\lambda_i^\pm(T_N)$ and $i = 1, 2$,

$$\begin{aligned} -Q_i(y) &\geq \lambda_i^-(T_N)\|y\|^2 \quad \text{for all } y \in E_i^-, \\ Q_i(y) &\geq \lambda_i^+(T_N)\|y\|^2 \quad \text{for all } y \in E_i^+. \end{aligned} \tag{3.2}$$

PROPOSITION 3.3. *Assume $(y_{1n}, y_{2n}, q_n) \in E \times E \times \Gamma$ are such that*

$$f(y_{1n}, y_{2n}, q_n) \to c, \quad \frac{\partial f}{\partial y_i}(y_{1n}, y_{2n}, q_n) \to 0, \quad i = 1, 2.$$

Then (y_{1n}, y_{2n}, q_n) is bounded in $E \times E \times \Gamma$ and, up to a subsequence, $y_{in} \to y_{i0}$ in E for $i = 1, 2$, $q_n \to q_0$ in L^∞, $q_n \rightharpoonup q_0$ in H^1. Moreover, (y_{10}, y_{20}, q_0) is a solution of (2.7) *for $i = 1, 2$.*

Furthermore, if $\nabla f(y_{1n}, y_{2n}, q_n) \to 0$, then, up to a subsequence, $(y_{1n}, y_{2n}, q_n) \to (y_{10}, y_{20}, q_0)$ and f satisfies the PS condition.

PROOF. Using (3.2), as in [10] we have the estimate

$$\begin{aligned}\varepsilon_n\|y_{1n}^-\|_{H^1} &\geq \left|\left\langle \frac{\partial f}{\partial y_1}(y_{1n},y_{2n},q_n), y_{1n}^-\right\rangle\right| \\ &\geq |Q_1(y_{1n}^-)| - \|\nabla\delta\|_\infty \|y_{1n}^-\|_{L^2}\left(\int_0^T V(q_n)^2\right)^{1/2} \\ &\geq \lambda_1^-(T)\|y_{1n}^-\|_{H^1}^2 - \|\nabla\delta\|_\infty\|V\|_\infty\sqrt{T}\|y_{1n}^-\|_{H^1}\end{aligned}$$

and the boundedness of $\|y_{1n}^-\|_{H^1}$ follows. In the same way we argue for y_{1n}^+, y_{2n}^- and y_{2n}^+ so that we obtain the boundedness of y_{1n} and y_{2n} in E. Since

$$\begin{aligned}\int_0^T \dot q_n^2 &= 2f(y_{1n},y_{2n},q_n) - Q_1(y_{1n}) - Q_2(y_{2n}) - 2\int_0^T (1+\delta(y_{1n},y_{2n}))V(q_n) \\ &\leq 2(c+1) + \max\{1,\omega_1^2\}\|y_{1n}\|_{H^1}^2 + \max\{1,\omega_2^2\}\|y_{2n}\|_{H^1}^2 \leq \text{const},\end{aligned}$$

and since $q_n(0) = 0$, we see that q_n is bounded in $H^1(0,T)$. We then deduce that, up to a subsequence, $q_n \to q_0$ in L^2, uniformly and weakly in H^1, and also $y_{in} \to y_{i0}$ in L^2, uniformly and weakly in H^1 for $i = 1, 2$. Since

$$\begin{aligned}\int_0^T |\dot y_{in} - \dot y_{i0}|^2 + \int_0^T |y_{in} - y_{i0}|^2 &= \int_0^T \dot y_{in}(\dot y_{in} - \dot y_{i0}) \\ &\quad - \int_0^T \dot y_{i0}(\dot y_{in} - \dot y_{i0}) + \int_0^T |y_{in} - y_{i0}|^2,\end{aligned}$$

recalling that $\int_0^T |y_{in} - y_{i0}|^2 \to 0$ as well as (by weak convergence) $\int_0^T \dot y_{i0}(\dot y_{in} - \dot y_{i0}) \to 0$, to show that $y_{in} \to y_{i0}$ in H^1 it is enough to prove that

$$\int_0^T \dot y_{in}(\dot y_{in} - \dot y_{i0}) \to 0.$$

Since

$$\begin{aligned}\int_0^T \dot y_{in}(\dot y_{in} - \dot y_{i0}) &= \left\langle \frac{\partial f}{\partial y_i}(y_{1n},y_{2n},q_n), y_{in} - y_{i0}\right\rangle + \omega_i^2 \int_0^T y_{in}(y_{in} - y_{i0}) \\ &\quad - \int_0^T \frac{\partial}{\partial y_i}\delta(y_{1n},y_{2n})V(q_n)(y_{in} - y_{i0}),\end{aligned}$$

the result follows because $y_{in} - y_{i0}$ is bounded, $\frac{\partial f}{\partial y_i}(y_{1n},y_{2n},q_n) \to 0$, $y_{in} \to y_{i0}$ in L^2 and the sequences y_{in} and $\frac{\partial}{\partial y_i}\delta(y_{1n},y_{2n})V(q_n)$ are bounded in L^∞. Then $y_{in} \to y_{i0}$ in H^1 for $i = 1, 2$.

Finally, if φ is any test function, we have

$$\begin{aligned}\left\langle \frac{\partial f}{\partial y_i}(y_{1n},y_{2n},q_n), \varphi\right\rangle &= \int_0^T (\dot y_{in}\dot\varphi - \omega_i^2 y_{in}\varphi) + \int_0^T \frac{\partial\delta}{\partial y_i}(y_{1n},y_{2n})V(q_n)\varphi \\ &\to \int_0^T (\dot y_i\dot\varphi - \omega_i^2 y_i\varphi) + \int_0^T \frac{\partial\delta}{\partial y_i}(y_1,y_2)V(q_0)\varphi, \qquad i = 1, 2,\end{aligned}$$

so that y_i is a weak solution of

$$\ddot{y}_i + \omega_i^2 y_i = \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q_0), \qquad i = 1, 2,$$

and, by standard arguments, also a classical solution. Therefore (y_{10}, y_{20}, q_0) is a solution of (2.7) for $i = 1, 2$. If also $\frac{\partial f}{\partial q}(y_{1n}, y_{2n}, q_n) \to 0$, using the same arguments, we have $q_n \to q_0$ in H^1 and (y_{10}, y_{20}, q_0) is a solution of (PT). □

We say that $h \in \mathcal{H}$ if

(1) $h : E_1^- \times E_2^- \to E \times E \times \Gamma$ is continuous,
(2) there are $R > 0$ and $q_h \in \Gamma$ such that

$$h(y_1, y_2) = (y_1, y_2, q_h) \quad \forall \|(y_1, y_2)\| \geq R.$$

Let us define

$$c(T_N) = \inf_{h \in \mathcal{H}} \sup_{(y_1, y_2) \in E_1^- \times E_2^-} f_N(h(y_1, y_2)). \tag{3.4}$$

To estimate $c(T_N)$ (see Lemma 3.6), we first prove the following inequality.

LEMMA 3.5. *For all $q \in \Gamma$ we have*

$$\int_0^T [\dot{q}^2/2 + V(q)] \geq \int_0^{2\pi} \sqrt{2V(s)}\, ds > 0.$$

PROOF. Let $\underline{q} \in \Gamma_\infty = \{q \in H^1_{\text{loc}}(\mathbb{R}) : q(-\infty) = 0,\ q(+\infty) = 2\pi\}$ be such that

$$\int_{\mathbb{R}} [\dot{\underline{q}}^2/2 + V(\underline{q})] = \min_q \int_{\mathbb{R}} [\dot{q}^2/2 + V(q)].$$

By energy conservation $\dot{\underline{q}}^2/2 - V(\underline{q}) = 0$, so that for all $q \in \Gamma$, we have

$$\int_0^T [\dot{q}^2/2 + V(q)] \geq \int_{\mathbb{R}} [\dot{\underline{q}}^2/2 + V(\underline{q})] = \int_{\mathbb{R}} 2V(\underline{q}).$$

From our assumptions on V it follows that $\dot{\underline{q}}(t) > 0$ and using the change of variables $s = \underline{q}(t)$ we have

$$\int_{\mathbb{R}} 2V(\underline{q})\, dt = \int_0^{2\pi} \frac{2V(s)}{\sqrt{2V(s)}}\, ds = \int_0^{2\pi} \sqrt{2V(s)}\, ds,$$

so that

$$\int_0^T [\dot{q}^2/2 + V(q)] \geq \int_0^{2\pi} \sqrt{2V(s)}\, ds > 0 \quad \forall q \in \Gamma. \qquad \square$$

LEMMA 3.6. *Let T_N satisfy* (2.9) *and $c(T_N)$ be defined as in* (3.4). *Then*

$$\int_0^{2\pi} \sqrt{2(1+\underline{\delta})V(s)}\, ds =: \underline{c} \leq c(T_N) \leq \bar{c} := 2\pi^2 + (1+\bar{\delta})\|V\|_\infty \quad \textit{for all } T_N.$$

PROOF. Let $\bar{q}_T$ be such that

$$\int_0^T [\dot{\bar{q}}_T^2/2 + (1+\bar{\delta})V(\bar{q}_T)] = \min_{q\in\Gamma} \int_0^T [\dot{q}^2/2 + (1+\bar{\delta})V(q)] = \bar{c}(T).$$

Then

$$\bar{c}(T) \leq \bar{c}(1) \leq 2\pi^2 + (1+\bar{\delta})\|V\|_\infty =: \bar{c}.$$

Letting $\bar{h}(y_1, y_2) = (y_1, y_2, \bar{q}_T)$, we have

$$\begin{aligned} c(T) &= \inf_{h\in\mathcal{H}} \sup_{(y_1,y_2)\in E_1^-\times E_2^-} f(h(y_1, y_2)) \leq \sup_{(y_1,y_2)\in E_1^-\times E_2^-} f(\bar{h}(y_1, y_2)) \\ &\leq \int_0^T [\dot{\bar{q}}_T^2/2 + (1+\bar{\delta})V(\bar{q}_T)] = \bar{c}(T) \leq \bar{c}. \end{aligned}$$

On the other hand, for any $h \in \mathcal{H}$, $h(y_1, y_2) = (h_1(y_1, y_2), h_2(y_1, y_2), h_3(y_1, y_2))$, consider the function $\bar{h} : E_1^- \times E_2^- \to E_1^- \times E_2^-$ defined by $\bar{h}(y_1, y_2) = (\pi_{E_1^-} h_1(y_1, y_2), \pi_{E_2^-} h_2(y_1, y_2))$. Since $\bar{h}_{|\partial B(0,R)} = \mathrm{Id}$ for all R large enough, there is $(\bar{y}_1, \bar{y}_2) \in E_1^- \times E_2^-$ such that $\bar{h}(\bar{y}_1, \bar{y}_2) = (0, 0)$, i.e.

$$h_1(\bar{y}_1, \bar{y}_2) \in E_1^+, \quad h_2(\bar{y}_1, \bar{y}_2) \in E_2^+.$$

Then, letting $q = h_3(\bar{y}_1, \bar{y}_2)$, we have, for all $h \in \mathcal{H}$,

$$\begin{aligned} \sup_{(y_1,y_2)\in E_1^-\times E_2^-} f(h(y_1, y_2)) &\geq f(h(\bar{y}_1, \bar{y}_2)) \geq \int_0^T [\dot{q}^2/2 + (1+\underline{\delta})V(q)]\, dt \\ &\geq \min_{q\in\Gamma} \int_0^T [\dot{q}^2/2 + (1+\underline{\delta})V(q)]\, dt = \underline{c}(T), \end{aligned}$$

and Lemma 3.5 yields

$$\underline{c}(T) \geq \int_0^{2\pi} \sqrt{2(1+\underline{\delta})V(s)}\, ds =: \underline{c} > 0. \qquad \square$$

PROPOSITION 3.7. *Let T_N satisfy* (2.9) *and $c(T_N)$ be defined as in* (3.4). *Then there is a critical point (y_{1N}, y_{2N}, q_N) for f_N at level $c(T_N)$ that solves problem* (PT). *Moreover, q_N has the following properties:*

(3.8) $\quad q_N(t) \in [0, 2\pi] \quad \forall t \in [0, T_N],$

(3.9) $\quad \dot{q}_N(0) = \dot{q}_N(T_N),$

(3.10) $\quad \displaystyle\int_0^{T_N} V(q_N) \leq \frac{\bar{c}}{1+\underline{\delta}-\alpha} = \frac{\bar{c}}{K}, \quad \int_0^{T_N} \dot{q}_N^2 \leq 2\bar{c}.$

PROOF. The existence of the critical point (y_{1N}, y_{2N}, q_N) at level $c(T_N)$ follows via the min-max principle, since f satisfies (PS) by Proposition 3.3 and by the estimates of Lemma 3.6.

To prove (3.8) let us introduce, for all T_N,

$$\begin{aligned}
&\Gamma_N^* = \{q \in \Gamma_N \mid q(s) \in [0, 2\pi]\ \forall s \in [0, T_N]\}\\
&\mathcal{H}^* = \{h \in \mathcal{H} \mid h(y_1, y_2) \in E \times E \times \Gamma_N^*\ \forall (y_1, y_2) \in E_1^- \times E_2^-\},\\
&c^*(T_N) = \inf_{h \in \mathcal{H}^*} \sup_{(y_1, y_2) \in E_1^- \times E_2^-} f_N(h(y_1, y_2)).
\end{aligned}$$

It is easy to show that

$$c^*(T_N) = c(T_N).$$

Indeed, $\mathcal{H}^* \subset \mathcal{H}$ implies that $c^*(T_N) \geq c(T_N)$. To prove the other inequality pick $h \in \mathcal{H}$ and let $h^* \in \mathcal{H}^*$ be defined as $h^*(y_1, y_2) = h(y_1, y_2)^*$, where $(y_1, y_2, q)^* = (y_1, y_2, q^*)$ and

$$q^*(t) = \begin{cases} q(t) & \text{if } 0 \leq q(t) \leq 2\pi,\\ 2\pi & \text{if } q(t) > 2\pi,\\ 0 & \text{if } q(t) < 0. \end{cases}$$

Then, since

$$(1 + \delta(y_1(t), y_2(t)))V(q^*(t)) \leq (1 + \delta(y_1(t), y_2(t)))V(q(t)) \quad \forall t \in [0, T],$$

we immediately see that

$$f_T(h^*(y_1, y_2)) \leq f_T(h(y_1, y_2)) \quad \forall (y_1, y_2) \in E_1^- \times E_2^-, \forall h \in \mathcal{H}$$

and

$$c^*(T) \leq c(T)$$

and also (3.8) follows.

Now let us show that (3.9) holds. Since (y_{1N}, y_{2N}, q_N) is a solution of (PT), by energy conservation, we have

$$\begin{aligned}
&\frac{\dot{y}_{1N}^2(0) + \omega_1^2 y_{1N}^2(0)}{2} + \frac{\dot{y}_{2N}^2(0) + \omega_2^2 y_{2N}^2(0)}{2} + \frac{\dot{q}_N^2(0)}{2}\\
&\qquad - (1 + \delta(y_{1N}(0), y_{2N}(0)))V(q_N(0))\\
&\quad = \frac{\dot{y}_{1N}^2(T_N) + \omega_1^2 y_{1N}^2(T_N)}{2} + \frac{\dot{y}_{2N}^2(T_N) + \omega_2^2 y_{2N}^2(T_N)}{2} + \frac{\dot{q}_N^2(T_N)}{2}\\
&\qquad - (1 + \delta(y_{1N}(T_N), y_{2N}(T_N)))V(q(T_N)).
\end{aligned}$$

Since $q_N(0) = 0$, $q_N(T_N) = 2\pi$, $V(0) = V(2\pi) = 0$ and by periodicity of y_{1N} and y_{2N}, we have

$$\dot{q}_N^2(0) = \dot{q}_N^2(T_N).$$

Since $q_N(t) \in [0, 2\pi]$ for all $t \in [0, T_N]$ we have $\dot{q}_N(0) \geq 0$, $\dot{q}_N(T) \geq 0$ and so

$$\dot{q}_N(0) = \dot{q}_N(T_N).$$

Finally, using the last estimate of Lemma 2.6, we have

$$\begin{aligned} c(T_N) &= \frac{1}{2}Q_1(y_{1N}) + \frac{1}{2}Q_2(y_{2N}) + \int_0^{T_N} [\dot{q}_N^2/2 + (1+\delta(y_{1N}, y_{2N}))V(q_N)] \\ &\geq \int_0^{T_N} \dot{q}_N^2/2 + (1+\underline{\delta}-\alpha)\int_0^{T_N} V(q_N), \end{aligned}$$

and, by the estimate on $c(T_N)$ in Lemma 3.6, (3.10) holds. □

4. Proof of Theorem 1.10

We say that $q(t)$ jumps from η to $2\pi-\eta$ in an interval $[\alpha,\beta]$ if $q(\alpha)=\eta$, $q(t)\in\,]\eta, 2\pi-\eta[$ for all $t\in]\alpha,\beta[$, $q(\beta)=2\pi-\eta$. Note that if $q(t)$ jumps in $[\alpha,\beta]$ from η to $2\pi-\eta$, then defining

$$\bar{q}(t) = \begin{cases} 0, & 0\le t\le \alpha-1, \\ \eta(t-\alpha+1), & \alpha-1\le t\le\alpha, \\ q(t), & \alpha\le t\le\beta, \\ 2\pi+\eta(t-\beta-1), & \beta\le t\le\beta+1, \\ 2\pi, & \beta+1\le t\le T, \end{cases}$$

and arguing as in Lemma 3.5, for any $B>0$ and $\eta\le\eta_0$ (η_0 given by (1.6)), η sufficiently small, we have

$$\begin{aligned} \text{(4.1)} \qquad \int_\alpha^\beta [\dot{q}^2/2 + BV(q)] &\geq \int_0^T [\dot{\bar{q}}^2/2 + BV(\bar{q})] - \eta^2 - BV(\eta) - BV(2\pi-\eta) \\ &\geq \int_0^{2\pi} \sqrt{2BV(s)}\,ds - \eta^2 - 2B\mu\eta^2 > 0. \end{aligned}$$

LEMMA 4.2. *Let $(y_1, y_2, q)\in E\times E\times\Gamma^*$ be a critical point for f_N at level $c(T_N)$ as in Proposition* 3.7 *and assume* (δ4) *holds. Then there exists* $0<\eta_1\le\eta_0$ (η_0 *given by* (1.6)) *such that for all* $0<\eta\le\eta_1$, *$q(t)$ jumps only once from η to $2\pi-\eta$. Moreover, if* $[\alpha,\beta]$ *is the interval where $q(t)$ jumps from η to $2\pi-\eta$, then $|\beta-\alpha|\le\bar{c}/KV_\eta$ with V_η as in* (1.4)*, and K as in* (1.13).

PROOF. Arguing by contradiction, let us assume that $q(t)$ jumps from η to $2\pi-\eta$ in two intervals, $[\alpha_1,\beta_1]$ and $[\alpha_2,\beta_2]$ with $\beta_1<\alpha_2$. Without loss of generality we can assume that

$$\text{(4.3)} \qquad \int_0^{\beta_1} V(q) \le \int_{\alpha_2}^T V(q).$$

Define

$$\bar{q}(t) = \begin{cases} q(t) & \text{if } t\in[0,\beta_1], \\ 2\pi+\eta(t-\beta_1-1) & \text{if } t\in[\beta_1,\beta_1+1], \\ 2\pi & \text{if } t\in[\beta_1+1, T], \end{cases}$$

and let $\bar{h} : E_1^- \times E_2^- \to E_1 \times E_2 \times \Gamma$ be defined by $\bar{h}(y_1^-, y_2^-) = (y_1^-, y_2^-, \bar{q})$ for all $(y_1^-, y_2^-) \in E_1^- \times E_2^-$.

For all $(y_1^-, y_2^-) \in E_1^- \times E_2^-$, using the last estimate of Lemma 2.6, we have

$$\begin{aligned} f(y_1, y_2, q) - f(y_1^-, y_2^-, \bar{q}) &\geq \frac{1}{2}Q(y_1) + \frac{1}{2}Q(y_2) \\ &\quad + \int_0^T [\dot{q}^2/2 + (1 + \delta(y_1, y_2))V(q)] - \int_0^T [\dot{\bar{q}}^2/2 + (1 + \delta(y_1^-, y_2^-))V(\bar{q})] \\ &\geq \int_0^T -\alpha V(q) + \int_0^{\beta_1} [\delta(y_1, y_2) - \delta(y_1^-, y_2^-)]V(q) \\ &\quad + \int_{\beta_1}^T [\dot{q}^2/2 + (1 + \delta(y_1, y_2))V(q)] - \int_{\beta_1}^{\beta_1+1} [\dot{\bar{q}}^2/2 + (1 + \delta(y_1^-, y_2^-))V(\bar{q})] \\ &\geq \int_0^{\beta_1} (\underline{\delta} - \alpha - \bar{\delta})V(q) + \int_{\beta_1}^T [\dot{q}^2/2 + (1 + \underline{\delta} - \alpha)V(q)] - \eta^2/2 - (1 + \bar{\delta})\mu\eta^2. \end{aligned}$$

From assumption (δ4) it follows that $\underline{\delta} - \alpha - \bar{\delta} > -(1 + \underline{\delta} - \alpha)$; and thanks to (4.3) we have

$$\int_0^{\beta_1} (\underline{\delta} - \alpha - \bar{\delta})V(q) > -(1 + \underline{\delta} - \alpha)\int_0^{\beta_1} V(q) \geq -(1 + \underline{\delta} - \alpha)\int_{\alpha_2}^T V(q).$$

Therefore, using also (4.1) we have

$$\begin{aligned} f(y_1, y_2, q) - f(y_1^-, y_2^-, \bar{q}) &\geq \int_{\beta_1}^{\alpha_2} [\dot{q}^2/2 + (1 + \underline{\delta} - \alpha)V(q)] - \eta^2[1 + \mu(1 + \bar{\delta})] \\ &\geq \int_0^{2\pi} \sqrt{2(1 + \underline{\delta} - \alpha)V(s)}\, ds \\ &\quad - \eta^2 - 2(1 + \underline{\delta} - \alpha)\mu\eta^2 - \eta^2[1 + \mu(1 + \bar{\delta})]. \end{aligned}$$

Then, choosing η_1 sufficiently small, we have

$$f(y_1, y_2, q) - f(y_1^-, y_2^-, \bar{q}) > 0, \quad \forall \eta \leq \eta_1, \forall (y_1^-, y_2^-) \in E_1^- \times E_2^-,$$

which is a contradiction, because (y_1, y_2, q) is a critical point at level $c(T)$. Finally, by (3.10) we have

$$\frac{\bar{c}}{K} \geq \int_0^{T_N} V(q_N) \geq \int_\alpha^\beta V(q_N) \geq V_\eta|\beta - \alpha|. \qquad \square$$

PROOF OF THEOREM 1.10. Let (y_{1N}, y_{2N}, q_N) be a critical point at level $c(T_N)$ which is a solution of (PT), given by Proposition 3.7. Fix any $\eta \leq \eta_1$ and let $[\alpha_N, \beta_N]$ denote the unique (by Lemma 4.2) interval where q_N jumps from η to $2\pi - \eta$. Let $\tau_N \in [\alpha_N, \beta_N]$ be such that $q_N(\tau_N) = \pi$ and $q_N(t) \leq \pi$ for all $t \leq \tau_N$. Since $|\beta_N - \alpha_N| \leq \bar{c}/KV_\eta$ and $T_N \to +\infty$, we have either $\tau_N \to +\infty$ or $T_N - \tau_N \to +\infty$. Let us first analyze the case where both $\tau_N \to +\infty$ and $T_N - \tau_N \to +\infty$. We define the function $\tilde{q}_N$ in the interval $[-\tau_N, T_N - \tau_N]$ as

$$\tilde{q}_N(t) = q_N(t + \tau_N) \quad \forall N \in \mathbb{N}.$$

By definition $\tilde{q}_N(t) \in [0, 2\pi]$ for all $t \in [-\tau_N, T_N - \tau_N]$ and $\tilde{q}_N(0) = \pi$ for all $N \in \mathbb{N}$. Moreover, by Proposition 3.7 we have

$$\int_{-\tau_N}^{T_N - \tau_N} \dot{\tilde{q}}_N^2 = \int_0^{T_N} \dot{q}_N^2 \leq 2\bar{c}.$$

Then, for any fixed $a < b \in \mathbb{R}$, since both $\tau_N \to +\infty$ and $T_N - \tau_N \to +\infty$, we have $\tilde{q}_N \in H^1(a, b)$ for all N large enough and

$$\|\tilde{q}_N\|^2_{H^1(a,b)} \leq 2\bar{c} + (b - a)4\pi^2.$$

Therefore, up to a subsequence, $\tilde{q}_N \rightharpoonup q$ in $H^1(a, b)$, $\tilde{q}_N \to q$ uniformly in $[a, b]$ and

$$\int_{\mathbb{R}} \dot{q}^2 = \sup_{a<b} \int_a^b \dot{q}^2 \leq \sup_{a<b} \liminf_{N\to+\infty} \int_a^b \dot{\tilde{q}}_N^2 \leq 2\bar{c},$$
$$\int_{\mathbb{R}} V(q) = \sup_{a<b} \int_a^b V(q) \leq \sup_{a<b} \liminf_{N\to+\infty} \int_a^b V(\tilde{q}_N) \leq \frac{\bar{c}}{1 + \underline{\delta} - \alpha} = \frac{\bar{c}}{K}.$$

Since $V(q) = 0$ only for $q = 2k\pi$, $k \in \mathbb{Z}$, and since q_N jumps only once from η to $2\pi - \eta$ for all $N \in \mathbb{N}$, we have

$$\lim_{t\to-\infty} q(t) = 0, \quad \lim_{t\to+\infty} (q(t) - 2\pi) = 0, \quad \lim_{t\to\pm\infty} \dot{q}(t) = 0.$$

Now let us analyze the case where only one of τ_N and $T_N - \tau_N$ diverges. We can assume that, up to a subsequence, $T_N - \tau_N$ diverges and $\tau_N < T_N/2$. Define the function $\tilde{q}_N$ in $[-(T_N + \tau_N)/2, (T_N - \tau_N)/2]$ by

$$\tilde{q}_N(t) = \begin{cases} q_N(t + T_N + \tau_N) - 2\pi & \text{if } t \in [-(T_N + \tau_N)/2, -\tau_N], \\ q_N(t + \tau_N) & \text{if } t \in [-\tau_N, (T_N - \tau_N)/2]. \end{cases}$$

Then for all N, $\tilde{q}_N(0) = \pi$, $\tilde{q}_N(t) \in [-2\pi + \eta, \pi]$ for all $t \in [-(T_N + \tau_N)/2, 0]$, and $\tilde{q}_N(t) \in [\eta, 2\pi]$ for all $t \in [0, (T_N - \tau_N)/2]$. Then, arguing as in the first case, for $a < b$ in $\mathbb{R}$ we have, up to a subsequence, $\tilde{q}_N \rightharpoonup q$ in $H^1(a, b)$, $\tilde{q}_N \to q$ uniformly in $[a, b]$, $\|q\|_\infty \leq 2\pi$ and

$$\int_{\mathbb{R}} \dot{q}^2 \leq 2\bar{c}, \qquad \int_{\mathbb{R}} V(q) \leq \frac{\bar{c}}{K}. \tag{4.4}$$

Since $V(q) = 0$ only for $q = 2k\pi$ and since for all $N \in \mathbb{N}$, $\tilde{q}_N \in [-2\pi + \eta, \pi]$ for all $t \in [-(T_N + \tau_N)/2, 0]$, and $\tilde{q}_N \in [\eta, 2\pi]$ for all $t \in [0, (T_N - \tau_N)/2]$, we deduce also in this case that

$$\lim_{t\to-\infty} q(t) = 0, \quad \lim_{t\to+\infty} (q(t) - 2\pi) = 0, \quad \lim_{t\to\pm\infty} \dot{q}(t) = 0.$$

Now define, for $i = 1, 2$,

$$\tilde{y}_{iN}(t) = \begin{cases} y_{iN}(t + T_N + \tau_N) & \text{if } t \in [-(T_N + \tau_N)/2, -\tau_N], \\ y_{iN}(t + \tau_N) & \text{if } t \in [-\tau_N, (T_N - \tau_N)/2]. \end{cases}$$

In view of Lemma 2.11 and (4.4) we know that $\tilde{y}_{iN}(t)$ is bounded in $H^1(a, b)$, so that, up to a subsequence, $\tilde{y}_{iN} \rightharpoonup y_i$ in $H^1(a, b)$ and $\tilde{y}_{iN} \to y_i$ uniformly in $L^\infty(a, b)$ for $i = 1, 2$. We can now pass to the limit in the equations

$$\begin{aligned} \ddot{\tilde{q}}_N &= (1 + \delta(\tilde{y}_{1N}, \tilde{y}_{2N}))V'(\tilde{q}_N), \\ \ddot{\tilde{y}}_{1N} + \omega_1^2 \tilde{y}_{1N} &= \frac{\partial}{\partial y_1}\delta(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N), \\ \ddot{\tilde{y}}_{2N} + \omega_2^2 \tilde{y}_{2N} &= \frac{\partial}{\partial y_2}\delta(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N), \end{aligned}$$

to deduce that (y_1, y_2, q) is a solution of

$$\begin{aligned} \ddot{q} &= (1 + \delta(y_1, y_2))V'(q), \\ \ddot{y}_1 + \omega_1^2 y_1 &= \frac{\partial}{\partial y_1}\delta(y_1, y_2)V(q), \\ \ddot{y}_2 + \omega_2^2 y_2 &= \frac{\partial}{\partial y_2}\delta(y_1, y_2)V(q), \end{aligned}$$

in the interval $[a, b]$ and hence also in $\mathbb{R}$.

Thus, as observed at the beginning of Section 2, conditions (1.3) are satisfied. Finally, by energy conservation, since $\dot{q}(\pm\infty) = 0$, also condition (1.2) holds. □

5. Proof of Theorem 1.11

LEMMA 5.1. *Let (y_{1N}, y_{2N}, q_N) be a critical point for f_N at level $c(T_N)$ given by Proposition* 3.7 *and assume that* ($\delta 4'$) *holds. Then for all* $0 < \eta \le \eta_0$ (η_0 *given by* (1.6)) *there exist* $0 < \tau_1 < \tau_2 < T_N$ *such that*

$$\begin{aligned} &0 \le q_N(t) \le \eta && \forall t \in [0, \tau_1], \\ &q_N(t) \in [\eta, 2\pi - \eta] && \forall t \in [\tau_1, \tau_2], \\ &2\pi - \eta \le q_N(t) \le 2\pi && \forall t \in [\tau_2, T_N]. \end{aligned}$$

PROOF. Let $\eta \le \eta_0$, let $\tau_1 = \inf\{s \in [0, T] \mid q(s) > \eta\}$ and $\tau_2 = \sup\{s \in [0, T] \mid q(s) < 2\pi - \eta\}$. If the lemma does not hold, then there is $\tau_1' \in (\tau_1, T]$ such that $q(\tau_1') = \eta$ (or there is $\tau_2' \in [0, \tau_2)$ such that $q(\tau_2') = 2\pi - \eta$; we will only discuss the first case). Then $q(t)$ reaches a maximum at $\tau_1'' \in (\tau_1, \tau_1')$, hence $\ddot{q}(\tau_1'') \le 0$. But

$$\ddot{q}(\tau_1'') = (1 + \delta(y_1(\tau_1''), y_2(\tau_1'')))V'(q(\tau_1''))$$

implies, by (V4), that $q(\tau_1'') \ge \bar{\eta}$. Then there exists an interval where q jumps from $\bar{\eta}/2$ to $\bar{\eta}$ and an interval where q jumps from $\bar{\eta}$ to $\bar{\eta}/2$. In each of these intervals, say $[a, b]$,

$$\frac{\bar{\eta}}{2} = \int_a^b \dot{q} \le \left(\int_a^b \dot{q}^2\right)^{1/2} \sqrt{b - a}$$

so that for $V_{\bar\eta/2}$ as in (1.4) we obtain

$$\int_a^b [\dot q^2/2 + (1+\underline{\delta}-\alpha)V(q)] \geq \frac{\bar\eta^2}{8(b-a)} + (1+\underline{\delta}-\alpha)V_{\bar\eta/2}(b-a) \geq \bar\eta\sqrt{\frac{1+\underline{\delta}-\alpha}{2}V_{\bar\eta/2}}.$$

Therefore

$$\int_{\tau_1}^{\tau_1'} [\dot q^2/2 + (1+\delta(y_1,y_2)-\alpha)V(q)] \geq 2\bar\eta\sqrt{\frac{1+\underline{\delta}-\alpha}{2}V_{\bar\eta/2}}. \tag{5.2}$$

Now we define a new function $\bar q \in \Gamma^*$ by setting

$$\bar q(t) = \begin{cases} 0, & 0 \leq t \leq \tau_1' - \tau_1, \\ q(t-\tau_1'+\tau_1), & \tau_1'-\tau_1 \leq t \leq \tau_1', \\ q(t), & \tau_1' \leq t \leq T. \end{cases}$$

We also introduce $\bar h$ defined as $\bar h(y_1^-, y_2^-) = (y_1^-, y_2^-, \bar q)$ for all $(y_1^-, y_2^-) \in E_1^- \times E_2^-$. Clearly $\bar h \in \mathcal{H}^*$, so that, since $f_T(y_1, y_2, q) = c(T)$,

$$0 \leq \sup_{(y_1^-,y_2^-)\in E_1^-\times E_2^-} f_T(\bar h(y_1^-, y_2^-)) - f_T(y_1, y_2, q).$$

On the other hand, using (5.2) and arguing as in the proof of Lemma 4.2 (see also [10, Lemma 11]) we have, for all $(y_1^-, y_2^-) \in E_1^- \times E_2^-$,

$$\begin{aligned} &f_T(y_1,y_2,q) - f_T(y_1^-, y_2^-, \bar q) \\ &\quad\geq \int_0^{\tau_1} [\dot q^2/2 + (1+\underline{\delta}-\alpha)V(q)] + \int_{\tau_1}^{\tau_1'} [\dot q^2/2 + (1+\underline{\delta}-\alpha)V(q)] \\ &\qquad + \int_{\tau_1'}^{T} (\underline{\delta}-\bar\delta-\alpha)V(q) - \int_{\tau_1'-\tau_1}^{\tau_1'} [\dot{\bar q}^2/2 + (1+\delta(y_1^-, y_2^-))V(\bar q)] \\ &\quad\geq -(\bar\delta-\underline{\delta}+\alpha)\int_0^T V(q) + 2\bar\eta\sqrt{\frac{1+\underline{\delta}-\alpha}{2}V_{\bar\eta/2}}. \end{aligned}$$

Then, using the estimate (3.10) and by definition of $\bar c$ (see Lemma 3.6), we have, for all $(y_1^-, y_2^-) \in E_1^- \times E_2^-$,

$$\begin{aligned} &f_T(y_1,y_2,q) - f_T(y_1^-, y_2^-, \bar q) \\ &\quad\geq -(\bar\delta-\underline{\delta}+\alpha)\frac{\bar c}{1+\underline{\delta}-\alpha} + 2\bar\eta\sqrt{\frac{1+\underline{\delta}-\alpha}{2}V_{\bar\eta/2}} \\ &\quad\geq -\frac{\bar\delta-\underline{\delta}+\alpha}{1+\underline{\delta}-\alpha}[2\pi^2 + (1+\bar\delta)\|V\|_\infty] + 2\bar\eta\sqrt{\frac{1+\underline{\delta}-\alpha}{2}V_{\bar\eta/2}}. \end{aligned}$$

Then, thanks to ($\delta 4'$), for all $(y_1^-, y_2^-) \in E_1^- \times E_2^-$ we have

$$f_T(y_1^-, y_2^-, \bar{q}) - f_T(y_1, y_2, q) \le -\bar{\eta}\sqrt{\frac{1+\underline{\delta}-\alpha}{2}} V_{\bar{\eta}/2},$$

contradiction. □

LEMMA 5.3. *Let (y_{1N}, y_{2N}, q_N) be a critical point for f_N at level $c(T_N)$ given by Proposition 3.7 and assume ($\delta 4'$) holds. For $0 < \eta \le \eta_0$, let τ_N^1 and τ_N^2 be given by Lemma 5.1. Then*

$$\tau_N^2 - \tau_N^1 \le \frac{\bar{c}}{(1+\underline{\delta}-\alpha)V_\eta}, \tag{5.4}$$

with V_η as in (1.4);

$$w(t) \equiv \eta \frac{\sinh\sqrt{\bar{a}}t}{\sinh\sqrt{\bar{a}}\tau_N^1} \le q_N(t) \le \eta \frac{\sinh\sqrt{\underline{a}}t}{\sinh\sqrt{\underline{a}}\tau_N^1} \equiv z(t)$$

for all $t \in [0, \tau_N^1]$ and

$$\tilde{w}(t) \equiv \eta \frac{\sinh\sqrt{\bar{a}}(T_N - t)}{\sinh\sqrt{\bar{a}}(T_N - \tau_N^2)} \le 2\pi - q_N(t) \le \eta \frac{\sinh\sqrt{\underline{a}}(T_N - t)}{\sinh\sqrt{\underline{a}}(T_N - \tau_N^2)} \equiv \tilde{z}(t) \tag{5.5}$$

for all $t \in [\tau_N^2, T_N]$, where $\bar{a} = 2\mu(1+\bar{\delta})$ and $\underline{a} = (\mu/2)(1+\underline{\delta})$. Moreover, $\tau_N^1 \to +\infty$ and $T_N - \tau_N^2 \to +\infty$ as $N \to +\infty$.

PROOF. We give only a sketch of the proof, more details can be found in [10, Lemmas 13–15 and Remark 14]. Estimate (5.4) is an easy consequence of (3.10). Thanks to Lemma 5.1 we can use a maximum principle argument to obtain exponential estimates on q_N. Then, using the estimates on q_N and (3.9), we conclude that both τ_N^1 and $T_N - \tau_N^2$ diverge. □

In the following proposition we prove the first part of Theorem 1.11.

PROPOSITION 5.6. *Let (y_{1N}, y_{2N}, q_N) be as in Proposition 3.7 and assume ($\delta 4'$) holds. Then for all $N \in \mathbb{N}$ there is $\tau_N \in [\tau_N^1, \tau_N^2]$ such that, up to a subsequence,*

$$q_N(\cdot - \tau_N) \to q, \quad y_{1N}(\cdot - \tau_N) \to y_1, \quad y_{2N}(\cdot - \tau_N) \to y_2,$$

where (y_1, y_2, q) is a solution of problem (1.1) satisfying (1.3). Furthermore, $q(t) \in [0, 2\pi]$ for all t.

PROOF. Fix $\eta \le \eta_0$. Then, by Lemmas 5.1 and 5.3 we can find τ_N^1, τ_N^2 such that

$$\begin{aligned} &q_N(t) \in [0, \eta] && \forall t \in [0, \tau_N^1], \\ &q_N(t) \in [\eta, 2\pi - \eta] && \forall t \in [\tau_N^1, \tau_N^2], \\ &q_N(t) \in [2\pi - \eta, 2\pi] && \forall t \in [\tau_N^2, T_N], \\ &|\tau_N^2 - \tau_N^1| \le \frac{\bar{c}}{(1+\underline{\delta}-\alpha)V_\eta}. \end{aligned}$$

Let τ_N be the τ_N^1 corresponding to η_0, and

$$\tilde{q}_N(t) = q_N(t+\tau_N), \qquad t \in [-\tau_N, T_N - \tau_N],$$

so that $\tilde{q}_N(0) = \eta_0$ for all N. Also define

$$\tilde{y}_{1N}(t) = y_{1N}(t+\tau_N), \qquad \tilde{y}_{2N}(t) = y_{2N}(t+\tau_N) \qquad \forall t \in [-\tau_N, T_N - \tau_N].$$

Arguing as in the proof of Theorem 1.10 we find that $\tilde{q}_N \to q$ in L^∞_{loc}, $\tilde{y}_{iN} \to y_i$ in L^∞_{loc} ($i = 1, 2$), and (y_1, y_2, q) is solution of (1.1) satisfying (1.3); moreover, $q(t) \in [0, 2\pi]$ for all t. $\square$

In the following proposition we conclude the proof of Theorem 1.11.

PROPOSITION 5.7. *Assume* ($\delta 4'$) *holds. Take* $\varphi_1 \in (0, 2\pi)$ *and* φ_2 *chosen according to* ($\omega 1$)–($\omega 2$). *Then there exist* R_1, R_2 *and a solution* $(y_1(t), y_2(t), q(t))$ *of* (1.1) *satisfying* (1.3) *with*

$$R_{i\pm} = R_i, \qquad f_{i+} - f_{i-} = \varphi_i$$

for $i = 1, 2$. *If* (E) *holds, then* $\omega_1^2 R_1^2 + \omega_2^2 R_2^2 > 0$.

PROOF. Let T_N satisfy (2.9), let (y_{1N}, y_{2N}, q_N) be the solution of (PT) given by Proposition 3.7 and let (y_1, y_2, q) be the solution of (1.1) satisfying (1.3) obtained in Proposition 5.6 as the limit of (y_{1N}, y_{2N}, q_N) for $N \to +\infty$.

Fix $\varepsilon > 0$; let $\eta \leq \eta_0$ such that

$$\frac{1}{\omega_i} \|\nabla \delta\| \frac{4\mu\eta^2}{\sqrt{\underline{a}}} < \frac{\varepsilon}{4}, \qquad \forall i = 1, 2,$$

and consider τ_N such that $q_N(\tau_N) = \eta$ and $q_N(t) \geq \eta$ for all $t \geq \tau_N$.

As in Proposition 5.6, we define

$$\tilde{q}_N(t) = q_N(t+\tau_N), \qquad \tilde{y}_{iN}(t) = y_{iN}(t+\tau_N),$$

for all $t \in [-\tau_N, T_N - \tau_N]$, $i = 1, 2$. By (2.13), (2.14), $\tilde{y}_{iN}$ has the following expression for suitable constants A_{iN} and μ_{iN}:

$$\begin{aligned} \tilde{y}_{iN}(t) = {} & \frac{1}{\omega_i} \int_{-\tau_N}^{t} \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N}) V(\tilde{q}_N) \sin \omega_i (t-s)\, ds \\ & + A_{iN} \cos(\omega_i t + \omega_i \tau_N + \mu_{iN}), \end{aligned} \tag{5.8}$$

or

$$\begin{aligned} \tilde{y}_{iN}(t) = {} & -\frac{1}{\omega_i} \int_{t}^{T_N - \tau_N} \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N}) V(\tilde{q}_N) \sin \omega_i (t-s)\, ds \\ & + A_{iN} \cos(\omega_i t + \omega_i \tau_N + \mu_{iN} - \varphi_{iN}). \end{aligned} \tag{5.9}$$

We claim that there exists $N_1 \in \mathbb{N}$ such that for all $N \geq N_1$, all $t \in [-\tau_N, T_N - \tau_N]$ and $i = 1, 2$ we have (with the notation $\tilde{y}_N = (\tilde{y}_{1N}, \tilde{y}_{2N})$, $y = (y_1, y_2)$)

$$\frac{1}{\omega_i}\left|\int_{-\tau_N}^{t} \frac{\partial\delta}{\partial y_i}(\tilde{y}_N)V(\tilde{q}_N)\sin\omega_i(t-s)\,ds - \int_{-\infty}^{t}\frac{\partial\delta}{\partial y_i}(y)V(q)\sin\omega_i(t-s)\,ds\right| < \varepsilon$$

and

$$\frac{1}{\omega_i}\left|\int_{t}^{T_N-\tau_N} \frac{\partial\delta}{\partial y_i}(\tilde{y}_N)V(\tilde{q}_N)\sin\omega_i(t-s)\,ds - \int_{t}^{+\infty}\frac{\partial\delta}{\partial y_i}(y)V(q)\sin\omega_i(t-s)\,ds\right| < \varepsilon.$$

We give a proof only of the first inequality, the other can be proved in the same way. Since $\tilde{q}_N(-\tau_N) = 0$ and $\tilde{q}_N(T_N - \tau_N) = 2\pi$ we extend $\tilde{q}_N$ by setting

$$\tilde{q}_N(t) = \begin{cases} 0 & \forall t \leq -\tau_N, \\ 2\pi & \forall t \geq T_N - \tau_N, \end{cases}$$

and in view of (5.8) and (5.9) we extend $\tilde{y}_{iN}$ by setting

$$\tilde{y}_{iN}(t) = \begin{cases} A_{iN}\cos(\omega_i t + \omega_i\tau_N + \mu_{iN}) & \forall t \leq -\tau_N, \\ A_{iN}\cos(\omega_i t + \omega_i\tau_N + \mu_{iN} - \varphi_{iN}) & \forall t \geq T_N - \tau_N. \end{cases}$$

With these extensions the claim follows if we prove that

$$\frac{1}{\omega_i}\left|\int_{-\infty}^{t}\left[\frac{\partial\delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) - \frac{\partial\delta}{\partial y_i}(y_1, y_2)V(q)\right]\sin\omega_i(t-s)\,ds\right| < \varepsilon$$

for all $t \in \mathbb{R}$. Denote by $[a, b]$ the unique interval where q jumps from η to $2\pi - \eta$. Since $\tilde{q}_N \to q$ in L^∞_{loc}, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have $\tilde{q}_N(t) \leq \eta$ for all $t \leq a - 1$ and $\tilde{q}_N(t) \geq 2\pi - \eta$ for all $t \geq b + 1$. Then, using the exponential estimate given by Lemma 5.3, we have

$$\int_{-\infty}^{a-1} V(\tilde{q}_N) + \int_{b+1}^{+\infty} V(\tilde{q}_N) \leq \frac{4\mu\eta^2}{\sqrt{\underline{a}}}, \qquad \int_{-\infty}^{a-1} V(q) + \int_{b+1}^{+\infty} V(q) \leq \frac{4\mu\eta^2}{\sqrt{\underline{a}}}.$$

Let us consider the case $t > b + 1$ (the other cases being simpler). In view of the previous inequalities and by the choice of η we have, for all $N \geq N_0$,

$$\begin{aligned}
&\frac{1}{\omega_i}\left|\int_{-\infty}^{t}\left[\frac{\partial\delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) - \frac{\partial\delta}{\partial y_i}(y_1, y_2)V(q)\right]\sin\omega_i(t-s)\,ds\right| \\
&\quad\leq \frac{1}{\omega_i}\|\nabla\delta\|_\infty\int_{-\infty}^{a-1}[V(\tilde{q}_N) + V(q)] + \frac{1}{\omega_i}\|\nabla\delta\|_\infty\int_{b+1}^{t}[V(\tilde{q}_N) + V(q)] \\
&\qquad + \frac{1}{\omega_i}\int_{a-1}^{b+1}\left|\frac{\partial\delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) - \frac{\partial\delta}{\partial y_i}(y_1, y_2)V(q)\right| ds \\
&\quad\leq \frac{1}{\omega_i}\|\nabla\delta\|_\infty\frac{8\mu\eta^2}{\sqrt{\underline{a}}} + \frac{1}{\omega_i}\int_{a-1}^{b+1}\left|\frac{\partial\delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) - \frac{\partial\delta}{\partial y_i}(y_1, y_2)V(q)\right| ds \\
&\quad< \frac{\varepsilon}{2} + \frac{1}{\omega_i}\int_{a-1}^{b+1}\left|\frac{\partial\delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) - \frac{\partial\delta}{\partial y_i}(y_1, y_2)V(q)\right| ds.
\end{aligned}$$

Since $\tilde{q}_N \to q$ and $\tilde{y}_{iN} \to y_i$ in $L^\infty(a-1, b+1)$, using the dominated convergence theorem we find that

$$\int_{a-1}^{b+1} \left| \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N}) V(\tilde{q}_N) - \frac{\partial \delta}{\partial y_i}(y_1, y_2) V(q) \right| ds \to 0 \quad \text{as } N \to \infty.$$

Thus, there exists $N_1 \geq N_0$ such that for all $N \geq N_1$ we have

$$\frac{1}{\omega_i} \int_{a-1}^{b+1} \left| \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N}) V(\tilde{q}_N) - \frac{\partial \delta}{\partial y_i}(y_1, y_2) V(q) \right| ds < \frac{\varepsilon}{2}, \qquad i = 1, 2,$$

and the claim is proved.

Using (2.3) and (5.8) we deduce for all $t \in [-2\pi/\omega_i, 4\pi/\omega_i]$ the estimate

$$\begin{aligned}
&|R_{i-} \cos(\omega_i t + f_{i-}) - A_{iN} \cos(\omega_i t + \omega_i \tau_N + \mu_{iN})| \\
&\quad = \left| y_i(t) - \frac{1}{\omega_i} \int_{-\infty}^{t} \frac{\partial \delta}{\partial y_i}(y_1, y_2) V(q) \sin \omega_i (t-s)\, ds - \tilde{y}_{iN}(t) \right. \\
&\qquad \left. + \frac{1}{\omega_i} \int_{-\tau_N}^{t} \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N}) V(\tilde{q}_N) \sin \omega_i (t-s)\, ds \right| \\
&\quad \leq \frac{1}{\omega_i} \left| \int_{-\infty}^{t} \frac{\partial \delta}{\partial y_i}(y_1, y_2) V(q) \sin \omega_i (t-s)\, ds \right. \\
&\qquad \left. - \int_{-\tau_N}^{t} \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N}) V(\tilde{q}_N) \sin \omega_i (t-s)\, ds \right| + |y_i(t) - \tilde{y}_{iN}(t)| \\
&\quad < \varepsilon + |y_i(t) - \tilde{y}_{iN}(t)|.
\end{aligned}$$

Since $\tilde{y}_{iN} \to y_i$ in L^∞_{loc}, there exists $N_2 \geq N_1$ such that for all $N \geq N_2$ we have

$$|R_{i-} \cos(\omega_i t + f_{i-}) - A_{iN} \cos(\omega_i t + \omega_i \tau_N + \mu_{iN})| < 2\varepsilon \qquad \forall t \in [-2\pi/\omega_i, 4\pi/\omega_i], \tag{5.10}$$

and, arguing in the same way, we also have

$$|R_{i+} \cos(\omega_i t + f_{i+}) - A_{iN} \cos(\omega_i t + \omega_i \tau_N + \mu_{iN} - \varphi_{iN})| < 2\varepsilon \qquad \forall t \in [-2\pi/\omega_i, 4\pi/\omega_i]. \tag{5.11}$$

Rewriting (5.10) for $t = s - \varphi_{iN}/\omega_i$ we have, for all $N \geq N_2$,

$$|R_{i-} \cos(\omega_i s + f_{i-} - \varphi_{iN}) - A_{iN} \cos(\omega_i s + \omega_i \tau_N + \mu_{iN} - \varphi_{iN})| < 2\varepsilon \quad \forall s \in [0, 2\pi/\omega_i].$$

Putting together this estimate and (5.11) we obtain, for all $N \geq N_2$,

$$|R_{i-} \cos(\omega_i s + f_{i-} - \varphi_{iN}) - R_{i+} \cos(\omega_i s + f_{i+})| < 4\varepsilon \quad \forall s \in [0, 2\pi/\omega_i]. \tag{5.12}$$

Therefore, recalling that $\varphi_{1N} \equiv \varphi_1$ for all N, (5.12) becomes

$$|R_{1-} \cos(\omega_1 s + f_{1-} - \varphi_1) - R_{1+} \cos(\omega_1 s + f_{1+})| < 4\varepsilon \quad \forall s \in [0, 2\pi/\omega_1]\ \forall N \geq N_2;$$

since ε was arbitrarily chosen, this immediately implies that

$$R_{1-} = R_{1+} \quad \text{and} \quad f_{1+} \equiv f_{1-} - \varphi_1 \bmod 2\pi.$$

Moreover, since $\varphi_{2N} \to \varphi_2$, there exists $N_3 \geq N_2$ such that for all $N \geq N_3$ we have

$$\begin{aligned}
&|R_{2-}\cos(\omega_2 s + f_{2-} - \varphi_2) - R_{2+}\cos(\omega_2 s + f_{2+})| \\
&\qquad \leq |R_{2-}\cos(\omega_2 s + f_{2-} - \varphi_2) - R_{2-}\cos(\omega_2 s + f_{2-} - \varphi_{2N})| \\
&\qquad\quad + |R_{2-}\cos(\omega_2 s + f_{2-} - \varphi_{2N}) - R_{2+}\cos(\omega_2 s + f_{2+})| \\
&\qquad < \varepsilon + 4\varepsilon \quad \forall s \in [0, 2\pi/\omega_1]\ \forall N \geq N_3;
\end{aligned}$$

since ε was arbitrarily chosen, this implies that

$$R_{2-} = R_{2+} \quad \text{and} \quad f_{2+} \equiv f_{2-} - \varphi_2 \bmod 2\pi. \qquad \square$$

6. Proof of Theorem 1.14

In this section we will use the notation already introduced (see (1.13), (2.10)) and we will consider a sequence T_N satisfying (2.9).

Let us define

$$\bar{\Gamma}_N = \left\{ q \in \Gamma_N : \int_0^{T_N} V(q) \leq \frac{\bar{c}}{K} \right\};$$

we recall that for all N, $c(T_N) \leq \bar{c} := 2\pi^2 + (1 + \bar{\delta})\|V\|_\infty$.

Remark 6.1. By Lemma 2.11, if, for $i = 1, 2$, y_i is a solution of

$$\begin{aligned}
&\ddot{y}_i + \omega_i^2 y_i = \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q), \\
&y_i(0) - y_i(T_N) = \dot{y}_i(0) - \dot{y}_i(T_N) = 0,
\end{aligned}$$

with $q \in \bar{\Gamma}_N$, then

$$\begin{aligned}
&\|y_i\|_\infty \leq \frac{C_i^N}{\omega_i}\|\nabla\delta\|_\infty \bar{K}, \qquad i = 1, 2, \\
&|Q_i(y_i)| \leq \frac{C_i^N}{\omega_i}\|\nabla\delta\|_\infty^2 \bar{K}^2, \quad i = 1, 2.
\end{aligned} \tag{6.2}$$

Also Proposition 3.7 implies that any critical point (y_{1N}, y_{2N}, q_N) of the functional f_{T_N} at level $c(T_N)$ is such that $q_N \in \bar{\Gamma}_N$, and thus estimates (6.2) hold for y_{1N} and y_{2N}.

Lemma 6.3. *Let (y_{1N}, y_{2N}, q_N) be a critical point at level $c(T_N)$ as in Proposition 3.7. Then*

$$\left| c(T_N) - \int_0^{T_N} [\dot{q}_N^2/2 + (1 + \delta(0,0))V(q_N)] \right| \leq \frac{3}{2}\|\nabla\delta\|_\infty^2 \bar{K}^2 \left(\frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2} \right).$$

PROOF. In view of Remark 6.1, the estimates (6.2) hold and we have

$$\begin{aligned}
&\left| c(T_N) - \int_0^{T_N} [\dot q_N^2/2 + (1+\delta(0,0))V(q_N)] \right| \\
&\quad = \left| \frac{1}{2} Q_1(y_{1N}) + \frac{1}{2} Q_2(y_{2N}) + \int_0^{T_N} [\delta(y_{1N}, y_{2N}) - \delta(0,0)] V(q_N) \right| \\
&\quad \le \frac{1}{2} |Q_1(y_{1N})| + \frac{1}{2} |Q_2(y_{2N})| + \|\nabla\delta\|_\infty (\|y_{1N}\|_\infty + \|y_{2N}\|_\infty) \int_0^{T_N} V(q_N) \\
&\quad \le \frac{1}{2} \|\nabla\delta\|_\infty^2 \bar K^2 \left(\frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2} \right) + \|\nabla\delta\|_\infty^2 \bar K^2 \left(\frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2} \right) \\
&\quad = \frac{3}{2} \|\nabla\delta\|_\infty^2 \bar K^2 \left(\frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2} \right). \qquad \square
\end{aligned}$$

Let us define $\gamma_0(T_N)$ and q_{0N} such that

(6.4)
$$\gamma_0(T_N) = \min_{q\in\Gamma} \int_0^{T_N} [\dot q^2/2 + (1+\delta(0,0))V(q)] = \int_0^{T_N} [\dot q_{0N}^2/2 + (1+\delta(0,0))V(q_{0N})].$$

Arguing as in Lemma 3.6 it is easy to show that

$$\gamma_0(T_N) \le 2\pi^2 + (1+\delta(0,0))\|V\|_\infty \le \bar c.$$

LEMMA 6.5. *If $(y_1, y_2) \in E \times E$ is such that*

$$\|y_i\|_\infty \le \mathcal{V} \frac{1+\delta(0,0)}{\|\nabla\delta\|_\infty \gamma_0(T_N)}, \qquad i = 1, 2,$$

then

$$\int_0^{T_N} [\dot q_{0N}^2/2 + (1+\delta(y_1, y_2))V(q_{0N})] \le \gamma_0(T_N) + 2\mathcal{V};$$

moreover, if also $|Q_i(y_i)| \le 2\mathcal{V}$, $i = 1, 2$, then

$$|f(y_1, y_2, q_{0N})| \le \gamma_0(T_N) + 4\mathcal{V}.$$

PROOF. We have

$$\begin{aligned}
&\int_0^{T_N} [\dot q_{0N}^2/2 + (1+\delta(y_1, y_2))V(q_{0N})] \\
&\qquad = \gamma_0(T_N) + \int_0^{T_N} [\delta(y_1, y_2) - \delta(0,0)] V(q_{0N}) \\
&\qquad \le \gamma_0(T_N) + \|\nabla\delta\|_\infty (\|y_1\|_\infty + \|y_2\|_\infty) \int_0^{T_N} V(q_{0N}).
\end{aligned}$$

Then, by definition of $\gamma_0(T_N)$ and using the assumptions we obtain

$$\begin{aligned}
\int_0^{T_N} [\dot q_{0N}^2/2 + (1+\delta(y_1, y_2))V(q_{0N})] &\le \gamma_0(T_N) + 2\mathcal{V} \frac{1+\delta(0,0)}{\gamma_0(T_N)} \frac{\gamma_0(T_N)}{1+\delta(0,0)} \\
&= \gamma_0(T_N) + 2\mathcal{V},
\end{aligned}$$

and the first inequality is proved. Next

$$\begin{aligned}|f(y_1, y_2, q_{0N})| &\le \frac{1}{2}|Q_1(y_1)| + \frac{1}{2}|Q_2(y_2)| + \int_0^{T_N} [\dot{q}_{0N}^2/2 + (1+\delta(y_1, y_2))V(q_{0N})] \\ &\le 2\mathcal{V} + \gamma_0(T_N) + 2\mathcal{V} = \gamma_0(T_N) + 4\mathcal{V},\end{aligned}$$

and the second inequality is also proved. □

LEMMA 6.6. *There exists* $\chi > 0$ *such that*

$$\max\left\{\left\|\frac{\partial f}{\partial y_1}(y_1, y_2, q)\right\|_\infty, \left\|\frac{\partial f}{\partial y_2}(y_1, y_2, q)\right\|_\infty\right\} \ge \chi$$

for all $(y_1, y_2, q) \in E \times E \times \bar{\Gamma}$ *satisfying*

$$|f(y_1, y_2, q)| \le C_1 = 3\bar{c}$$

and at least one of the following four inequalities:

$$\begin{aligned}&\|y_1\|_\infty \ge \frac{2C_1^N}{\omega_1}\|\nabla\delta\|_\infty \bar{K}, \qquad |Q_1(y_1)| \ge \frac{2C_1^N}{\omega_1}(\|\nabla\delta\|_\infty \bar{K})^2, \\ &\|y_2\|_\infty \ge \frac{2C_2^N}{\omega_2}\|\nabla\delta\|_\infty \bar{K}, \qquad |Q_2(y_2)| \ge \frac{2C_2^N}{\omega_2}(\|\nabla\delta\|_\infty \bar{K})^2.\end{aligned} \tag{6.7}$$

PROOF. By contradiction, assume that there exists $(y_{1n}, y_{2n}, q_n) \in E \times E \times \bar{\Gamma}$ satisfying

$$|f(y_{1n}, y_{2n}, q_n)| \le C_1, \qquad \frac{\partial f}{\partial y_i}(y_{1n}, y_{2n}, q_n) \to 0 \quad \text{for all } i = 1, 2,$$

and at least one of the inequalities in (6.7). By Proposition 3.3, up to a subsequence, $y_{in} \to y_i$ in E, $q_n \to q$ in L^∞, (y_1, y_2, q) is a solution of (2.7) and satisfies at least one of the inequalities in (6.7). Moreover, $q \in \bar{\Gamma}_N$, since $V(q_n) \to V(q)$ almost everywhere and by the Fatou lemma

$$\int_0^T V(q) \le \liminf_{n\to+\infty} \int_0^T V(q_n) \le \bar{K}.$$

Thus we get a contradiction with Remark 6.1. □

LEMMA 6.8. *Assume*

$$\frac{C_{\varphi_i}}{\omega_i}\|\nabla\delta\|_\infty \bar{K} \max\{1, \|\nabla\delta\|_\infty \bar{K}\} < \frac{\mathcal{V}}{2\bar{K}}, \qquad i = 1, 2.$$

Then for N *large enough,*

$$\gamma_0(T_N) - 3\mathcal{V} \le c(T_N) \le \gamma_0(T_N) + 8\mathcal{V}.$$

PROOF. Let (y_{1N}, y_{2N}, q_N) be a critical point at level $c(T_N)$. By Lemma 6.3 and by definition of $\gamma_0(T_N)$ we have

$$\begin{aligned} c(T_N) &\geq \int_0^{T_N} [\dot{q}_N/2 + (1+\delta(0,0))V(q_N)] - \frac{3}{2}\left(\frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2}\right)\|\nabla\delta\|_\infty^2 \bar{K}^2 \\ &\geq \gamma_0(T_N) - \frac{3}{2}\left(\frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2}\right)\|\nabla\delta\|_\infty^2 \bar{K}^2. \end{aligned}$$

Since $C_1^N = C_{\varphi_1}$ and $C_2^N \to C_{\varphi_2}$, for N large we have

$$c(T_N) \geq \gamma_0(T_N) - 3\mathcal{V}.$$

In order to prove the other inequality we will construct an admissible path $h = (h_1, h_2, q_{0N}) \in \mathcal{H}$ along which the value of the functional f is less than $\gamma_0(T_N) + 8\mathcal{V}$. To show the existence of such functions h_1, h_2 we will deform, using a suitable pseudo-gradient vector field, the identity map $\mathrm{Id}\colon E_1^- \times E_2^- \to E_1^- \times E_2^-$.

Let $\varphi\colon \mathbb{R} \to [0,1]$ and $\psi\colon [0,+\infty) \to [0,1]$ be defined by

$$\varphi(s) = \begin{cases} 0, & s \leq 0, \\ \dfrac{2s}{\gamma_0(T_N) + 4\mathcal{V}}, & 0 \leq s \leq (\gamma_0(T_N) + 4\mathcal{V})/2, \\ 1, & (\gamma_0(T_N) + 4\mathcal{V})/2 \leq s \leq C_1 := 3\bar{c}, \\ C_1 + 1 - s, & C_1 \leq s \leq C_1 + 1, \\ 0, & s \geq C_1 + 1, \end{cases}$$

$$\psi(s) = \begin{cases} 0, & 0 \leq s \leq 1, \\ s - 1, & 1 \leq s \leq 2, \\ 1, & s \geq 2, \end{cases}$$

and define the vector field $v\colon E \times E \to E \times E$ by

$$\begin{aligned} v_i(y_1, y_2) = -\Bigg[&\psi\left(\frac{\omega_1\|y_1\|_\infty}{C_1^N\|\nabla\delta\|_\infty \bar{K}}\right) + \psi\left(\frac{\omega_2\|y_2\|_\infty}{C_2^N\|\nabla\delta\|_\infty \bar{K}}\right) + \psi\left(\frac{\omega_1|Q_1(y_1)|}{C_1^N\|\nabla\delta\|_\infty^2 \bar{K}^2}\right) \\ &+ \psi\left(\frac{\omega_2|Q_2(y_2)|}{C_2^N\|\nabla\delta\|_\infty^2 \bar{K}^2}\right)\Bigg] \frac{\varphi(f(y_1, y_2, q_{0N}))\frac{\partial f}{\partial y_i}(y_1, y_2, q_{0N})}{1 + \left\|\frac{\partial f}{\partial y_i}(y_1, y_2, q_{0N})\right\|}. \end{aligned}$$

Since v is a bounded locally Lipschitz function of (y_1, y_2), the Cauchy problem

$$\begin{cases} \dfrac{d\eta}{ds}(s, y_1, y_2) = v(\eta(s, y_1, y_2)), \\ \eta(0, y_1, y_2) = (y_1, y_2), \end{cases}$$

has a unique solution for every $(y_1, y_2) \in E \times E$, defined on $[0, +\infty)$.

We claim that, setting $\tau_3 = (C_1 - \gamma_0(T_N))(1+\chi)/\chi^2$ (χ given by Lemma 6.6), we have

$$f(\eta_1(\tau_3, y_1, y_2), \eta_2(\tau_3, y_1, y_2), q_{0N}) \leq \gamma_0(T_N) + 8\mathcal{V}$$

for all (y_1, y_2) such that $f(y_1, y_2, q_{0N}) \le C_1$. First of all,

$$\begin{aligned}
&\frac{df}{ds}(\eta_1(s, y_1, y_2), \eta_2(s, y_1, y_2), q_{0N}) \\
&\quad = \left\langle \frac{\partial f}{\partial y_1}(\eta_1, \eta_2, q_{0N}), \frac{d\eta_1}{ds}(s, y_1, y_2) \right\rangle + \left\langle \frac{\partial f}{\partial y_2}(\eta_1, \eta_2, q_{0N}), \frac{d\eta_2}{ds}(s, y_1, y_2) \right\rangle \\
&\quad = -[\psi + \psi + \psi + \psi]\varphi \left[\frac{\| \frac{\partial f}{\partial y_1} \|^2}{1 + \| \frac{\partial f}{\partial y_1} \|} + \frac{\| \frac{\partial f}{\partial y_2} \|^2}{1 + \| \frac{\partial f}{\partial y_2} \|} \right] \le 0,
\end{aligned}$$

and hence $f(\eta(s, y_1, y_2), q_{0N})$ is a nonincreasing function of s and the claim follows for all (y_1, y_2) such that $f(y_1, y_2, q_{0N}) \le \gamma_{0N} + 8\mathcal{V}$.

Take now any $(y_1, y_2) \in E \times E$ such that

$$\gamma_0(T_N) + 8\mathcal{V} < f(y_1, y_2, q_{0N}) \le C_1.$$

Assume, by contradiction, that

$$f(\eta_1(s, y_1, y_2), \eta_2(s, y_1, y_2), q_{0N}) > \gamma_0(T_N) + 8\mathcal{V}, \qquad \forall s \in [0, \tau_3].$$

Fix $s \in [0, \tau_3]$. If $\|\eta_i(s, y_1, y_2)\|_\infty \ge (2C_i^N/\omega_i)\|\nabla\delta\|_\infty \bar{K}$ for $i = 1$ or 2 then by Lemma 6.6 we have

$$\max\left\{ \left\| \frac{\partial f}{\partial y_1}(\eta_1, \eta_2, q_{0N}) \right\|_\infty, \left\| \frac{\partial f}{\partial y_2}(\eta_1, \eta_2, q_{0N}) \right\|_\infty \right\} \ge \chi$$

and, by definition of ψ,

$$\psi\left(\frac{\omega_i \|\eta_i(s, y_1, y_2)\|_\infty}{C_i^N \|\nabla\delta\|_\infty \bar{K}} \right) = 1.$$

Otherwise $\|\eta_i(s, y_1, y_2)\|_\infty < (2C_i^N/\omega_i)\|\nabla\delta\|_\infty \bar{K}$ for $i = 1, 2$. Using the assumption and the definition of $\bar{K}$, we obtain, for $i = 1, 2$ and N sufficiently large,

$$\begin{aligned}
\|\eta_i\|_\infty &< \frac{2C_i^N}{\omega_i}\|\nabla\delta\|_\infty^2 \bar{K}^2 \frac{1}{\|\nabla\delta\|_\infty \bar{K}} < \frac{\mathcal{V}}{\bar{K}} \frac{1}{\|\nabla\delta\|_\infty \bar{K}} \\
&\le \frac{\mathcal{V}}{\bar{c}} \frac{1}{\|\nabla\delta\|_\infty} \frac{1 + \underline{\delta} - \alpha}{\bar{c}} < \frac{\mathcal{V}}{\gamma_0(T_N)} \frac{1}{\|\nabla\delta\|_\infty}(1 + \delta(0, 0)),
\end{aligned}$$

so that the first conclusion of Lemma 6.5 holds for $(\eta_1(s, y_1, y_2), \eta_2(s, y_1, y_2))$. Then if

$$|Q_1(\eta_1(s, y_1, y_2))| < \frac{2C_1^N}{\omega_1}\|\nabla\delta\|_\infty^2 \bar{K}^2$$

(the same argument applies if $|Q_2(\eta_2(s, y_1, y_2))| < (2C_2^N/\omega_2)\|\nabla\delta\|_\infty^2 \bar{K}^2$), for N

sufficiently large we have

$$\begin{aligned}\frac{1}{2}Q_2(\eta_2(s,y_1,y_2)) &= f(\eta_1(s,y_1,y_2),\eta_2(s,y_1,y_2),q_{0N})\\ &\quad -\frac{1}{2}Q_1(\eta_1(s,y_1,y_2)) - \int_0^T [\dot q_{0N}^2/2 + (1+\delta(\eta_1,\eta_2))V(q_{0N})]\\ &> \gamma_0(T_N) + 8\mathcal{V} - \frac{C_1^N}{\omega_1}\|\nabla\delta\|_\infty^2 \bar K^2 - \gamma_0(T_N) - 2\mathcal{V}\\ &= 6\mathcal{V} - \frac{C_1^N}{\omega_1}\|\nabla\delta\|_\infty^2\bar K^2 \ge 6\mathcal{V} - \frac{\mathcal{V}}{2\bar K} \ge 5\mathcal{V} \ge 5\frac{C_2^N}{\omega_2}\|\nabla\delta\|_\infty^2\bar K^2,\end{aligned}$$

that is,

$$Q_2(\eta_2(s,y_1,y_2)) \ge 10\frac{C_2^N}{\omega_2}\|\nabla\delta\|_\infty^2\bar K^2,$$

so that by Lemma 6.6 we have

$$\max\left\{\left\|\frac{\partial f}{\partial y_1}(\eta_1,\eta_2,q_{0N})\right\|_\infty, \left\|\frac{\partial f}{\partial y_2}(\eta_1,\eta_2,q_{0N})\right\|_\infty\right\} \ge \chi$$

and, by definition of ψ,

$$\psi\left(\frac{\omega_2|Q_2(\eta_2(s,y_1,y_2))|}{C_2^N\|\nabla\delta\|_\infty^2\bar K^2}\right) = 1.$$

Therefore we always have

$$\max\left\{\left\|\frac{\partial f}{\partial y_1}(\eta_1,\eta_2,q_{0N})\right\|_\infty, \left\|\frac{\partial f}{\partial y_2}(\eta_1,\eta_2,q_{0N})\right\|_\infty\right\} \ge \chi$$

and

$$\begin{aligned}&\psi\left(\frac{\omega_1\|\eta_1(s,y_1,y_2)\|_\infty}{C_1^N\|\nabla\delta\|_\infty\bar K}\right) + \psi\left(\frac{\omega_2\|\eta_2(s,y_1,y_2)\|_\infty}{C_2^N\|\nabla\delta\|_\infty\bar K}\right)\\ &\qquad + \psi\left(\frac{\omega_1|Q_1(\eta_1(s,y_1,y_2))|}{C_1^N\|\nabla\delta\|_\infty^2\bar K^2}\right) + \psi\left(\frac{\omega_2|Q_2(\eta_2(s,y_1,y_2))|}{C_2^N\|\nabla\delta\|_\infty^2\bar K^2}\right) \ge 1.\end{aligned}$$

We also have, for all $s \in [0,\tau_3]$,

$$\varphi(f(\eta_1(s,y_1,y_2),\eta_2(s,y_1,y_2),q_{0N})) = 1.$$

Then

$$\begin{aligned}C_1 - \gamma_0(T_N) - 8\mathcal{V} &> f(y_1,y_2,q_{0N}) - f(\eta_1(\tau_3,y_1,y_2),\eta_2(\tau_3,y_1,y_2),q_{0N})\\ &= -\int_0^{\tau_3}\frac{df}{ds}(\eta_1(s,y_1,y_2),\eta_2(s,y_1,y_2),q_{0N})\,ds\\ &\ge \frac{\chi^2}{1+\chi}\tau_3 = C_1 - \gamma_0(T_N),\end{aligned}$$

a contradiction which proves the claim.

We now define

$$h\colon E_1^- \times E_2^- \to E \times E \times \bar{\Gamma}, \tag{6.9}$$
$$(y_1, y_2) \mapsto h(y_1, y_2) = (\eta_1(\tau_3, y_1, y_2), \eta_2(\tau_3, y_1, y_2), q_{0N}).$$

There exists L large such that

$$f(y_1, y_2, q_{0N}) < 0 \quad \forall \|(y_1, y_2)\| \geq L,$$

and hence, by definition of (η_1, η_2),

$$h(y_1, y_2) = (y_1, y_2, q_{0N}) \quad \forall \|(y_1, y_2)\| \geq L,$$

which shows that $h \in \mathcal{H}$. Finally, since $f(y_1, y_2, q_{0N}) \leq C_1$ for all $(y_1, y_2) \in E_1^- \times E_2^-$, we have

$$f(h(y_1, y_2)) \leq \gamma_0(T_N) + 8\mathcal{V} \quad \forall (y_1, y_2) \in E_1^- \times E_2^-, \tag{6.10}$$

and

$$c(T_N) \leq \gamma_0(T_N) + 8\mathcal{V}. \qquad \square$$

LEMMA 6.11. *Let $(y_{1N}, y_{2N}, q_N) \in E \times E \times \bar{\Gamma}$ be a critical point at level $c(T_N)$ and assume*

$$\frac{C_{\varphi_i}}{\omega_i}\|\nabla\delta\|_\infty \bar{K} \max\{1, \|\nabla\delta\|_\infty \bar{K}\} < \frac{\mathcal{V}}{2\bar{K}}, \qquad i = 1, 2.$$

Then, for N large, there exists $\eta_1 \leq \eta_0$ such that for all $0 < \eta \leq \eta_1$, there exist $0 < \tau_1^N < \tau_2^N < T_N$ such that

$$\begin{aligned} &0 \leq q_N(t) \leq \eta && \forall t \in [0, \tau_1^N],\\ &q_N(t) \in [\eta, 2\pi - \eta] && \forall t \in [\tau_1^N, \tau_2^N],\\ &2\pi - \eta \leq q_N(t) \leq 2\pi && \forall t \in [\tau_2^N, T_N]. \end{aligned}$$

PROOF. In the proof we will omit the superscripts and subscripts N for brevity. Let $\eta \leq \eta_1$, let $\tau_1 = \inf\{s \in [0, T] \mid q(s) > \eta\}$ and $\tau_2 = \sup\{s \in [0, T] \mid q(s) < 2\pi - \eta\}$. If the lemma does not hold, then arguing as in Lemma 5.1, we deduce that there is $\tau_1' \in (\tau_1, T]$ such that $q(\tau_1') = \eta$ and

$$\int_{\tau_1}^{\tau_1'} [\dot{q}^2/2 + (1 + \delta(0, 0))V(q)] \geq 2\bar{\eta}\sqrt{\frac{1 + \delta(0, 0)}{2}}\, V_{\bar{\eta}/2}. \tag{6.12}$$

Now we define a new function $\bar{q} \in \bar{\Gamma}$ by setting

$$\bar{q}(t) = \begin{cases} 0, & 0 \leq t \leq \tau_1' - 1,\\ \eta t - \eta\tau_1' + \eta, & \tau_1' - 1 \leq t \leq \tau_1',\\ q(t), & \tau_1' \leq t \leq T. \end{cases}$$

In view of Remark 6.1 we have, for N sufficiently large,

$$\frac{1}{2}Q_1(y_1) + \frac{1}{2}Q_2(y_2) \geq -\frac{1}{2}\left(\frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2}\right)\|\nabla\delta\|_\infty^2 \bar{K}^2 > -\mathcal{V},$$

and

$$\begin{aligned}\int_0^T [\delta(y_1, y_2) - \delta(0,0)]V(q) &\geq -\|\nabla\delta\|_\infty(\|y_1\|_\infty + \|y_2\|_\infty)\int_0^T V(q)\\ &\geq -\left(\frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2}\right)\|\nabla\delta\|_\infty^2 \bar{K}^2 > -2\mathcal{V}.\end{aligned}$$

Then, by the previous two estimates, we have

$$\begin{aligned}c(T) &= \frac{1}{2}Q_1(y_1) + \frac{1}{2}Q_2(y_2) + \int_0^T [\dot{q}^2/2 + (1+\delta(y_1,y_2))V(q)]\\ &> -\mathcal{V} + \int_0^T [\dot{q}^2/2 + (1+\delta(0,0))V(q)] + \int_0^T [\delta(y_1,y_2) - \delta(0,0)]V(q)\\ &> \int_0^T [\dot{q}^2/2 + (1+\delta(0,0))V(q)] - 3\mathcal{V}.\end{aligned} \tag{6.13}$$

By (6.12) and by definition of $\mathcal{V}$ (see (1.13)), we have

$$\begin{aligned}&\int_0^T [\dot{q}^2/2 + (1+\delta(0,0))V(q)]\\ &\geq 2\bar{\eta}\sqrt{\frac{1+\delta(0,0)}{2}}V_{\bar{\eta}/2} + \int_{\tau_1'}^T [\dot{q}^2/2 + (1+\delta(0,0))V(q)]\\ &= 24\mathcal{V} + \int_0^T [\dot{\bar{q}}^2/2 + (1+\delta(0,0))V(\bar{q})] - \int_{\tau_1'-1}^{\tau_1'} [\dot{\bar{q}}^2/2 + (1+\delta(0,0))V(\bar{q})]\\ &\geq 24\mathcal{V} + \gamma_0(T) - \eta^2/2 - (1+\delta(0,0))V(\eta)\\ &\geq 24\mathcal{V} + \gamma_0(T) - \eta^2[1/2 + \mu(1+\delta(0,0))].\end{aligned} \tag{6.14}$$

Then, putting together (6.13) and (6.14) and using Lemma 6.8 we obtain

$$\begin{aligned}c(T) &> -3\mathcal{V} + 24\mathcal{V} + \gamma_0(T) - \eta^2[1/2 + \mu(1+\delta(0,0))]\\ &\geq -3\mathcal{V} + 24\mathcal{V} + c(T) - 8\mathcal{V} - \eta^2[1/2 + \mu(1+\delta(0,0))]\\ &= 13\mathcal{V} - \eta^2[1/2 + \mu(1+\delta(0,0))] + c(T)\\ &> \bar{\eta}\sqrt{\frac{1+\delta(0,0)}{2}}V_{\bar{\eta}/2} - \eta^2[1/2 + \mu(1+\delta(0,0))] + c(T).\end{aligned}$$

Therefore, since $\eta \leq \eta_1$, choosing η_1 small enough, we get the contradiction $c(T) > c(T)$. □

PROOF OF THEOREM 1.14. Let $(y_{1N}, y_{2N}, q_N) \in E \times E \times \bar{\Gamma}$ be a critical point at level $c(T_N)$. For $0 < \eta \leq \eta_1$, let τ_1^N and τ_2^N be given by Lemma 6.11. Then, we can repeat the same arguments of Lemma 5.3, Proposition 5.6 and Proposition 5.7, and the theorem is proved. □

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