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Statistical mechanics. — *Correlation inequalities for spin glass in one dimension*, by PIERLUIGI CONTUCCI and FRANCESCO UNGUENDOLI, communicated by Sandro Graffi on 14 March 2008.

ABSTRACT. — We prove two inequalities for the direct and truncated correlation functions for the nearest-neighbour one-dimensional Edwards–Anderson model with symmetric quenched disorder. The second inequality has the opposite sign of the GKS inequality of type II. In the non-symmetric case with positive average we show that while the direct correlation keeps its sign the truncated one changes sign when crossing a suitable line in the parameter space. That line separates the regions satisfying the second GKS inequality and the one proved here.

KEY WORDS: Spin glasses; correlation inequalities; one-dimensional systems.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 35J65.

1. INTRODUCTION AND RESULTS

In a recent paper [CL] a correlation inequality was proved for spin systems with quenched symmetric random interaction in arbitrary dimension, extending a previous result for the Gaussian case [CG1]. That inequality yields results for spin glasses similar to those obtained for ferromagnetic systems from the first GKS inequality [Gr1, Gr2, KS], e.g. it gives monotonicity of the pressure in the volume and bounds on the surface pressure. Other inequalities have been considered: in particular the extension to non-symmetric interactions and possible versions of a second type GKS inequality. Within ferromagnetic systems the GKS inequality of the second type has indeed many consequences among which monotonicity in the volume of the correlation functions and relations in different lattices between critical temperatures and exponents.

In this work we study the $d = 1$ case (the spin chain) with nearest neighbour interaction. In the same spirit of the GKS systems no assumption of translation invariance is made on the interaction distributions. Such mild hypotheses make our result non-trivial because it does not rely on a general exact solution, which can be obtained only at the thermodynamic limit and only for some specific distributions of the disorder. For $d = 1$ we prove that both the inequality of the first type does extend to the non-symmetric case and an inequality of the second type indeed holds in the symmetric case. A similar result with a complete proof of inequalities of type I and II has been obtained so far only along the Nishimori line [CMN, MNC]. The theorem is obtained by exploiting the fact that for $d = 1$ the partition function expressed as sum over loops contains at most two terms. This fact does not generalise to higher dimensions.

Let us consider a spin chain with periodic boundary condition

$$H(\sigma, J) = -\sum_{i=1}^{N} J_i \sigma_i \sigma_{i+1}$$

with $\sigma_{N+1} = \sigma_1$. Here, as usual, $\sigma_i = \pm 1$, $i = 1, \ldots, N$. The random variables J_i have independent distributions $p^{(i)}(J_i)$. We consider three different hypotheses, which will be called system I, II and III in the remaining part of the paper:

I.

$$p^{(i)}(|J_i|) \geq p^{(i)}(-|J_i|), \quad \forall i, \forall |J_i| \in \mathbb{R}^+. \tag{1.1}$$

II. The J_i are symmetric around a positive mean $\mu_i > 0$:

$$p^{(i)}(\mu_i + |J_i|) = p^{(i)}(\mu_i - |J_i|), \quad \forall i, \forall |J_i| \in \mathbb{R}^+. \tag{1.2}$$

In the case of discrete variables: $J_i = \mu_i \pm J^{(i)}$, $p^{(i)}(\mu_i + J^{(i)}) = p^{(i)}(\mu_i - J^{(i)}) = 1/2$, we assume that $J^{(i)} > \mu_i$ (see below for further explanations) and we introduce the notations:

$$a_i = \mu_i + J^{(i)}, \quad -b_i = \mu_i - J^{(i)}, \quad a_i, b_i > 0.$$

III. The J_i are discrete variables taking on values $\pm J^{(i)}$ with $J^{(i)} > 0$ such that

$$\alpha := \prod_i (p_i - q_i) \geq 0 \quad \text{where} \quad p_i = p^{(i)}(J^{(i)}),\ q_i = p^{(i)}(-J^{(i)}).$$

Let ω_h be the thermal average of $\sigma_h \sigma_{h+1}$, $\omega_{h,k}$ that of $\sigma_h \sigma_{h+1} \sigma_k \sigma_{k+1}$, and Av[$\cdot$] the average over the quenched disorder.

Our main results are:

PROPOSITION 1.1. *For all three systems,*

$$\mathrm{Av}[J_h \omega_h] > 0, \quad \forall h = 1, \ldots, N. \tag{1.3}$$

PROPOSITION 1.2. *For systems I and III with $\alpha = 0$,*

$$\mathrm{Av}[J_h J_k (\omega_{hk} - \omega_h \omega_k)] < 0, \quad \forall h, k = 1, \ldots, N,\ h \neq k. \tag{1.4}$$

PROPOSITION 1.3. *For system III with $\alpha > 0$, for every l there exists a curve $\alpha(J^{(l)})$ in the $(J^{(l)}, \alpha)$ quadrant such that the quantity*

$$\mathrm{Av}[J_h J_k (\omega_{hk} - \omega_h \omega_k)] \tag{1.5}$$

changes its sign from negative to positive when crossing the curve $\alpha(J^{(l)})$ by increasing α and such that on the curve $\alpha(J^{(l)})$,

$$\mathrm{Av}[J_h J_k (\omega_{hk} - \omega_h \omega_k)] = 0, \quad \forall h, k = 1, \ldots, N,\ h \neq k. \tag{1.6}$$

Moreover $\mathrm{Av}[J_h J_k (\omega_{hk} - \omega_h \omega_k)]$ *is increasing in α along the* $J^{(l)} = \text{const}$ *lines.*

2. PROOFS

We start by proving the following lemmata.

LEMMA 2.1. *System III can be rewritten as*

$$H(\tau, K) = -K_N \tau_N \tau_{N-1} - \sum_{i=1}^{N-1} J^{(i)} \tau_i \tau_{i+1} \tag{2.7}$$

with

$$K_N = J^{(N)} \prod_{i=1}^{N} \operatorname{sgn}(J_i) = \pm J^{(N)}. \tag{2.8}$$

Setting $P = \operatorname{prob}(K_N = J^{(N)})$ *and* $Q = \operatorname{prob}(K_N = -J^{(N)})$ *we have*

$$P = \frac{1 + \prod_i (p_i - q_i)}{2}, \tag{2.9}$$

$$Q = \frac{1 - \prod_i (p_i - q_i)}{2}. \tag{2.10}$$

LEMMA 2.2. *Consider system II with discrete variables and assume that* $\mu_h = 0$ *for at least one h. Such a system can be rewritten as*

$$H(\tau, K) = -\sum_{i=1}^{N} K_i \tau_i \tau_{i+1},$$

where

$$K_h = J_h = \pm a_h, \quad a_h > 0, \tag{2.11}$$

$$K_i = \begin{cases} a_i > 0, \\ b_i > 0, \end{cases} \tag{2.12}$$

the two cases having probability $1/2$.

PROOF OF LEMMA 2.1. The proof is based on the gauge transformation $\alpha_j = \prod_{1 \le i < j} \operatorname{sgn}(J_i)$ for $2 \le j \le N$, $\alpha_1 = 1$. Set $\tau_i = \alpha_i \sigma_i$. H is given by (2.7) with $K_N = J^{(N)} \prod_{i=1}^N \operatorname{sgn}(J_i)$. We now have to compute the new probability measure for $\prod_{i=1}^N \operatorname{sgn}(J_i)$. We obtain

$$\operatorname{Av}\Big[\prod_{i=1}^{N} \operatorname{sgn}(J_i)\Big] = \prod_{i=1}^{N} \operatorname{Av}[\operatorname{sgn}(J_i)] = \prod_{i=1}^{N} (p_i - q_i) = P - Q.$$

PROOF OF LEMMA 2.2. Group the bond configurations in pairs that only differ by the sign of J_h and gauge transform them using the same transformation of Lemma 2.1. What we obtain is

$$K_i = |J_i| > 0, \quad K_h = J_h = \pm J^{(h)},$$

Moreover, since $p(K^{(h)}) = p(J^{(h)})$ or $p(K^{(h)}) = p(-J^{(h)})$, $p(K^{(l)}) = 1/2$ for all l.

Introduce, for system III, the following shorthand notations:

$$C_i := \cosh(K^{(i)}) = \cosh(J^{(i)}), \qquad S_i := \sinh(K^{(i)}) = \sinh(J^{(i)}).$$

PROOF OF PROPOSITION 1.1. The partition function and the correlation of an N-spin chain with periodic boundary conditions can be written as

$$Z = \prod_i C_i + \prod_i S_i, \tag{2.13}$$

$$\omega_h = \frac{1}{Z}\Big[S_h \prod_{i \neq h} C_i + C_h \prod_{i \neq h} S_i\Big]. \tag{2.14}$$

SYSTEM III: Using Lemma 2.1 one has

$$\begin{aligned} \mathrm{Av}_{\{J\}}[J_h\omega_h] &= \mathrm{Av}_{(K_h)}[K_h\omega_h] = K^{(h)}\{P\omega|_{k_h=k^{(h)}} - Q\omega|_{k_h=-k^{(h)}}\} \\ &= K^{(h)}\{Q[\omega|_{k_h=k^{(h)}} - \omega|_{k_h=-k^{(h)}}] + (P-Q)\omega|_{k_h=k^{(h)}}\} \geq 0 \end{aligned} \tag{2.15}$$

due to the first Griffiths inequality for ferromagnetic systems.

SYSTEM I: The discrete distribution is a special case of system III. The continuous case can be obtained by writing the full probability distribution as a product of a discrete part with symmetric values with respect to zero, and a continuos one from 0 and ∞. More explicitly, considering a function f,

$$\mathrm{Av}_J[f(J)] = \int_0^{+\infty} (f(J)p(J) + f(-J)p(-J))\, dJ,$$

and writting $\mathrm{av}_{\{J\}}[\cdot]$ for the discrete average one has

$$\mathrm{Av}_{\{J\}}[J_h\omega_h] = \int_0^{+\infty} \cdots \int_0^{+\infty} \mathrm{av}_{\{J\}}[J_h\omega_h]\, dJ_1 \cdots dJ_N.$$

By (1.1) and (2.15) the quantity $\mathrm{av}_{\{J\}}[\cdot]$ is positive, and the positivity of the whole average follows immediately.

SYSTEM II: Since the pressure is a convex function of the μ_i's (the second derivative is a variance) we can prove our theorem for $\mu_h = 0$. In the discrete case, using Lemma 2.2 we observe that the average over the K_i for $i \neq h$ is on positive values and does not affect the sign. From $\mu_h = 0$ it follows that $P = Q = 1/2$ and we obtain the conclusion from (2.15).

The result for the continuous case is obtained from the discrete one as for system I.

PROOF OF PROPOSITION 1.2. For discrete variables (system III), using the standard hyperbolic expansion

$$\omega_{hk} = \frac{1}{Z}\Big[S_h S_k \prod_{i \neq h,k} C_i + C_h C_k \prod_{i \neq h,k} S_i\Big]$$

we obtain

$$\begin{aligned}
\omega_{hk} - \omega_h\omega_k &= \frac{1}{Z^2}\Big\{\Big(S_hS_k\prod_{i\neq h,k} C_i + C_hC_k\prod_{i\neq h,k} S_i\Big)\Big(\prod_i C_i + \prod_i S_i\Big) \\
&\quad - \Big(S_h\prod_{i\neq h} C_i + C_h\prod_{i\neq h} S_i\Big)\Big(S_k\prod_{i\neq k} C_i + C_k\prod_{i\neq k} S_i\Big)\Big\} \\
&= \frac{1}{Z^2}\Big\{\prod_{i\neq h,k} C_iS_i\cdot(C_h^2C_k^2 + S_h^2S_k^2 - C_h^2S_k^2 - S_h^2C_k^2)\Big\} \\
&= \frac{\prod_{i\neq h,k} C_iS_i}{(\prod_i C_i + \prod_i S_i)^2}.
\end{aligned}$$

If at least one of the random variables is symmetric we have $P = Q = 1/2$; using Lemma 2.1 one has

$$\begin{aligned}
(2.16)\qquad & \mathrm{Av}[K_hK_k(\omega_{hk} - \omega_h\omega_k)] \\
&= J^{(k)}\cdot \mathrm{Av}_{(K_N)}\Big[\frac{K_h\cdot\prod_{i\neq h,k} C_iS_i}{(\prod_i C_i + \prod_i S_i)^2}\Big] \\
&= J^{(k)}J^{(h)}\prod_{i\neq h,k} C_iS_i\cdot\frac{1}{2}\Big\{\frac{1}{(\prod_i C_i + \prod_i S_i)^2} - \frac{1}{(\prod_i C_i - \prod_i S_i)^2}\Big\} \\
&= -2\,J^{(k)}J^{(h)}\prod_{i\neq h,k} C_iS_i\cdot\frac{\prod_i C_iS_i}{(\prod_i C_i^2 - \prod_i S_i^2)^2} < 0.
\end{aligned}$$

The extension to the continuous case is as above.

PROOF OF PROPOSITION 1.3. Let $\alpha > 0$ or equivalently $P = (1+\alpha)/2 > 1/2 > Q = (1-\alpha)/2$. As in (2.16), we obtain

$$\begin{aligned}
& \mathrm{Av}[K_hK_k(\omega_{hk} - \omega_h\omega_k)] \\
&= J^{(k)}J^{(h)}\prod_{i\neq h,k} C_iS_i\cdot\frac{P(\prod C_i - \prod S_i)^2 - Q(\prod C_i + \prod S_i)^2}{(\prod C_i^2 - \prod S_i^2)^2} \\
&= \frac{J^{(k)}J^{(h)}\prod_{i\neq h,k} C_iS_i}{(\prod_i C_i^2 - \prod_i S_i^2)^2}\cdot\Big\{(P-Q)\Big(\prod_i C_i^2 + \prod_i S_i^2\Big) - 2\prod_i C_iS_i\Big\}.
\end{aligned}$$

The sign of the above expression is, by inspection, the same as that of the curly parentheses. Set

$$g(\alpha;\{J\}) := \alpha\Big(\prod_i C_i^2 + \prod_i S_i^2\Big) - 2\prod_i C_iS_i.$$

One obtains:

- $\alpha = 0$ (zero mean spin glass) $\Rightarrow g(\alpha;\{J\}) < 0$;
- $\alpha = 1$ (ferromagnetic) $\Rightarrow g(\alpha;\{J\}) = (\prod_i C_i - \prod_i S_i)^2 > 0$;

- for all $J^{(l)}$, $g(\alpha; \{J\})$ is an increasing function of α;
- $\mathrm{Av}_{(K_h)}[K_h K_k(\omega_{hk} - \omega_h\omega_k)] = 0$ on the $(J^{(l)}, \alpha)$ plane curve with $J^{(l)} > 0$ and $0 \le \alpha \le 1$ defined by

$$\alpha(J^{(l)}) = \frac{2C_l S_l \prod_{i\neq l} C_i S_i}{C_l^2 \prod_{i\neq l} C_i^2 + S_l^2 \prod_{i\neq l} S_i^2}. \tag{2.17}$$

The proof of the inequalities for one-dimensional systems with free boundary conditions or for tree-like lattices is trivial since, due to the absence of loops, the partition function factorizes as

$$\mathbf{Z} = 2^N \prod_i \cosh(\lambda_i J_i),$$

and consequently the first inequality is satisfied even without taking the average and the second inequality reduces obviously to the equality to zero.

3. Comments

We have proved that a one-dimensional spin glass system satisfies a family of correlation inequalities without the assumption of translation invariance for the interaction distribution. The first inequality extends a similar one proved in [CL] for any lattice and any interaction with zero mean value. Here we have shown that the inequality is stable under suitable deformations of the zero mean hypotheses. The inequality of type II proved here shows that in the zero mean case the truncated correlation function has the opposite sign of the standard GKS inequality, i.e. the case of interactions with zero variance and positive mean. We have moreover identified the line crossing at which the truncated correlation changes its sign. It would be interesting to establish if an inequality of type (1.4) is also satisfied in higher dimensions (see [KNA]). In fact, as a straightforward computation shows in the Gaussian case, if such an inequality holds then the overlap expectation would be monotonic in the volume and several regularity properties would follow (see [CG2]). We also mention that the inequality (1.4) does not hold in general topologies as was shown to us by Hal Tasaki for a Bernoulli spin chain with an extra bond connecting two non-adjacent sites. Moreover, a similar violation of (1.4) can be obtained in the case where the disorder, still having zero average, is non-symmetric.

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