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Mathematical physics. — *A criterion for the reality of the spectrum of PT-symmetric Schrödinger operators with complex-valued periodic potentials*, by EMANUELA CALICETI and SANDRO GRAFFI, communicated by S. Graffi on 12 June 2008.

ABSTRACT. — Consider in $L^2(\mathbb{R})$ the Schrödinger operator family $H(g) := -d_x^2 + V_g(x)$ depending on the real parameter g, where $V_g(x)$ is a complex-valued but PT-symmetric periodic potential. An explicit condition on V is obtained which ensures that the spectrum of $H(g)$ is purely real and band shaped; furthermore, a further condition is obtained which ensures that the spectrum contains at least a pair of complex analytic arcs.

KEY WORDS: PT-symmetry; real spectrum; periodic potentials.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 34L40, 47E05, 81Q10, 81Q15.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

There is currently an intense and ever increasing activity on an aspect of quantum theory known as PT-symmetric quantum mechanics (see e.g. [BBMe], [BBM], [Be], [BBJ], [CJT], [CJN], [LZ], [Sp], [ZCBR]). Mathematically speaking, in the simplest, one-dimensional case one deals with the stationary Schrödinger equation

$$H\psi := \left(-\frac{d^2}{dx^2} + V\right)\psi = E\psi, \tag{1.1}$$

where the potential $V(x)$ can be complex-valued but is invariant under the combined action of the linear parity operation P, $P\psi(x) = \psi(-x)$, and the (anti)linear "time-reversal" symmetry, i.e. the complex-conjugation operation $T\psi(x) = \overline{\psi}(x)$; that is, $\overline{V}(-x) = V(x)$. The basic mathematical problem is to determine under what conditions, if any, on the complex PT-symmetric potential V the spectrum of the corresponding Schrödinger operator is purely real.

Here we deal with this problem in the context of periodic potentials on $\mathbb{R}$, already considered in [Ah], [BDM], [Ce], [CR], [Jo], [Sh]. Without loss of generality, the period is assumed to be 2π. If V is periodic and real-valued it is well known (see e.g. [BS]) that, under mild regularity assumptions, the spectrum is absolutely continuous on $\mathbb{R}$ and band shaped.

It is then natural to ask whether or not there exist classes of PT-symmetric, complex periodic potentials generating Schrödinger operators with real band spectrum. This question has been examined in [Ah], [BDM], [Ce], [CR], [Jo], by a combination of numerical and WKB techniques, in several particular examples. It was later proved in [Sh] that the above arguments cannot exclude the occurrence of complex spectra, and actually a condition has been isolated under which H admits complex spectrum consisting of a disjoint union of analytic arcs [Sh].

Our first result is the explicit determination a class of PT-symmetric, complex periodic potentials admitting real band spectrum. Denote by $\mathcal{T}(u)$ the non-negative quadratic form in $L^2(\mathbb{R})$ with domain $H^1(\mathbb{R})$ defined by the kinetic energy:

$$\mathcal{T}(u) := \int_{\mathbb{R}} |u'|^2 \, dx, \qquad u \in H^1(\mathbb{R}). \tag{1.2}$$

Let q be a real-valued, tempered distribution. Assume:

(1) q is a 2π-periodic, P-symmetric distribution belonging to $H^{-1}_{\text{loc}}(\mathbb{R})$;
(2) $W : \mathbb{R} \to \mathbb{C}$ belongs to $L^\infty(\mathbb{R})$ and is PT-symmetric, $\overline{W(-x)} = -W(x)$;
(3) q generates a real quadratic form $\mathcal{Q}(u)$ in $L^2(\mathbb{R})$ with domain $H^1(\mathbb{R})$;
(4) $\mathcal{Q}(u)$ is relatively bounded with respect to $\mathcal{T}(u)$ with relative bound $b < 1$, i.e. there are $b < 1$ and $a > 0$ such that

$$\mathcal{Q}(u) \leq bT(u) + a\|u\|^2. \tag{1.3}$$

Under these assumptions the real quadratic form

$$\mathcal{H}_0(u) := \mathcal{T}(u) + \mathcal{Q}(u), \qquad u \in H^1(\mathbb{R}), \tag{1.4}$$

is closed and bounded below in $L^2(\mathbb{R})$. We denote by $H(0)$ the corresponding self-adjoint operator. This is the self-adjoint realization of the formal differential expression (note the abuse of notation)

$$H(0) = -\frac{d^2}{dx^2} + q(x).$$

Under these circumstances it is known (see e.g. [AGHKH]) that the spectrum of $H(0)$ is continuous and band shaped. For $n = 1, 2, \dots$ we denote by

$$B_{2n} := [\alpha_{2n}, \beta_{2n}], \quad B_{2n+1} := [\beta_{2n+1}, \alpha_{2n+1}]$$

the bands of $H(0)$, and by $\Delta_n :=]\beta_{2n}, \beta_{2n+1}[,]\alpha_{2n+1}, \alpha_{2(n+2)}[$ the gaps between the bands. Here

$$0 \leq \alpha_0 \leq \beta_0 \leq \beta_1 \leq \alpha_1 \leq \alpha_2 \leq \beta_2 \leq \beta_3 \leq \alpha_3 \leq \alpha_4 \leq \cdots .$$

The maximal multiplication operator by W is continuous in L^2, and therefore so is the quadratic form $\langle u, Wu\rangle$. It follows that the quadratic form family

$$\mathcal{H}_g(u) := \mathcal{T}(u) + \mathcal{Q}(u) + g\langle u, Wu\rangle, \quad u \in H^1(\mathbb{R}), \tag{1.5}$$

is closed and sectorial in $L^2(\mathbb{R})$ for any $g \in \mathbb{C}$.

We denote by $H(g)$ the uniquely associated m-sectorial operator in $L^2(\mathbb{R})$. This is the realization of the formal differential operator family

$$H(g) = -\frac{d^2}{dx^2} + q(x) + gW(x).$$

By definition, $H(g)$ is a holomorphic family of operators of type B in the sense of Kato for $g \in \mathbb{C}$; by (1) it is also PT-symmetric for $g \in \mathbb{R}$. Our first result deals with its spectral properties.

THEOREM 1.1. *Let all gaps of $H(0)$ be open, that is, $\alpha_n < \beta_n < \alpha_{n+1}$ for all $n \in \mathbb{N}$, and let there exist $d > 0$ such that*

$$\frac{1}{2} \inf_{n\in\mathbb{N}} \Delta_n =: d > 0. \tag{1.6}$$

If

$$|g| < \frac{d^2}{2(1+d)\|W\|_\infty} =: \overline{g} \tag{1.7}$$

then there exist

$$0 \le \alpha_0(g) < \beta_0(g) < \beta_1(g) < \alpha_1(g) < \alpha_2(g) < \beta_2(g) < \cdots$$

such that

$$\sigma(H(g)) = \Big(\bigcup_{n\in\mathbb{N}} B_{2n}(g)\Big) \cup \Big(\bigcup_{n\in\mathbb{N}} B_{2n+1}(g)\Big), \tag{1.8}$$

where, as above

$$B_{2n}(g) := [\alpha_{2n}(g), \beta_{2n}(g)], \quad B_{2n+1}(g) := [\beta_{2n+1}(g), \alpha_{2n+1}(g)].$$

REMARK. The theorem states that for $|g|$ small enough the spectrum of the non-self-adjoint operator $H(g)$ remains real and band shaped. The proof is critically dependent on the validity of the lower bound (1.6). Therefore it cannot apply to smooth potentials $q(x)$, in which case the gaps vanish as $n \to \infty$. Actually, we have the following

EXAMPLE. A locally $H^{-1}(\mathbb{R})$ distribution $q(x)$ fulfilling the above conditions is

$$q(x) = \sum_{n\in\mathbb{Z}} \delta(x - 2\pi n),$$

the periodic δ function. Here we have

$$\mathcal{Q}(u) = \sum_{n\in\mathbb{Z}} |u(2\pi n)|^2 = \int_{\mathbb{R}} q(x)|u(x)|^2\,dx, \quad u \in H^1(\mathbb{R}).$$

This example is known as the Kronig–Penney model in the one-electron theory of solids. Let us verify that condition (1.3) is satisfied. As is known, this follows from the inequality (see e.g. [Ka, §VI.4.10])

$$|u(2\pi n)|^2 \le \epsilon \int_{2\pi n}^{2\pi(n+1)} |u'(y)|^2\,dy + \delta \int_{2\pi n}^{2\pi(n+1)} |u(y)|^2\,dy,$$

where ϵ can be chosen arbitrarily small for δ large enough. In fact, if $u \in H^1(\mathbb{R})$ this inequality yields

$$\begin{aligned}\int_{\mathbb{R}} q(x)|u(x)|^2\,dx = \sum_{n\in\mathbb{Z}} |u(2\pi n)|^2 &\le \epsilon \int_{\mathbb{R}} |u'(y)|^2\,dy + \delta \int_{\mathbb{R}} |u(y)|^2\,dy \\ &= \epsilon\mathcal{T}(u) + \delta\|u\|^2, \quad u \in H^1(\mathbb{R}),\end{aligned}$$

which in turn entails the closedness of $\mathcal{T}(u) + \mathcal{Q}(u)$ defined on H^1 by the standard Kato criterion. The closedness and sectoriality of $\mathcal{H}_g(u)$ defined on $H^1(\mathbb{R})$ is an immediate consequence of the continuity of W as a maximal multiplication operator in L^2. For the verification of (1.6), see e.g. [AGHKH]. Hence any bounded PT-symmetric periodic perturbation of the Kronig–Penney potential has real spectrum for $g \in \mathbb{R}$ with $|g| < \overline{g}$, where $\overline{g}$ is defined by (1.7).

As a second result, we show that an elementary argument of perturbation theory allows us to sharpen the result of [Sh] about the existence of complex spectra for PT-symmetric periodic potentials.

Let indeed $W \in L^\infty(\mathbb{R}; \mathbb{C})$ be a 2π-periodic function. Then the continuity of W as a multiplication operator in $L^2(\mathbb{R})$ entails that the Schrödinger operator

$$K(g)u := -\frac{d^2u}{dx^2} + gWu, \quad u \in D(K) := H^2(\mathbb{R}),\ g \in \mathbb{R}, \tag{1.9}$$

is closed and has non-empty resolvent set. Consider the Fourier expansion of W:

$$W(x) = \sum_{n\in\mathbb{Z}} w_n e^{inx}, \quad w_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(x)e^{-inx}\,dx,$$

which converges pointwise almost everywhere in $[-\pi, \pi]$. Then we have

THEOREM 1.2. *Let $W(x)$ be PT-symmetric, i.e. $\overline{W(-x)} = W(x)$. Then*

(i) $\overline{w}_n = w_n$ *for all* $n \in \mathbb{Z}$*;*
(ii) *If there exist $k \in \mathbb{N}$, k odd, such that $w_k w_{-k} < 0$, then there is $\delta > 0$ such that for $|g| < \delta$ the spectrum of $K(g)$ contains at least a pair of complex conjugate analytic arcs.*

REMARKS. 1. This theorem sharpens the results of [Sh] in the sense that its assumptions are explicit because they involve only the given potential $W(x)$, while those of Theorem 3 and Corollary 4 of [Sh] involve some conditions on the Floquet discriminant of the equation $K(g)\psi = E\psi$. This requires some a priori information on the solutions of the equation itself.

2. Explicit examples of potentials fulfilling the above conditions are

$$W(x) = i\sin^{2k+1} nx, \quad k = 0, 1, \ldots,\ n \text{ odd}.$$

For $g = 1$ these potentials have been considered in [BDM], where it is claimed that the spectrum is purely real. A more careful examination by using [Sh] shows that the appearance of complex spectra cannot be excluded.

2. PROOF OF THE STATEMENTS

Let us first state an elementary remark as a lemma. Incidentally, this also proves Theorem 1.2(i).

LEMMA 2.1. *Let $f \in L^\infty(\mathbb{R};\mathbb{C})$ be 2π-periodic and PT-symmetric, $\bar{f}(-x) = f(x)$. Consider its Fourier coefficients*

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx.$$

Then $\bar{f}_n = f_n$ for all $n \in \mathbb{Z}$.

PROOF. The assertion is an immediate consequence of the Carleson–Fefferman theorem, which states the pointwise convergence of the Fourier expansion

$$\bar{f}(-x) = \sum_{n\in\mathbb{Z}} \bar{f}_n e^{inx} = \sum_{n\in\mathbb{Z}} f_n e^{inx} = f(x)$$

almost everywhere in $[0, 2\pi]$.

To prove Theorem 1.1, let us first recall that by the Floquet–Bloch theory (see e.g. [BS], [Ea]), $\lambda \in \sigma(H(g))$ if and only if the equation $H(g)\psi = \lambda\psi$ has a non-constant bounded solution. In turn, all bounded solutions have the (Bloch) form

$$\psi_p(x;\lambda,g) = e^{ipx}\phi_p(x;\lambda,g), \tag{2.1}$$

where $p \in\ ]-1/2, 1/2] := B$ (the Brillouin zone) and ϕ_p is 2π-periodic. It is indeed immediately checked that $\psi_p(x;\lambda,g)$ solves $H(g)\psi = \lambda\psi$ if and only if $\phi_p(x;\lambda,g)$ is a solution of

$$H_p(g)\phi_p(x;\lambda,g) = \lambda\phi_p(x;\lambda,g).$$

Here $H_p(g)$ is the operator in $L^2(0, 2\pi)$ given by

$$H_p(g)u = \left(-i\frac{d}{dx} + p\right)^2 u + qu + gWu, \qquad u \in D(H_p(g)), \tag{2.2}$$

with periodic boundary conditions; its realization will be recalled below. More precisely, denote by S^1 the one-dimensional torus, i.e. the interval $[-\pi,\pi]$ with the endpoints identified. By assumptions (1) and (2) above, the restriction of q to S^1, still denoted q by a standard abuse of notation, belongs to $H^{-1}(S^1)$ and generates a real quadratic form $\mathcal{Q}_p(u)$ in $L^2(S^1)$ with domain $H^1(S^1)$. By assumption (3), $\mathcal{Q}_p(u)$ is relatively bounded, with relative bound zero, with respect to

$$\mathcal{T}_p(u) := \int_{-\pi}^{\pi} [-iu' + pu][i\overline{u}' + p\overline{u}]\,dx, \qquad D(\mathcal{T}_p(u)) = H^1(S^1), \tag{2.3}$$

so that the real semibounded form $\mathcal{H}_p^0(u) := \mathcal{T}_p(u) + \mathcal{Q}_p(u)$ defined on $H^1(S^1)$ is closed. The corresponding self-adjoint operator in $L^2(S^1)$ is the self-adjoint realization of the formal differential expression (note again the abuse of notation)

$$H_p(0) = -\frac{d^2}{dx^2} - 2ip\frac{d}{dx} + p^2 + q.$$

As above, the form $\mathcal{H}_p(g)(u) := \mathcal{H}_p^0(u) + \langle u, Wu \rangle$ defined on $H^1(S^1)$ is closed and sectorial in $L^2(S^1)$. Let $H_p(g)$ be the associated m-sectorial operator in $L^2(S^1)$. On $u \in D(H_p(g))$ the action of the operator $H_p(g)$ is specified by (2.2); moreover, $H_p(g)$ has compact resolvent. Let

$$\sigma(H_p(g)) := \{\lambda_n(g; p) : n = 0, 1, \dots\}$$

denote the spectrum of $H_p(g)$, with $p \in]-1/2, 1/2]$, $|g| < \overline{g}$. By the above remarks we have

$$\sigma(H(g)) = \bigcup_{p \in]-1/2,1/2]} \sigma(H_p(g)) = \{\lambda_n(g; p) : n = 0, 1, \dots;\ p \in]-1/2, 1/2]\}.$$

To prove the reality of $\sigma(H(g))$, $|g| < \overline{g}$, it is therefore enough to prove the reality of all eigenvalues $\lambda_n(g; p)$ for $n = 0, 1, \dots$ and $p \in]-1/2, 1/2]$.

To this end, let us further recall the construction of the bands for $g = 0$: it can be proved that under the present conditions all eigenvalues $\lambda_n(0; p)$ are simple for all $p \in [-1/2, 1/2]$; the functions $\lambda_n(0; p)$ are continuous and even in $[-1/2, 1/2]$ with respect to p, so that one can restrict to $p \in [0, 1/2]$; the functions $\lambda_{2k}(0; p)$ are strictly increasing on $[0, 1/2]$ while the functions $\lambda_{2k+1}(0; p)$ are strictly decreasing, $k = 0, 1 \dots$. Set

$$\alpha_k = \lambda_k(0; 0), \quad \beta_k = \lambda_k(0; 1/2).$$

Then

$$\alpha_0 < \beta_0 < \beta_1 < \alpha_1 < \alpha_2 < \beta_2 < \beta_3 < \cdots .$$

The intervals $[\alpha_{2n}, \beta_{2n}]$ and $[\beta_{2n+1}, \alpha_{2n+1}]$ coincide with the ranges of $\lambda_{2n}(0; p)$ and $\lambda_{2n+1}(0; p)$, respectively, and represent the bands of $\sigma(H(0))$; the intervals

$$\Delta_n :=]\beta_{2n}, \beta_{2n+1}[,\ ]\alpha_{2n+1}, \alpha_{2(n+2)}[$$

are the gaps between the bands.

The monotonicity of the functions $\lambda_n(0; p)$ and assumption (1.6) entail

$$\inf_n \min_{p \in [0,1/2]} |\lambda_n(0; p) - \lambda_{n+1}(0; p)| \geq 2d. \tag{2.4}$$

Let us now state the following preliminary result:

PROPOSITION 2.2. (i) *Let $g \in \overline{\mathcal{D}}$, where $\mathcal{D}$ is the disk $\{g : |g| < \overline{g}\}$. For any n, there is a function $\lambda_n(g; p) : \overline{\mathcal{D}} \times [0, 1/2] \to \mathbb{C}$, holomorphic in g and continuous in p, such that $\lambda_n(g; p)$ is a simple eigenvalue of $\mathcal{H}_p(g)$ for all $(g, p) \in \overline{\mathcal{D}} \times [0, 1/2]$.*

(ii)

$$\sup_{g \in \overline{\mathcal{D}}, p \in [0,1/2]} |\lambda_n(g; p) - \lambda_n(0; p)| < d/2.$$

(iii) *If $g \in \mathbb{R} \cap \overline{\mathcal{D}}$ then all eigenvalues $\lambda_n(g; p)$ are real.*

(iv) *If $g \in \mathbb{R} \cap \overline{\mathcal{D}}$ then $\sigma(\mathcal{H}_p(g)) \equiv \{\lambda_n(g; p)\}_{n=0}^{\infty}$.*

Assuming the validity of this proposition the proof of Theorem 1.1 is immediate.

PROOF OF THEOREM 1.1. Since the functions $\lambda_n(g;p)$ are real and continuous for $g \in \mathbb{R} \cap \overline{\mathcal{D}}$, $p \in [0, 1/2]$, we can define

$$\alpha_{2n}(g) := \min_{p\in[0,1/2]} \lambda_n(g;p); \qquad \beta_{2n}(g) := \max_{p\in[0,1/2]} \lambda_{2n}(g;p), \tag{2.5}$$

$$\alpha_{2n+1}(g) := \max_{p\in[0,1/2]} \lambda_{2n+1}(g;p); \quad \beta_{2n+1}(g) := \min_{p\in[0,1/2]} \lambda_{2n+1}(g;p). \tag{2.6}$$

Then

$$\sigma(H(g)) = \bigcup_{n=0}^{\infty} B_n(g),$$

where the bands $B_n(g)$ are defined, in analogy with the $g = 0$ case, by

$$B_{2n}(g) := [\alpha_{2n}(g), \beta_{2n}(g)], \qquad B_{2n+1}(g) := [\beta_{2n+1}(g), \alpha_{2n+1}(g)]$$

By Proposition 2.2(ii) we have, for all $n = 0, 1, \ldots$ and $g \in \overline{D}$,

$$\begin{aligned}\alpha_{2n}(g) - d/2 = \lambda_{2n}(0;0) - d/2 \le \lambda_{2n}(0;p) - d/2 \\ \le \lambda_{2n}(g;p) \le \lambda_{2n}(0;1/2) + d/2 = \beta_{2n} + d/2.\end{aligned}$$

This yields

$$B_{2n}(g) \subset [\alpha_{2n} - d/2, \beta_{2n} + d/2].$$

By an analogous argument,

$$B_{2n+1}(g) \subset [\beta_{2n+1} - d/2, \alpha_{2n+1} + d/2].$$

Therefore the bands are pairwise disjoint, because the gaps

$$\Delta_n(g) :=]\beta_{2n}(g), \alpha_{2n}(g)[,\ ]\alpha_{2n+1}(g), \beta_{2n+1}(g)[$$

are all open and their widths are no smaller than d. In fact, by (1.6) we have

$$|\alpha_n - \alpha_{n+1}| \ge 2d, \qquad |\beta_n - \beta_{n+1}| \le 2d.$$

This concludes the proof of the theorem.

We now prove separately the assertions of Proposition 2.2.

PROOF OF PROPOSITION 2.2(i), (ii). Since the maximal multiplication operator by W is continuous in $L^2(S^1)$ with norm $\|W\|_\infty$, the operator family $\mathcal{H}_p(g)$ is a type-A holomorphic family with respect to $g \in \mathbb{C}$, uniformly with respect to $p \in [0, 1/2]$. Hence we can directly apply regular perturbation theory (see e.g. [Ka]): the perturbation expansion near any eigenvalue $\lambda_n(0;p)$ of $\mathcal{H}_p(0)$ exists and is convergent for $g \in \overline{\mathcal{D}}$ to a simple eigenvalue $\lambda_n(g;p)$ of $H_p(g)$:

$$\lambda_n(g;p) = \lambda_n(0;p) + \sum_{s=1}^{\infty} \lambda_n^s(0;p) g^s, \qquad g \in \overline{\mathcal{D}}. \tag{2.7}$$

The convergence radius $r_n(p)$ is no smaller than $\overline{g}$. Hence $\overline{g}$ represents a lower bound for $r_n(p)$ independent of n and p. Moreover, $\lambda_n^s(0;p)$ is continuous for all $p \in [0, 1/2]$, and hence the same is true for the sum $\lambda_n(g;p)$. This proves (i).

To prove (ii), recall that the coefficients $\lambda_n^s(0;p)$ have the majorization (see [Ka, §II.3])

$$(2.8) \qquad |\lambda_n^s(0;p)| \le \left(\frac{2\|W\|_\infty}{\inf_k \min_{[0,1/2]} |\lambda_k(0;p) - \lambda_{k\pm1}(0;p)|}\right)^s \le (\|W\|_\infty/d)^s.$$

Therefore, by (2.7) and (1.7),

$$|\lambda_n(g;p) - \lambda_n(0;p)| \le \frac{|g|\,\|W\|_\infty/d}{1-|g|\,\|W\|_\infty/d} = \frac{|g|\,\|W\|_\infty}{d-|g|\,\|W\|_\infty} < \frac{d}{2},$$

whence the stated majorization follows.

Proof of Proposition 2.2(iii). As is known, and anyway very easy to verify, the PT-symmetry of an operator entails that the eigenvalues are either real or complex conjugate. By standard regular perturbation theory (see e.g. [Ka, §VII.2]) any eigenvalue $\lambda_n(0;p)$ of $H_p(0)$ is stable with respect to $H_p(g)$; since $\lambda_n(0;p)$ is simple, for g suitably small there is one and only one eigenvalue $\lambda_n(g;p)$ of $H_p(g)$ near $\lambda_n(0;p)$, and $\lambda_n(g;p) \to \lambda_n(0;p)$ as $g \to 0$. This excludes the existence of the complex conjugate eigenvalue $\overline{\lambda_n}(g;p)$ distinct from $\lambda_n(g;p)$. Thus for $g \in \mathbb{R}$ with $|g|$ suitably small, $\lambda_n(g;p)$ is real. This entails the reality of series expansion (2.7) for g small and hence for all $g \in \overline{\mathcal{D}}$. This in turn implies the reality of $\lambda_n(g;p)$ for all $g \in \overline{\mathcal{D}}$.

Proof of Proposition 2.2(iv). We repeat here the argument introduced in [CGS], [CG] to prove the analogous result in different contexts. We give all details to make the paper self-contained. We have seen that for any $r \in \mathbb{N}$ the Rayleigh–Schrödinger perturbation expansion associated with the eigenvalue $\lambda_r(g;p)$ of $H_p(g)$ which converges to $\lambda_r(0;p)$ as $g \to 0$, has radius of convergence no smaller than $\overline{g}$. Hence, for all $g \in \mathbb{R}$ such that $|g| < \overline{g}$, $H_p(g)$ admits a sequence of real eigenvalues $\lambda_r(g;p)$, $r \in \mathbb{N}$. We want to prove that for $|g| < \overline{g}$, $g \in \mathbb{R}$, $H_p(g)$ has no other eigenvalues. Thus all its eigenvalues are real. To this end, for any $r \in \mathbb{N}$ let $\mathcal{Q}_r$ denote the disk centered at $\lambda_r(0;p)$ with radius d. Then if $g \in \mathbb{R}$, $|g| < \overline{g}$, and $\lambda(g)$ is an eigenvalue of $H_p(g)$, we have

$$\lambda(g) \in \bigcup_{r\in\mathbb{N}} \mathcal{Q}_r.$$

In fact, defining

$$R_0(z) := (H_p(0) - z)^{-1}$$

for any $z \notin \bigcup_{r\in\mathbb{N}} \mathcal{Q}_r$ we have

$$(2.9) \quad \|gWR_0(z)\| \le |g|\,\|W\|_\infty \|R_0(z)\| < \overline{g}\|W\|_\infty [\mathrm{dist}(z,\sigma(H_0))]^{-1} \le \frac{\overline{g}\|W\|_\infty}{d} < 1.$$

The last inequality in (2.9) follows directly from the definition (1.7) of $\overline{g}$. Thus, $z \in \rho(H_p(g))$ and

$$R(g,z) := (H_p(g) - z)^{-1} = R_0(z)[1 + gWR_0(z)]^{-1}.$$

Now let $g_0 \in \mathbb{R}$ be fixed with $|g| < \overline{g}$. Without loss of generality we assume that $g_0 > 0$. Let $\lambda(g_0)$ be a given eigenvalue of $H_p(g_0)$. Then $\lambda(g_0)$ must be contained in the interior (and not on the boundary) of $\mathcal{Q}_{n_0}$ for some $n_0 \in \mathbb{N}$. Moreover, if m_0 is the multiplicity of $\lambda(g_0)$, then for g close to g_0 there are m_0 eigenvalues (counting multiplicities) $\lambda^{(s)}(g)$, $s = 1, \dots, m_0$, of $H_p(g)$ which converge to $\lambda(g_0)$ as $g \to g_0$ and each function $\lambda^{(s)}(g)$ represents a branch of one or several holomorphic functions which have at most algebraic singularities at $g = g_0$ (see [Ka, Thm. VII.1.8]). Let us now consider any one of such branches $\lambda^{(s)}(g)$ for $0 < g < g_0$, suppressing the index s from now on. First of all we notice that, by continuity, $\lambda(g)$ cannot lie outside $\mathcal{Q}_{n_0}$ for g close to g_0. Moreover, if we denote by Γ_t the circle centered at $\lambda_{n_0}(0; p)$ with radius t, $0 < t \le d$, we have, for $z \in \Gamma_t$ and $0 < g \le g_0$,

$$\|gWR_0(z)\| \le g\|W\|_\infty[\mathrm{dist}(z, \sigma(H_p(0)))]^{-1} \le g\|W\|_\infty/t\,. \tag{2.10}$$

Then $t > g\|W\|_\infty$ implies $z \notin \sigma(H_p(g))$, i.e. if $z \in \sigma(H_p(g)) \cap \Gamma_t$ then $t \le g\|W\|_\infty < g_0\|W\|_\infty < \overline{g}\|W\|_\infty < d$. Hence we observe that as $g \to g_0^-$, $\lambda(g)$ is contained in the disk centered at $\lambda_{n_0}(0; p)$ with radius $g\|W\|_\infty$. Suppose that the holomorphic function $\lambda(g)$ is defined on the interval $]g_1, g_0]$ with $g_1 > 0$. We will show that it can be continued up to $g = 0$, and in fact up to $g = -\overline{g}$. From what has been established so far the function $\lambda(g)$ is bounded as $g \to g_1^+$. Thus, by the well known stability properties of the eigenvalues of analytic families of operators, $\lambda(g)$ must converge to an eigenvalue $\lambda(g_1)$ of $H_p(g_1)$ as $g \to g_1^+$ and $\lambda(g_1)$ is contained in the disk centered at $\lambda_{n_0}(0; p)$ with radius $g_1\|W\|_\infty$. Repeating the argument starting now from $\lambda(g_1)$, we can continue $\lambda(g)$ to a holomorphic function on an interval $]g_2, g_1]$, which has at most an algebraic singularity at $g = g_2$. We build in this way a sequence $g_1 > g_2 > \cdots$ which can accumulate only at $g = -\overline{g}$. In particular the function $\lambda(g)$ is piecewise holomorphic on $]-\overline{g}, \overline{g}[$. But while passing through $g = 0$, $\lambda(g)$ coincides with the eigenvalue $\lambda_r(g; p)$ generated by an unperturbed eigenvalue $\lambda_r(0; p)$ of $H_p(0)$ (namely $\lambda_{n_0}(0; p)$), which represents a real analytic function defined for $g \in]-\overline{g}, \overline{g}[$. Thus, $\lambda(g_0)$ arises from this function and is therefore real. This concludes the proof of (iv).

PROOF OF THEOREM 1.2. Consider the operator $K_p(g)$ acting in $L^2(S^1)$, defined on the domain $H^2(S^1)$. By the Floquet–Bloch theory recalled above, we have again

$$\sigma(K(g)) = \bigcup_{p \in]-1/2, 1/2]} \sigma(K_p(g)).$$

It is then enough to prove that there is $\eta > 0$ such that $K_p(g)$ has complex eigenvalues for $p \in]1/2 - \eta, 1/2]$. Since $K_p(g)$ is PT-symmetric, eigenvalues may occur only in complex-conjugate pairs. The eigenvalues of $K_p(0)$ are $\lambda_n(0; p) = (n + p)^2$, $n \in \mathbb{Z}$. The eigenvalue $\lambda_0(0, p) = p^2$ is simple for all $p \in [0, 1/2]$; any other eigenvalue is simple for $p \ne 0$, $p \ne 1/2$ and has multiplicity 2 for $p = 0$ or $p = 1/2$ because $n^2 = (-n)^2$ and $(n + 1/2)^2 = (-n - 1 + 1/2)^2$, $n = 0, 1, \dots$. For $p = 1/2$ a set of orthonormal eigenfunctions corresponding to the double eigenvalue $(n + 1/2)^2 = (-n - 1 + 1/2)^2$, $n = 0, 1, \dots$, is given by $\{u_n, u_{-n-1}\}$, where

$$u_n := \frac{1}{\sqrt{2\pi}} e^{inx}, \qquad n \in \mathbb{Z}.$$

Note that, for $0 < p < 1/2$, u_n and u_{-n-1} are the (normalized) eigenfunctions corresponding to the simple eigenvalues $(n+p)^2$, $(-n-1+p)^2$. Let now $n = k \in \mathbb{N}$, and consider the 2×2 matrix

$$T := \begin{pmatrix} \langle u_k, W u_k\rangle & \langle u_k, W u_{-k-1}\rangle \\ \langle u_{-k-1}, W u_k\rangle & \langle u_{-k-1}, W u_{-k-1}\rangle \end{pmatrix}.$$

A trivial computation yields

$$T := \begin{pmatrix} 0 & w_{2k+1} \\ w_{-2k-1} & 0 \end{pmatrix}$$

with purely imaginary eigenvalues $\mu^{\pm} = \pm i\sqrt{-w_{2k+1}w_{-2k-1}}$. By standard degenerate perturbation theory, for g small enough $\mathcal{H}_{1/2}(g)$ admits a pair of complex conjugate eigenvalues

$$\lambda_k^{\pm}(g, 1/2) = (k+1/2)^2 \pm ig\sqrt{-w_{2k+1}w_{-2k-1}} + O(g^2). \tag{2.11}$$

Under the above assumptions, for any fixed $g \in \mathbb{R}$ the operator family $p \mapsto K_p(g)$ is type-A holomorphic in the sense of Kato (see [Ka, §VII.2]) for all $p \in \mathbb{C}$ because its domain does not depend on p and the scalar products $\langle u, K_p(g)u\rangle$, $u \in H^1(S^1)$, are obviously holomorphic functions of $p \in \mathbb{C}$. This entails the continuity with respect to p of the eigenvalues $\lambda_n(g; p)$. Therefore, for $|g|$ suitably small, there is $\eta(g) > 0$ such that

$$\operatorname{Im} \lambda_k^{\pm}(g; p) \neq 0, \quad 1/2 - \eta \leq p \leq 1/2.$$

It follows (see e.g. [BS], [Ea]) that the complex arcs $\mathcal{E}_k^{\pm} := \operatorname{Range}(\lambda_k^{\pm}(g; p))$, $p \in [1/2 - \eta(g), 1/2]$, lie in the spectrum of $K(g)$. This concludes the proof of the theorem.

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