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Mathematical analysis. — *Hardy–Sobolev inequalities and hyperbolic symmetry*, by DANIELE CASTORINA, ISABELLA FABBRI, GIANNI MANCINI and KUNNATH SANDEEP, communicated on 12 June 2008.

ABSTRACT. — We discuss uniqueness and nondegeneracy of extremals for some weighted Sobolev inequalities and give some applications to Grushin and scalar curvature type equations. The main theme is hyperbolic symmetry.

KEY WORDS: Nonlinear PDE; hyperbolic symmetry; Hardy–Sobolev inequalities.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 35J60; Secondary 35B05, 35A15.

1. INTRODUCTION

In this note we announce some results (see [3] and [11] for details) concerning positive extremals for optimal Hardy–Sobolev–Maz'ya inequalities (see [12])

$$\sqrt{S_t^\mu}\left(\int_{\mathbb{R}^k\times\mathbb{R}^h}\frac{|u|^p}{|y|^t}\,dy\,dz\right)^{1/p}\le\left(\int_{\mathbb{R}^k\times\mathbb{R}^h}\left[|\nabla u|^2-\mu\frac{u^2}{|y|^2}\right]dy\,dz\right)^{1/2},\tag{1.1}$$

where $(y,z)\in\mathbb{R}^k\times\mathbb{R}^h$, $k,h\in\mathbb{N}$, $N=k+h$, $p>2$ and $p\le 2N/(N-2)$ if $N\ge 3$, $t=N-(N-2)p/2$. Inequality (1.1) holds for all $\mu\le((k-2)/2)^2$ and $u\in C_0^\infty((\mathbb{R}^k\setminus\{0\})\times\mathbb{R}^h)$. If $k=1$, we mean $\mathbb{R}^+$ instead of $\mathbb{R}$.

Existence of minimizers for (1.1) has been established in [2] in the special case $\mu=0$, $k\ge 2$ and subsequently in [14] and [15] in other cases.

In case $k\ge 2$ and $\mu=0$, cylindrical symmetry, regularity and decay properties of (positive) extremals have been established in [5]. Actually, in case $p=2(N-1)/(N-2)$ extremals have been completely identified therein: they are given, up to dilation and translation in z, by

$$U(y,z)=\left[\frac{(N-2)(k-1)}{(1+|y|)^2+|z|^2}\right]^{(N-2)/2}.\tag{1.2}$$

Cylindrical symmetry has been established in case $0\le\mu\le((k-2)/2)^2$ in [8].

We address here uniqueness and nondegeneracy of positive extremals. We work out this problem by studying the Euler–Lagrange equation for (1.1):

$$-\Delta u=\mu\frac{u}{|y|^2}+\frac{|u|^{p-2}u}{|y|^t}\quad\text{in }\mathbb{R}^N\quad(\text{in }\mathbb{R}^+\times\mathbb{R}^h\text{ if }k=1).\tag{1.3}$$

The starting point, to be discussed in Section 2, where a symmetry result is also presented, is the connection we discovered between (1.3) and the following equation on $\mathbb{H}^n$, the $n = h + 1$-dimensional hyperbolic space:

$$\Delta_{\mathbb{H}^n} v + \lambda v + v^{p-1} = 0, \tag{1.4}$$

where $\Delta_{\mathbb{H}^n}$ is the Laplace–Beltrami operator in $\mathbb{H}^n$. In Sections 3–4, we present uniqueness results for (1.4), which apply to (1.3) and to critical Grushin equations as well and then we prove nondegeneracy in some cases.

In Section 4 we apply these results to scalar curvature type equations.

2. Hyperbolic symmetry

We deal with symmetric entire solutions of (1.3), i.e. solutions $u \in \mathcal{D}_\mu$, the closure, with respect to the norm given by the r.h.s. in (1.1), of the space of cylindrically symmetric $C_0^\infty((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^h)$ functions. Actually, $\mathcal{D}_\mu$ is the class where solutions of (1.3) have been found (see [14], [8], [15], [11]).

If $\mu < ((k-2)/2)^2$ and $k \neq 2$ then $\mathcal{D}_\mu = D^1(\mathbb{R}^N)$. However, $\mathcal{D}_{((k-2)/2)^2}$ is larger than $D^1(\mathbb{R}^N)$. In fact, $u \in \mathcal{D}_{((k-2)/2)^2}$ iff $\int_{\mathbb{R}^k \times \mathbb{R}^h} |y|^{2-k} |\nabla |y|^{(k-2)/2} u|^2 \, dy\, dz < +\infty$.

So, let $u \in \mathcal{D}_\mu$ be a positive solution of (1.3). Set

$$u = u(|y|, z), \qquad (Hu)(r, z) := r^{(N-2)/2} u(r, z).$$

The crucial observation is the following. Let $n := h + 1$. Then Hu solves

$$-\Delta_{\mathbb{H}^n} v := -(r^2 \Delta v - (n-2) r v_r) = \lambda v + v^{p-1} \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^h, \tag{2.1}$$

where $\lambda = \mu + ((n-1)^2 - (k-2)^2)/4$ and $\Delta_{\mathbb{H}^n}$ denotes the Laplace–Beltrami operator on (the half-space model for) $\mathbb{H}^n$. In addition, the map H is energy preserving, in the sense we are going to specify. First, recall that in the half-space model for $\mathbb{H}^n$ the Sobolev norm of $v \in H^1(\mathbb{H}^n)$ is

$$|||v|||^2 := \int_{\mathbb{H}^n} [|\nabla_{\mathbb{H}^n} v|^2 + v^2] \, dV_{\mathbb{H}^n} = \int_{\mathbb{R}^+ \times \mathbb{R}^h} [r^2 |\nabla v|^2 + v^2] \frac{dr\, dz}{r^{h+1}}.$$

Now, given $v \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^h)$, let

$$(T_k v)(y, z) := |y|^{-(N-2)/2} v(|y|, z), \qquad y \in \mathbb{R}^k,\ z \in \mathbb{R}^{n-1}.$$

Then

$$\omega_k \int_{\mathbb{H}^n} [|\nabla_{\mathbb{H}^n} v|^2 - \lambda v^2] \, dV_{\mathbb{H}^n} = \int_{\mathbb{R}^k \times \mathbb{R}^h} \left[|\nabla T_k v|^2 - \mu \frac{(T_k v)^2}{|y|^2} \right] dy\, dz, \tag{2.2}$$

where $\lambda = \mu + ((n-1)^2 - (k-2)^2)/4$. From (2.2) and (1.1) we get the (sharp) Poincaré and Sobolev inequalities

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} v|^2 \, dV_{\mathbb{H}^n} \geq \frac{(n-1)^2}{4} \int_{\mathbb{H}^n} v^2 \, dV_{\mathbb{H}^n} \quad \forall v \in H^1(\mathbb{H}^n), \tag{2.3}$$

$$\tilde{S}_{n,p} \left(\int_{\mathbb{H}^n} v^p \, dV_{\mathbb{H}^n} \right)^{2/p} \leq \int_{\mathbb{H}^n} \left[|\nabla_{\mathbb{H}^n} v|^2 - \frac{(n-1)^2}{4} v^2 \right] dV_{\mathbb{H}^n}, \tag{2.4}$$

where, in (2.4), $2 < p \leq 2n/(n-2)$ if $n \geq 3$ and $p > 2$ if $n = 2$.

In particular, $(\int_{\mathbb{H}^n} [|\nabla_{\mathbb{H}^n} v|^2 - \lambda v^2] \, dV_{\mathbb{H}^n})^{1/2}$ is a norm on $C_0^\infty(\mathbb{H}^n)$ if $\lambda \leq (n-1)^2/4$. We let $\mathcal{H}_\lambda$ be the closure of $C_0^\infty(\mathbb{H}^n)$ with respect to this norm. From (2.2) we see that T_k extends to an isometry between $\mathcal{H}_\lambda$ and $\mathcal{D}_\mu$.

THEOREM 2.1. *Let $p > 2$ if $n = 2$, and $2 < p \leq 2n/(n-2)$ if $n \geq 3$. Let $v \in \mathcal{H}_\lambda$ be a positive solution of* (2.1). *Then v has hyperbolic symmetry, i.e., for some $x_0 \in \mathbb{H}$, $v(x)$ depends only on the distance between x_0 and x in $\mathbb{H}^n$.*

The proof of Theorem 2.1 is based on a hyperbolic version of the moving plane method (cf. [1]) in connection with (2.4).

3. HYPERBOLIC SYMMETRY AND UNIQUENESS

THEOREM 3.1. *Let $\lambda \leq 2p/(p+2)^2$ if $n = 2$, and $\lambda \leq (n-1)^2/4$ if $n \geq 3$. Then* (2.1) *has at most one positive solution in $\mathcal{H}_\lambda$.*

COROLLARY 3.2. *Positive symmetric extremals of* (1.1) *are unique if $\mu \leq (k-2)^2/4$ and $h \geq 2$, $k \geq 1$. If $h = 1$, we have to assume $\mu \leq (k-2)^2/4 - \frac{1}{4}(\frac{p-2}{p+2})^2$.*

To prove Theorem 3.1 we can assume, by Theorem 2.1 and in the ball model for $\mathbb{H}^n$, that $v \in \mathcal{H}_\lambda$ is a positive radial solution in $\{\xi \in \mathbb{R}^n : |\xi| < 1\}$ of

$$\left[\frac{1 - |\xi|^2}{2} \right]^2 \Delta v + (n-2) \left[\frac{1 - |\xi|^2}{2} \right] \langle \nabla v, \xi \rangle + \lambda v + v^{p-1} = 0. \tag{3.1}$$

In hyperbolic polar coordinates $t = \log \frac{1+|\xi|}{1-|\xi|}$, $w(t) := v(\tanh \frac{t}{2})$, (3.1) reads

$$w'' + (n-1)(\coth t) w' + \lambda w + w^{p-1} = 0, \qquad w'(0) = 0. \tag{3.2}$$

By means of an auxiliary energy, inspired by Kwong's work (see [9]), and Sturm comparison arguments we first prove

PROPOSITION 3.3. *Let $\lambda \leq (n-1)^2/4$ and $p \leq 2^*$ if $n \geq 3$. If $n = 2$ assume $\lambda \leq 2p/(p+2)^2$. Then the Dirichlet problem*

$$\begin{aligned} &\psi'' + (n-1)(\coth t)\psi' + \lambda \psi + \psi^{p-1} = 0, \\ &\psi'(0) = 0, \quad \psi(T) = 0, \quad \psi(t) > 0 \quad \forall t \in [0, T), \end{aligned} \tag{3.3}$$

has at most one solution, and no solution if $n \geq 3$, $p = 2^*$, $\lambda = n(n-2)/4$.

The next step is to establish precise asymptotic decay.

LEMMA 3.4. *Let $n \geq 2$, $p > 2$, $\lambda \leq (n-1)^2/4$. Let $v \in \mathcal{H}_\lambda$ be a positive radial solution of* (3.1) *and $w(t) := v(\tanh \frac{t}{2})$. Then*

$$\lim_{t\to+\infty} \frac{\log w^2}{t} = \lim_{t\to+\infty} \frac{\log w'^2}{t} = -[n-1+\sqrt{(n-1)^2-4\lambda}].$$

Using Proposition 3.3 and Lemma 3.4 we can prove Theorem 3.1.

We now derive from Theorem 3.1 uniqueness of cylindrically symmetric positive extremals of the weighted Sobolev inequality (see [13])

$$\begin{aligned} \|u\|_\alpha^2 &:= \int_{\mathbb{R}^N} (|\nabla_y u|^2 + (\alpha+1)^2|y|^{2\alpha}|\nabla_z u|^2)\, dy\, dz \\ &\geq \tilde{S}\left(\int_{\mathbb{R}^N} |u|^{2Q/Q-2}\, dy\, dz\right)^{(Q-2)/Q} \qquad \forall u \in C_0^\infty(\mathbb{R}^N), \end{aligned} \tag{3.4}$$

where $N := k + h,\ k, h \geq 1$, $(y, z) \in \mathbb{R}^k \times \mathbb{R}^h$, $\alpha > 0,\ Q := k + h(1+\alpha)$. The associated Euler–Lagrange equation is the critical Grushin-type equation

$$-\Delta_y u - (\alpha+1)^2|y|^{2\alpha}\Delta_z u = |u|^{4/(Q-2)}u, \quad u \in \mathcal{D}^\alpha(\mathbb{R}^N), \tag{3.5}$$

where $\mathcal{D}^\alpha(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to $\|u\|_\alpha$ (see [13] and [7]). A cylindrically symmetric positive solution u of (3.5) gives, via the change of variables

$$v(r,z) := (\alpha+1)^{-(Q-2)/2} r^{(Q-2)/2(1+\alpha)} u(r^{1/(1+\alpha)}, z),$$

a solution of

$$-\Delta_{\mathbb{H}^n} v - \frac{1}{4}\left[h^2 - \left(\frac{k-2}{\alpha+1}\right)^2\right]v = v^{(Q+2)/(Q-2)}.$$

In addition, a direct computation gives for every $u \in C_0^\infty((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^h)$, $u = u(|y|, z)$ and $n = h+1$,

$$\omega_k(\alpha+1)\int_{\mathbb{H}^n}\left[|\nabla_{\mathbb{H}^n} v|^2 - \frac{h^2}{4}v^2\right]dV_{\mathbb{H}^n} = (\alpha+1)^{-(Q-2)}\left[\|u\|_\alpha^2 - \frac{(k-2)^2}{4}\int_{\mathbb{R}^N}\frac{u^2}{|y|^2}\right].$$

This gives a Hardy inequality (see also [4]), which in turn implies, when $k \geq 3$, that $v \in H^1(\mathbb{H}^n)$ iff $\|u\|_\alpha < \infty$ ((3.4) can be similarly derived from (2.4)).

So, as another application of Theorem 3.1, we get the following improvement (in case $k \geq 2$) of a uniqueness result in [13]:

PROPOSITION 3.5. *Let $k \neq 2$ and $h \geq 1$, or $k = 2$ and $h \geq 2$. Then there is at most one cylindrically symmetric positive entire solution of* (3.5).

4. HYPERBOLIC SYMMETRY AND NONDEGENERACY

We now show how to prove nondegeneracy of positive solutions for

$$-\Delta u = \frac{u^{N/(N-2)}}{|y|}, \quad u \in D^1(\mathbb{R}^N). \tag{4.1}$$

Positive solutions of (4.1) are given by $U_{\lambda,\zeta}(y,z) := \lambda^{(N-2)/2} U(\lambda y, \lambda z + \zeta)$, $\lambda > 0$, $\zeta \in \mathbb{R}^h$, U as in (1.2). Taking derivatives in λ and ζ, we see that

$$\Psi_0 := \frac{1 - |y|^2 - |z|^2}{[(1+|y|)^2 + |z|^2]^{N/2}} \quad \text{and} \quad \Psi_j := \frac{z_j}{[(1+|y|)^2 + |z|^2]^{N/2}} \tag{4.2}$$

for $j = 1, \dots, h$ are solutions for the linearization of (4.1) at U:

$$-\Delta \Psi = \frac{N(k-1)}{(1+|y|)^2 + |z|^2} \Psi, \quad \Psi \in D^1(\mathbb{R}^N). \tag{4.3}$$

Nondegeneracy is the content of the following:

THEOREM 4.1. *Let Ψ be a solution of* (4.3) *and Ψ_j be as in* (4.2). *Then*

$$\exists c_0, \dots, c_h \in \mathbb{R} : \quad \Psi = \sum_{j=0}^{h} c_j \Psi_j .$$

Qualitative properties of solutions of (4.3) are first required.

PROPOSITION 4.2. *Let Ψ be a solution of* (4.3). *Then Ψ is smooth in $\{y \neq 0\}$ and Hölder continuous up to $\{y = 0\}$. Furthermore,*

(i) $|\Psi(x)| \leq c/(1+|x|^{N-2})$ *for some $c > 0$,*
(ii) *Ψ is radially symmetric in y.*

Thanks to (ii), we can rewrite (4.3) as an equation for the hyperbolic laplacian on the ball model. Let M denote the Möbius map (the standard hyperbolic isometry between the half-space and the ball model)

$$M(r,z) := \left(\frac{1 - r^2 - |z|^2}{(1+r)^2 + |z|^2}, \frac{2z}{(1+r)^2 + |z|^2} \right).$$

Then $\Phi(\xi) := (H\Psi)(M\xi)$, $\xi \in \mathbb{R}^n$, $|\xi| < 1$, is in $H^1(\mathbb{H}^n)$ and solves

$$\begin{aligned} -\Delta_{\mathbb{H}^n} \Phi &:= -\left[\frac{1-|\xi|^2}{2}\right]^2 \Delta \Phi - (n-2)\left[\frac{1-|\xi|^2}{2}\right]\langle \nabla \Phi, \xi \rangle \\ &= \frac{h^2 - (k-2)^2}{4} \Phi + \frac{N(k-1)}{4}(1 - |\xi|^2)\Phi. \end{aligned} \tag{4.4}$$

In particular, (4.4) has the solutions

$$\Phi_j(\xi) = \left(\frac{1-|\xi|^2}{4}\right)^{(N-2)/2} \xi_j, \qquad j = 0, \ldots, h.$$

Now the proof of Theorem 4.1 ends in two steps. If Φ solves (4.3), then

STEP 1. $\exists c_0, \ldots, c_h \in \mathbb{R}$ such that $\Phi^r := \Phi - \sum_{j=0}^{h} c_j \Phi_j$ is radial.
STEP 2. (4.4) has no nontrivial radial solution in $H^1(\mathbb{H}^n)$.

We sketch the proof of Step 2. Let $z(\rho) := \Phi^r(\sqrt{\rho})(\frac{1-\rho}{2})^{-(N-2)/2}$ for $\rho \in (0,1)$. Proposition 4.2(ii) implies z is bounded in (0, 1), while, easily,

$$\rho(1-\rho)z'' + \left[\frac{h+1}{2} - \left(\frac{h+2k-3}{2} + 1\right)\rho\right]z' + \frac{k-1}{2}z = 0 \tag{4.5}$$

in (0, 1). Now, (4.5) is *Gauss's hypergeometric equation*. Its solutions are given explicitly by hypergeometric functions and one can deduce that $z \neq 0$ implies z unbounded on (0, 1). Thus $z \equiv 0$.

5. SCALAR CURVATURE TYPE EQUATIONS

The Webster scalar curvature problem on the unit sphere in $\mathbb{C}^n$, a CR analogue of the Nirenberg problem, is to find positive solutions of

$$-\Delta_{H^n} u(\xi) = \mathcal{R}(\xi) u(\xi)^{(Q+2)/(Q-2)} \quad \text{in } H^n, \tag{5.1}$$

where $Q := 2n+2$, $H^n = \mathbb{R}^{2n} \times \mathbb{R}$ is the Heisenberg group and Δ_{H^n} is the Heisenberg sublaplacian.

In case $\mathcal{R}(\xi) = 1 + \varepsilon\mathcal{K}(\xi)$, (5.1) has been studied in [10], and in case $\mathcal{R}(\xi) = \mathcal{R}(|Z|, t)$, $\xi = (Z, t) \in \mathbb{R}^{2n} \times \mathbb{R}$, it has been studied in [6]. In this regard, we recall (see [13]) that the Heisenberg sublaplacian acts on cylindrically symmetric functions as the Grushin operator:

$$\text{if} \quad u(Z,t) = u(|Z|, t) \quad \text{then} \quad \Delta_{H^n} u = \Delta_Z u + 4|Z|^2 u_{tt},$$

where Δ_Z is the laplacian in $\mathbb{R}^{2n}$. More general critical Grushin equations appear in [7], related to the Yamabe problem in groups of Heisenberg type:

$$-\Delta_y u - 4|y|^2 \Delta_z u = u^{(Q+2)/(Q-2)} \tag{5.2}$$

for $(y, z) \in \mathbb{R}^{2n} \times \mathbb{R}^h$ and $h \geq 1$. Here $Q = 2n + 2h$. So, (5.1) belongs to the class

$$-\Delta_y u - 4|y|^2 \Delta_z u = K(y, z) u^{(Q+2)/(Q-2)}. \tag{5.3}$$

Under symmetry assumptions, (5.3) is related to equation (4.1), in the following sense.

Let $N = n + 1 + h$ and consider the equation

$$(5.4) \qquad -\Delta u = \phi(y,z)\frac{u^{N/(N-2)}}{|y|}, \qquad (y,z) \in \mathbb{R}^{n+1} \times \mathbb{R}^h.$$

LEMMA 5.1. *Let $K(y,z) = K(|y|,z)$. Set $\phi(y,t) := \frac{1}{4}K(\sqrt{|y|},z)$ and assume $u = u(|y|,z)$ solves* (5.4). *Then $v(y,t) := u(|y|^2,t)$ solves* (5.3).

On the base of Sections 3–4, several results can be proved for (5.4). We present some of them, somehow global in nature, applied to (5.1).

THEOREM 5.1. *Let $\mathcal{R} = \mathcal{R}(|Z|,t)$, with $\mathcal{R} \equiv \mathcal{R}(\infty) = 1$ on $\{Z = 0\}$. Suppose that either* $\inf \mathcal{R}(\xi) > 2^{-2-1/n}$, *or $\mathcal{R}(Z,t) = 1 + \rho(Z,t)$ where $\rho \in C_c(H^n \setminus \{Z = 0\})$. Then* (5.1) *has a solution.*

We solve (5.4) by global variational methods. For $u \in D^1(\mathbb{R}^N)$, let

$$(5.5) \qquad J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{N-2}{2(N-1)}\int_{\mathbb{R}^N} \phi(x)\frac{|u|^{2(N-1)/(N-2)}}{|y|}\,dx.$$

A blow-up analysis can be carried out for Palais–Smale sequences of J. Thanks to the uniqueness result for (4.1), precise information on how blow-up can occur can be obtained assuming $\phi(0,z) \equiv \phi(\infty) = 1$. Denoting by $S = S_1^0$ the best constant in (1.1) with $\mu = 0$ and $t = 1$, we can prove in this case

LEMMA 5.2. *Either J has a positive critical point, or Palais–Smale sequences at levels $\beta \in (S^{N-1}/2(N-1), S^{N-1}/(N-1))$ cannot blow up.*

The nature of blow-up implies low sublevels are disconnected and this leads to a min-max level $\beta > S^{N-1}/2(N-1)$ while the assumptions in Theorems 5.1 allow proving that $\beta < S^{N-1}/(N-1)$, a rather surprising result under the second one.

Finally, nondegeneracy of the critical points of J established in Section 4 allows one to perform a finite-dimensional reduction, in the spirit of [10]. Let

$$(*) \qquad \begin{aligned} &N \geq 4, \quad \phi = 1 + \varepsilon\varphi, \quad \psi(z) := \varphi(0,z), \quad z \in \mathbb{R}^h, \\ &\text{the limits } \lim_{x\to 0}\phi\left(\frac{x}{|x|^2}\right) \text{ and } \lim_{x\to 0}\nabla_z\phi\left(\frac{x}{|x|^2}\right) \neq 0 \text{ exist.} \end{aligned}$$

THEOREM 5.2. *Assume* $(*)$. *Let $\psi(z)$ have a finite number of nondegenerate critical points ζ_j, $j = 1,\dots,m$, of index $m_j = m(\psi;\zeta_j)$ such that*

$$\Delta^*(\zeta_j) := (k-1)\Delta_y\varphi(0,\zeta_j) + (2k+h-3)\Delta_z\varphi(0,\zeta_j) \neq 0 \quad \forall j.$$

Then (5.4) *has a solution for* $|\varepsilon|$ *small provided*

$$\sum_{\{j\,:\,\Delta^*(\zeta_j)>0\}} (-1)^{m_j} \neq 1.$$

As for (5.1), let $\mathcal{K} = \mathcal{K}(|Z|, t)$ be smooth, such that $\lim_{|Z|^2+t^2\to\infty} \mathcal{K}$ exists and

$$\frac{\partial^i \mathcal{K}}{\partial r^i}(0, t) \equiv 0 \quad \forall i = 1, \ldots, 4, \quad \exists c \neq 0 : \quad t^2 \mathcal{K}_t(0, t) \underset{|t|\to\infty}{\longrightarrow} c.$$

COROLLARY 5.3. *Let* $\mathcal{R} := 1 + \varepsilon\mathcal{K}$*,* $\mathcal{K}$ *as above and such that* $\mathcal{K}(0, t)$ *has only a finite number of, nondegenerate, critical points. Then* (5.1) *has a solution for* ε *small, provided* $\mathcal{K}(0, t)$ *has at least two minima.*

REFERENCES

[1] L. ALMEIDA - L. DAMASCELLI - Y. GE, *A few symmetry results for nonlinear elliptic PDE on noncompact manifolds*. Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002), 313–342.

[2] M. BADIALE - G. TARANTELLO, *A Sobolev–Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics*. Arch. Ration. Mech. Anal. 163 (2002), 259–293.

[3] D. CASTORINA - I. FABBRI - G. MANCINI - K. SANDEEP, *Hardy–Sobolev inequalities, hyperbolic symmetry and scalar curvature type equations*. J. Differential Equations, to appear.

[4] L. D'AMBROSIO, *Hardy inequalities related to Grushin type operators*. Proc. Amer. Math. Soc. 132 (2003), 725–734.

[5] I. FABBRI - G. MANCINI - K. SANDEEP, *Classification of solutions of a critical Hardy–Sobolev operator*. J. Differential Equations 224 (2006), 258–276.

[6] V. FELLI - F. UGUZZONI, *Some existence results for the Webster scalar curvature problem in presence of symmetry*. Ann. Mat. Pura Appl. 183 (2004), 469–493.

[7] N. GAROFALO - D. VASSILEV, *Symmetry properties of positive entire solutions of Yamabe-type equations on groups of Heisenberg type*. Duke Math. J. 106 (2001), 411–448.

[8] M. GAZZINI - R. MUSINA, *Hardy–Sobolev–Maz'ya inequalities: symmetry and breaking symmetry of extremal functions*. Preprint.

[9] M. K. KWONG - Y. LI, *Uniqueness of radial solutions of semilinear elliptic equations*. Trans. Amer. Math. Soc. 333 (1992), 339–363.

[10] A. MALCHIODI - F. UGUZZONI, *A perturbation result for the Webster scalar curvature problem on the CR sphere*. J. Math. Pures Appl. 81 (2002), 983–997.

[11] G. MANCINI - K. SANDEEP, *On a semilinear elliptic equation in* $\mathbb{H}^n$. Ann. Scuola Norm. Sup. Pisa Cl. Sci., to appear.

[12] V. G. MAZ'YA, *Sobolev Spaces*. Springer Ser. Soviet Math., Springer, Berlin, 1985.

[13] R. MONTI - D. MORBIDELLI, *Kelvin transform for Grushin operators and critical semilinear equations*. Duke Math. J. 131 (2006), 167–202.

[14] R. MUSINA, *Ground state solutions of a critical problem involving cylindrical weights*. Nonlinear Anal. 68 (2008), 3972–3986.

[15] A. K. TERTIKAS - K. TINTAREV, *On existence of minimizers for the Hardy–Sobolev–Maz'ya inequality*. Ann. Mat. Pura Appl. 186 (2007), 645–662.

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D. Castorina, I. Fabbri, G. Mancini
Dipartimento di Matematica
Università degli Studi "Roma Tre"
Largo S. Leonardo Murialdo 1
00146 ROMA, Italy
daniele.castorina@dipmat.unipg.it
fabri@mat.uniroma3.it
mancini@mat.uniroma3.it

K. Sandeep
TIFR Centre for Applicable Mathematics
Sharadanagar, Chikkabommasandra
BANGALORE 560 064, India
sandeep@math.tifrbng.res.in