



**Partial differential equations.** — *Positive solutions of the Robin problem for semilinear elliptic equations on annuli*, by YU-XIA FU and QIU-YI DAI, communicated on 14 March 2008.

ABSTRACT. — Let  $n \geq 3$  and  $\Omega_R = \{x \in \mathbb{R}^n; R < |x| < 1\}$ . We consider the following Robin problem:

$$\begin{cases} -\Delta u = f(u), & x \in \Omega_R, \\ u > 0, & x \in \Omega_R, \\ \frac{\partial u}{\partial \nu} + \beta u = 0, & x \in \partial\Omega_R, \end{cases}$$

where  $\beta$  is a positive parameter and  $\nu$  is the unit outward vector normal to  $\partial\Omega_R$ .

Under the assumptions (F1)–(F5) in the introduction, we prove that the above problem has at most one solution when  $\beta$  is small enough. In addition to (F1)–(F5), if (A1) in the introduction is satisfied, then the above problem has at least  $k$  nonradial solutions when  $\beta$  is large enough.

KEY WORDS: Positive solutions; Robin problem; semilinear elliptic equations.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 35J65.

## 1. INTRODUCTION

Let  $n \geq 3$  and  $\Omega = \Omega_R = \{x \in \mathbb{R}^n; R < |x| < 1\}$ . We consider the following problem:

$$(1.1) \quad \begin{cases} -\Delta u = f(u), & x \in \Omega_R, \\ u > 0, & x \in \Omega_R, \\ \frac{\partial u}{\partial \nu} + \beta u = 0, & x \in \partial\Omega_R, \end{cases}$$

where  $\beta$  is a nonnegative parameter,  $\nu$  is the unit outward vector normal to  $\partial\Omega_R$ , and  $f \in C^1(\mathbb{R})$  is a nonnegative function.

The investigation of the structure of solution sets is one of the main topics in the study of elliptic partial differential equations. For problems like (1.1), there is vast literature on two special cases:  $\beta = 0$  and  $\beta = \infty$ .

When  $\beta = 0$ , problem (1.1) is called the Neumann problem. Under our assumptions, it is trivial, since by integration by parts, we can easily prove that (1.1) has no solution when  $\beta = 0$ .

When  $\beta = \infty$ , problem (1.1) is called the Dirichlet problem. There are many interesting results on the Dirichlet problem (1.1), especially for the case of  $f(u) = u^p$

and  $1 < p < (n + 2)/(n - 2)$ . For example, the compactness of the solution set was obtained in [6]. Uniqueness and nonuniqueness of radial solutions were discussed in [12, 13]. The existence of nonradial solutions was proven in [4]. Later on, a multiplicity result for nonradial solutions for the inner radius  $R$  very close to 1 was given in [10, 11] (to mention but a few results). For more results on the Dirichlet problem (1.1) on general domains, we refer to [2, 13, 14, 19] and the references therein.

In the present note, we focus on the case  $\beta \in (0, \infty)$ . In this case, (1.1) is called the Robin problem. Compared with the Dirichlet and Neumann problems, there are few results on the Robin problem (see, however, [8, 15]). The reason seems to be the general belief that results for the Robin problem should be similar to those for the Dirichlet problem and the methods used may be similar. However, recent research implies that this is not the case even for  $f(u) = \lambda u$ . When  $f(u) = \lambda u$ , it is well known that problem (1.1) has a positive solution only for  $\lambda = \lambda_1(\Omega)$ , where  $\lambda_1(\Omega)$  is the first eigenvalue of the Laplacian with the corresponding boundary condition. In the Dirichlet case, we know that  $\lambda_1(\Omega)$  is monotone with respect to the domain  $\Omega$ , i.e.  $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$  for  $\Omega_1 \subset \Omega_2$ . However, this is false for the Robin problem. A Faber–Krahn type inequality is valid for both the first eigenvalue of the Dirichlet problem and of the Robin problem (see [7]), but the proofs are very different: the Robin case is more difficult, since the usual symmetrization method ceases to be applicable (see [3, 5, 7]).

This note will give a new example which shows that the Robin problem is essentially different from the Dirichlet problem. For this purpose, we assume that

- (F1)  $f(s) > 0$  for  $s > 0$ ,
- (F2)  $sf'(s) \geq (1 + \sigma)f(s)$  for some  $1 < \sigma < 4/(n - 2)$ ,
- (F3)  $\lim_{s \rightarrow \infty} f(s)/s^p = C > 0$ ,  $1 < p < (n + 2)/(n - 2)$ ,
- (F4)  $f(s)/s$  is monotone on  $(0, \infty)$ ,
- (F5)  $\lim_{s \rightarrow 0} f'(s) = 0$ .

A typical example of  $f(s)$  which satisfies (F1)–(F5) is  $f(s) = s^p$  with  $1 < p < (n + 2)/(n - 2)$ . For a fixed integer  $k > 1$ , we denote by  $L$  the least common multiple of  $1, \dots, k$ , and by  $\omega_n$  the area of the unit sphere in  $\mathbb{R}^n$ . Let

$$C(n) = \left( \frac{2n\pi}{n+2} \right)^2 \left( \frac{n}{\omega_n} \right)^{4/n} \quad \text{and} \quad D(n, L) = \left( \frac{L(L+n-2)}{\sigma C(n)} \right)^{n/2}.$$

Then our assumption on the inner radius  $R$  of the annulus  $\Omega_R$  is

$$(A1) \quad R \geq \left( \frac{D(n, L)}{1 + D(n, L)} \right)^{1/n}.$$

By calculus of variation methods, one can easily prove that problem (1.1) has at least one solution under the assumptions (F1)–(F5). So we will ignore the question of pure existence for problem (1.1), and will be concerned with the uniqueness, symmetry and symmetry breaking of solutions. Our main results can be stated as follows.

**THEOREM 1.** *If (F1)–(F5) are satisfied, then there exists a positive number  $\beta_*$  such that problem (1.1) has at most one solution for any  $\beta \in (0, \beta_*)$ .*

REMARK 2. By the symmetry of the operator and the domain, we conclude from Theorem 1 that the solution of problem (1.1) is unique and radially symmetric when  $\beta \in (0, \beta_*)$ . This contrasts with the Dirichlet case, where many nonradial solutions can be obtained provided that (A1) is satisfied (see [10]).

THEOREM 3. *If (F1)–(F5) and (A1) are satisfied, then there exists a positive number  $\beta^*$  such that problem (1.1) has at least  $k$  nonradial positive solutions for any  $\beta \in (\beta^*, +\infty)$ .*

REMARK 4. Theorem 3 implies that the symmetry and uniqueness results for solutions of problem (1.1) fail to hold when  $\beta$  is large enough.

The paper is organized as follows. In Section 2, we study the uniqueness of solutions of problem (1.1). Section 3 is devoted to looking for nonradial solutions.

## 2. THE PROOF OF THEOREM 1

To prove Theorem 1, we need the following lemmas.

LEMMA 2.1. *If  $u$  is a nonnegative solution of the equation*

$$-\Delta u = u^p, \quad x \in \mathbb{R}^n,$$

*with  $1 < p < (n + 2)/(n - 2)$ , then  $u \equiv 0$ .*

Lemma 2.1 is proved in [6].

LEMMA 2.2. *For  $\beta$  small enough, there exists a number  $M > 0$  independent of  $\beta$  such that any solution  $u = u_\beta$  of problem (1.1) satisfies*

$$\|u\|_{L^\infty(\bar{\Omega}_R)} \leq M.$$

PROOF. Suppose that the conclusion is not true. Then there exists a sequence  $\beta_j \rightarrow 0$  as  $j \rightarrow \infty$ , a corresponding sequence of solutions  $u_j = u_{\beta_j}$  of problem (1.1) with  $\beta = \beta_j$ , and a sequence of points  $x_j$  in  $\bar{\Omega}_R$  such that

$$M_j = \|u_j\|_{L^\infty(\bar{\Omega}_R)} = u_j(x_j) \rightarrow \infty \quad \text{as } j \rightarrow \infty.$$

Consider the auxiliary function

$$v_j(y) = M_j^{-1} u_j(x_j + M_j^{(1-p)/2} y)$$

defined on  $\bar{\Omega}_j = M_j^{(p-1)/2}(\bar{\Omega}_R - x_j)$ . Then it is easy to verify that  $v_j(y)$  satisfies the following problem:

$$(2.1) \quad \begin{cases} -\Delta_y v_j = v_j^p \frac{f(M_j v_j)}{(M_j v_j)^p}, & y \in \Omega_j, \\ v_j(0) = 1, \\ \frac{\partial v_j}{\partial \nu} + M_j^{(1-p)/2} \beta_j v_j = 0, & y \in \partial \Omega_j. \end{cases}$$

If we denote by  $D$  either the whole space  $\mathbb{R}^n$  or the half-space  $\mathbb{R}_+^n$ , then  $\Omega_j \rightarrow D$  as  $j \rightarrow \infty$ .

Since  $|v_j(y)| \leq 1$ , by standard elliptic estimates there exists a positive constant  $C_1$  independent of  $j$  such that for any compact domain  $K \subset D$ , we have

$$\|v_j\|_{C^{2,\gamma}(K)} \leq C_1$$

for  $j$  large enough. Hence, up to a subsequence, we may assume that  $\{v_j\}$  converges uniformly on any compact domain of  $D$  to a function  $v$ . Furthermore, by (2.1) and the assumption (F3),  $v$  should satisfy

$$(2.2) \quad \begin{cases} -\Delta v = Cv^p, & y \in D, \\ v(0) = 1, \\ \frac{\partial v}{\partial \nu} = 0, & y \in \partial D \ (D = \mathbb{R}_+^n), \end{cases}$$

If  $D = \mathbb{R}^n$ , then  $\tilde{v} = C^{1/(p-1)}v$  satisfies

$$(2.3) \quad \begin{cases} -\Delta \tilde{v} = \tilde{v}^p, & y \in \mathbb{R}^n, \\ \tilde{v}(0) = C^{1/(p-1)}. \end{cases}$$

This contradicts Lemma 2.1.

If  $D = \mathbb{R}_+^n$ , then the function  $\tilde{v}$  defined by

$$\tilde{v}(y_1, \dots, y_n) = \begin{cases} C^{1/(p-1)}v(y_1, \dots, y_n) & \text{if } y \in \mathbb{R}_+^n, \\ C^{1/(p-1)}v(y_1, \dots, y_{n-1}, -y_n) & \text{if } y \in \mathbb{R}_-^n, \end{cases}$$

satisfies (2.3). This also contradicts Lemma 2.1.

**PROOF OF THEOREM 1.** We argue by contradiction. Suppose that the conclusion of Theorem 1 is false. Then there exists a sequence  $\beta_j \rightarrow 0^+$  as  $j \rightarrow \infty$  such that problem (1.1) with  $\beta = \beta_j$  has two different solutions  $u_j^{(1)}, u_j^{(2)}$ . We shall deduce that, up to a subsequence,

$$\lim_{j \rightarrow \infty} u_j^{(i)} = 0 \quad \text{uniformly on } \Omega_R \text{ for } i = 1, 2.$$

Indeed, by Lemma 2.2 and the standard elliptic estimate, there exists a positive constant  $C$  independent of  $j$  such that

$$\|u_j^{(i)}\|_{C^{2,\gamma}(\Omega_R)} \leq C \quad \text{for } i = 1, 2.$$

Hence, up to a subsequence, we may assume that  $u_j^{(i)}$  ( $i = 1, 2$ ) converges uniformly on  $\Omega_R$  to a function  $u^{(i)}$  ( $i = 1, 2$ ) as  $j \rightarrow \infty$ , and  $u^{(i)}$  ( $i = 1, 2$ ) is a solution of the following problem:

$$(2.4) \quad \begin{cases} -\Delta u = f(u), & x \in \Omega_R, \\ u(x) \geq 0, & x \in \Omega_R, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial \Omega_R. \end{cases}$$

Now,  $u^{(i)} \equiv 0$  ( $i = 1, 2$ ) follows from  $f(u) \geq 0$  and integration by parts.

Set

$$w_j(x) = \frac{u_j^{(1)}(x) - u_j^{(2)}(x)}{\|u_j^{(1)} - u_j^{(2)}\|_{L^\infty(\bar{\Omega}_R)}}.$$

We can verify that  $w_j(x)$  satisfies

$$(2.5) \quad \begin{cases} -\Delta w_j = f'(\xi_j)w_j, & x \in \Omega_R, \\ \|w_j\|_{L^\infty(\bar{\Omega}_R)} = 1, \\ \frac{\partial w_j}{\partial \nu} + \beta_j w_j = 0, & x \in \partial\Omega_R, \end{cases}$$

where  $\xi_j(x)$  is between  $u_j^{(1)}(x)$  and  $u_j^{(2)}(x)$ . Accordingly,  $\xi_j(x) \rightarrow 0$  as  $j \rightarrow \infty$ . Noticing that  $|w_j| \leq 1$ , by a standard elliptic estimate, and up to a subsequence, we may assume that  $w_j(x)$  converges uniformly on  $\Omega_R$  to a function  $w(x)$  as  $j \rightarrow \infty$ . Furthermore, since  $\xi_j(x) \rightarrow 0$ , by (F5) and (2.5), we know that  $w(x)$  satisfies

$$(2.6) \quad \begin{cases} -\Delta w = 0, & x \in \Omega_R, \\ \|w\|_{L^\infty(\bar{\Omega}_R)} = 1, \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega_R. \end{cases}$$

This implies that  $w(x) \equiv \pm 1$  on  $\Omega_R$ . Thus, there exists an integer  $N$  such that  $u_j^{(1)}(x) - u_j^{(2)}(x)$  has a fixed sign on  $\Omega_R$  for any  $j > N$ . Consequently, it follows from the assumption (F4) that  $f(u_j^{(1)})/u_j^{(1)} - f(u_j^{(2)})/u_j^{(2)}$  has a fixed sign on  $\Omega_R$  for any  $j > N$ .

On the other hand, by Green's formula, we have

$$\int_{\Omega_R} [f(u_j^{(1)})u_j^{(2)} - f(u_j^{(2)})u_j^{(1)}] = 0,$$

which implies that  $f(u_j^{(1)})/u_j^{(1)} - f(u_j^{(2)})/u_j^{(2)}$  should change sign on  $\Omega_R$  for any  $j$ . This contradiction completes the proof of Theorem 1.

### 3. THE PROOF OF THEOREM 3

This section is devoted to proving Theorem 3. To this end, we introduce some notations and definitions. Let  $O(n)$  be the set of all  $n \times n$  orthogonal matrices.

DEFINITION 3.1. *Let  $G$  be a subgroup of  $O(n)$ . The action of an element  $g$  in  $G$  on a function  $u : \Omega_R \rightarrow \mathbb{R}$  is defined by*

$$gu(x) = u(gx) \quad \forall x \in \Omega_R.$$

A function  $u$  is said to be invariant under  $G$  or  $G$ -symmetric if

$$gu(x) = u(x) \quad \text{for all } g \in G, x \in \Omega_R;$$

in this case, we write  $u \in \text{Inv } G$ .

DEFINITION 3.2. Two functions  $u$  and  $v$  on  $\Omega_R$  are said to be equivalent if there exists a  $g \in O(n)$  such that  $v(x) = gu(x)$  for all  $x \in \Omega_R$ .

It is easy to check that if  $u$  and  $v$  are equivalent, then  $u$  is a solution of problem (1.1) if and only if  $v$  is.

Let  $G_j$  be a rotation subgroup of  $O(2)$  defined by

$$G_j := \left\{ g \in O(2); g(x_1, x_2) = \left( x_1 \cos \frac{2\pi l}{j} + x_2 \sin \frac{2\pi l}{j}, -x_1 \sin \frac{2\pi l}{j} + x_2 \cos \frac{2\pi l}{j} \right), \right. \\ \left. (x_1, x_2) \in \mathbb{R}^2, l \text{ is any integer} \right\}.$$

We will solve problem (1.1) by finding positive critical points of the functional

$$J(u) = \frac{1}{2} \int_{\Omega_R} |\nabla u|^2 + \frac{1}{2} \int_{\partial\Omega_R} \beta u^2 - \int_{\Omega_R} F(u^+),$$

where  $F(u) = \int_0^u f(t) dt$  and  $u^+ = \max\{u, 0\}$ .

Instead of looking for positive critical points of  $J(u)$  on  $H^1(\Omega_R)$ , we look for them on the Nehari submanifold  $M$ , where

$$M = \{u \in H^1(\Omega_R); I(u) = 0 \text{ and } u \neq 0\}$$

and

$$I(u) = \int_{\Omega_R} |\nabla u|^2 + \int_{\partial\Omega_R} \beta u^2 - \int_{\Omega_R} u^+ f(u^+).$$

Let

$$V_0 = \{u \in M; u \in \text{Inv } O(n)\},$$

$$V_j = \{u \in M; u \in \text{Inv}(G_j \times O(n-2))\} \quad \text{for } j = 1, \dots, L.$$

It is obvious that  $V_0 \subset V_j$  for  $j = 1, \dots, L$ . We first prove that  $V_0 \neq \emptyset$  and  $V_j - V_0 \neq \emptyset$  for  $j = 1, \dots, L$ . The former follows from the mountain pass lemma (see [1]) and the symmetric criticality principle (see Theorem 1.28, pp. 18 in [16]). Indeed, if  $J(u)$  is viewed as a functional on  $H_r^1(\Omega_R) = \{u \in H^1(\Omega_R); u \in \text{Inv } O(n)\}$ , then it is easy to verify that  $J(u)$  is  $C^1$  and has a mountain pass structure. Moreover,  $J(u)$  satisfies the P-S condition due to (F2), (F3) and the compactness of the imbedding  $H_r^1(\Omega_R) \hookrightarrow L^{p+1}(\Omega_R)$ . Consequently, we can obtain a nontrivial radial solution of problem (1.1) by making use of the mountain pass lemma and the symmetric criticality principle. Hence,  $V_0 \neq \emptyset$ .

To prove  $V_j - V_0 \neq \emptyset$ , the following lemmas are needed.

LEMMA 3.3. There exists a constant  $C > 0$  independent of  $v$  such that for any  $v \in M$ , we have

$$\|v\|_{H^1(\Omega_R)} \geq C.$$

PROOF. For any  $v \in M$ , we have  $v \neq 0$  and

$$\int_{\Omega_R} |\nabla v|^2 + \int_{\partial\Omega_R} \beta v^2 = \int_{\Omega_R} v^+ f(v^+).$$

By (F3), we conclude that there exists a constant  $C_0$  such that

$$f(v^+) \leq C_0(v^+)^p.$$

Hence,

$$\int_{\Omega_R} |\nabla v|^2 + \int_{\partial\Omega_R} \beta v^2 \leq C_0 \int_{\Omega_R} (v^+)^{p+1}.$$

By the Sobolev inequality, we have

$$(3.1) \quad \int_{\Omega_R} |\nabla v|^2 + \int_{\partial\Omega_R} \beta v^2 \leq C_0 \|v^+\|_{H^1(\Omega_R)}^{p+1} \leq C_0 \|v^+\|_{H^1(\Omega_R)}^{p+1}.$$

If we denote by  $\lambda_1^\beta(\Omega_R)$  the first eigenvalue of

$$(3.2) \quad \begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \Omega_R, \\ \frac{\partial\varphi}{\partial\nu} + \beta\varphi = 0, & x \in \partial\Omega_R, \end{cases}$$

then

$$\int_{\Omega_R} |\nabla v|^2 + \int_{\partial\Omega_R} \beta v^2 \geq \lambda_1^\beta(\Omega_R) \int_{\Omega_R} v^2 \quad \text{for any } v \in H^1(\Omega_R).$$

Accordingly,

$$(3.3) \quad \begin{aligned} \|v\|_{H^1(\Omega_R)}^2 &= \int_{\Omega_R} |\nabla v|^2 + \int_{\Omega_R} v^2 \\ &\leq \int_{\Omega_R} |\nabla v|^2 + \frac{1}{\lambda_1^\beta(\Omega_R)} \int_{\Omega_R} |\nabla v|^2 + \frac{\beta}{\lambda_1^\beta(\Omega_R)} \int_{\partial\Omega_R} v^2 \\ &\leq \left(1 + \frac{1}{\lambda_1^\beta(\Omega_R)}\right) \left(\int_{\Omega_R} |\nabla v|^2 + \int_{\partial\Omega_R} \beta v^2\right). \end{aligned}$$

Combining (3.1) and (3.3), we obtain

$$\frac{\lambda_1^\beta(\Omega_R)}{1 + \lambda_1^\beta(\Omega_R)} \|v\|_{H^1(\Omega_R)}^2 \leq C_0 \|v\|_{H^1(\Omega_R)}^{p+1}.$$

This implies that

$$\|v\|_{H^1(\Omega_R)} \geq \left[ \frac{\lambda_1^\beta(\Omega_R)}{C_0(1 + \lambda_1^\beta(\Omega_R))} \right]^{1/(p-1)},$$

and the proof of Lemma 3.3 is complete.

Let

$$b_0 = \inf\{J(u); u \in V_0\}.$$

Then, as  $V_0 \hookrightarrow L^{p+1}(\Omega_R)$  is compact, there exists  $u_0 \in V_0$  such that  $J(u_0) = b_0$ . Moreover, by Lemma 3.3 and the symmetric criticality principle, we have  $u_0 > 0$ , and  $u_0$  is a solution of problem (1.1). In the following paragraphs, unless specially declared, we always use  $u_0$  to denote a minimizer of  $J(u)$  in  $V_0$ , that is,  $u_0$  is a function such that

$$J(u_0) = b_0 = \inf\{J(u); u \in V_0\}.$$

To continue our proof of  $V_j - V_0 \neq \emptyset$ , we consider the following linearized eigenvalue problem of problem (1.1) at  $u_0$ :

$$(3.4) \quad \begin{cases} -\Delta w - f'(u_0)w = \mu w, & x \in \Omega_R, \\ \frac{\partial w}{\partial \nu} + \beta w = 0, & x \in \partial\Omega_R. \end{cases}$$

In spherical coordinates, problem (3.4) is equivalent to

$$(3.5) \quad \begin{cases} -\varphi''(r) - \frac{n-1}{r}\varphi'(r) - \left\{f'(u_0) - \frac{\alpha_j}{r^2}\right\}\varphi(r) = \mu_{j,l}\varphi(r), & r \in (R, 1), \\ \varphi'(1) + \beta\varphi(1) = 0, \\ -\varphi'(R) + \beta\varphi(R) = 0, \end{cases}$$

where  $\alpha_j = j(j+n-2)$ ,  $j = 0, 1, \dots$ , and  $l = 1, 2, \dots$ . Note that the  $\alpha_j$  are the eigenvalues of  $-\Delta$  on  $S^{n-1}$ , the unit sphere. If  $\psi_j$  denote the eigenfunctions of  $-\Delta$  on  $S^{n-1}$  corresponding to  $\alpha_j$ , and  $\varphi_{j,l}$  are the eigenfunctions corresponding to  $\mu_{j,l}(u_0)$ , then the associated eigenfunctions  $w_{j,l}$  of problem (3.4) are given by  $w_{j,l} = \varphi_{j,l}\psi_j$ . In particular, for  $l = 1$ , if we denote  $\varphi_{j,1}$  by  $\varphi_j$ , then  $w_j = w_{j,1} = \varphi_j\psi_j$ . It is easy to check that  $w_0$  is radially symmetric.

To obtain a useful property of the eigenvalues of problem (3.5), we need the following lemma.

LEMMA 3.4. *Let  $\lambda_1^D(\Omega_R)$  be the first eigenvalue of*

$$(3.6) \quad \begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \Omega_R, \\ \varphi = 0, & x \in \partial\Omega_R. \end{cases}$$

Then

$$\lambda_1^D(\Omega_R) > \left[ \frac{2n\pi}{(n+2)(1-R^n)^{1/n}} \right]^2 \left[ \frac{n}{\omega_n} \right]^{4/n}.$$

PROOF. By Theorem 5.1, pp. 121 in [18] (or see [9]), we have

$$\lambda_1^D(\Omega_R) \geq \frac{4n\pi^2}{n+2} (n\omega_n^{-1})^{2/n} \frac{1}{|\Omega_R|^{2/n}}.$$

Since

$$|\Omega_R| = \frac{\omega_n}{n}(1-R^n),$$

we have

$$\lambda_1^D(\Omega_R) > \left[ \frac{2n\pi}{(n+2)(1-R^n)^{1/n}} \right]^2 \left[ \frac{n}{\omega_n} \right]^{4/n}.$$

This completes the proof of Lemma 3.4.



With the aid of Lemma 3.4, we can prove a useful property of the eigenvalues of problem (3.5).

LEMMA 3.5. *Assume that (F1), (F2), and (A1) are satisfied. Then there exists a positive number  $\beta^*$  such that  $\mu_{j,1}(u_0) < 0$  for any  $0 \leq j \leq L$ , and  $\beta > \beta^*$ .*

PROOF. By the Rayleigh–Ritz formula for the first eigenvalue, we have

$$\mu_{j,1} = \inf \left\{ \frac{Q_j(v)}{\int_R^1 r^{n-1} v^2}; v \in H^1(\Omega_R), v \text{ is radial} \right\},$$

where

$$Q_j(v) = - \int_R^1 \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial v}{\partial r} \right) v - \int_R^1 r^{n-1} \left( f'(u_0) - \frac{\alpha_j}{r^2} \right) v^2.$$

Since  $u_0$  is a solution of problem (1.1), we have

$$(3.7) \quad \int_{\Omega_R} |\nabla u_0|^2 + \int_{\partial\Omega_R} \beta u_0^2 = \int_{\Omega_R} u_0 f(u_0).$$

By (F2) and (3.7), we get

$$(3.8) \quad \begin{aligned} \omega_n Q_j(u_0) &= \int_{\partial\Omega_R} \beta u_0^2 + \int_{\Omega_R} |\nabla u_0|^2 - \int_{\Omega_R} f'(u_0) u_0^2 + \int_{\Omega_R} \frac{\alpha_j}{r^2} u_0^2 \\ &\leq -\sigma \left( \int_{\partial\Omega_R} \beta u_0^2 + \int_{\Omega_R} |\nabla u_0|^2 \right) + \frac{\alpha_j}{R^2} \int_{\Omega_R} u_0^2, \end{aligned}$$

where  $\omega_n$  is the area of  $S^{n-1}$ .

Let  $\lambda_1^\beta(\Omega_R)$  be the first eigenvalue of the eigenvalue problem (3.2). Then

$$\int_{\Omega_R} |u_0|^2 \leq \frac{1}{\lambda_1^\beta(\Omega_R)} \left( \int_{\partial\Omega_R} \beta u_0^2 + \int_{\Omega_R} |\nabla u_0|^2 \right).$$

From (3.8), we get

$$Q_j(u_0) \leq \frac{1}{\omega_n} \left( -\sigma + \frac{\alpha_j}{R^2 \lambda_1^\beta(\Omega_R)} \right) \left( \int_{\partial\Omega_R} \beta u_0^2 + \int_{\Omega_R} |\nabla u_0|^2 \right).$$

If  $j = 0$ , we have obviously  $Q_0(u_0) < 0$  and  $\mu_{0,1}(u_0) < 0$ .

If  $j \geq 1$ , we conclude that there exists a positive number  $\beta^*$  such that for any  $\beta \in (\beta^*, \infty)$ , we have

$$\lambda_1^\beta(\Omega_R) > \left( \frac{2n\pi}{n+2} \right)^2 \left( \frac{n}{\omega_n} \right)^{4/n} (1 - R^n)^{-2/n},$$

due to Lemma 3.4 and  $\lim_{\beta \rightarrow \infty} \lambda_1^\beta(\Omega_R) = \lambda_1^D(\Omega_R)$ . Consequently,

$$-\sigma + \frac{\alpha_j}{R^2 \lambda_1^\beta(\Omega_R)} < -\sigma + \frac{\alpha_j}{R^2 \left( \frac{2n\pi}{n+2} \right)^2 \left( \frac{n}{\omega_n} \right)^{4/n} (1 - R^n)^{-2/n}}.$$

Let

$$C(n) = \left( \frac{2n\pi}{n+2} \right)^2 \left( \frac{n}{w_n} \right)^{4/n}.$$

Then the right hand side of the above inequality becomes

$$-\sigma + \frac{\alpha_j}{R^2(1-R^n)^{-2/n}C(n)}.$$

Since

$$-\sigma + \frac{\alpha_j}{R^2(1-R^n)^{-2/n}C(n)} \leq 0$$

means

$$\left( \frac{R^n}{1-R^n} \right)^{2/n} \geq \frac{\alpha_j}{C(n)\sigma},$$

it follows from the hypothesis (A1) that  $Q_j(u_0) < 0$ . Hence,  $\mu_{j,1}(u_0) < 0$  for  $1 \leq j \leq L$ , and the proof of Lemma 3.5 is complete.

**LEMMA 3.6.** *Let  $(\rho, \theta)$  be the polar coordinates in  $\mathbb{R}^2$ . For  $n > 2$ , let  $\psi_j = \rho^j \cos j\theta$  ( $j \geq 1$ ) be the eigenfunction of  $-\Delta$  on  $S^{n-1}$  corresponding to  $\alpha_j$ . Then  $w_j = \varphi_j \psi_j \in \text{Inv}(G_j \times O(n-2))$ .*

**PROOF.** Since  $\varphi_j$  is an eigenfunction of problem (3.5),  $\varphi_j$  is radial and  $\varphi_j \in \text{Inv } O(n) \subset \text{Inv}(G_j \times O(n-2))$ . Moreover, by definition, the action of  $G_j$  on  $\psi_j$  is equivalent to rotating it through  $2\pi l/j$ , so we have

$$g\psi_j = \rho^j \cos j(\theta + 2\pi l/j) = \psi_j, \quad \forall g \in G_j.$$

Hence,  $w_j = \varphi_j \psi_j \in \text{Inv}(G_j \times O(n-2))$ .

Now, the conclusion  $V_j - V_0 \neq \emptyset$  follows immediately from the following lemma.

**LEMMA 3.7.** *Let  $w_0$  and  $w_j \in \text{Inv}(G_j \times O(n-2))$  be eigenfunctions of problem (3.1) associated to  $\mu_{0,1}$  and  $\mu_{j,1}$  ( $1 \leq j \leq L$ ) respectively, such that  $\int_{\Omega_R} w_0^2 = \int_{\Omega_R} w_j^2 = 1$ . Then there exist  $\varepsilon > 0$  and a smooth function  $\delta : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$  with  $\delta(0) = \delta'(0) = 0$  such that*

$$(3.9) \quad I(u_0 + \delta(t)w_0 + tw_j) = 0$$

and

$$(3.10) \quad J(u_0 + \delta(t)w_0 + tw_j) < J(u_0)$$

for  $t$  small enough.

**PROOF.** Define a function  $H : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$H(\delta, t) = I(u_0 + \delta w_0 + tw_j).$$

Obviously,  $H(0, 0) = 0$ , since  $u_0$  is a solution of problem (1.1). A simple calculation implies that

$$H(\delta, 0) = \delta \int_{\Omega_R} \{f(u_0) - f'(u_0)u_0\}w_0 + O(\delta^2).$$

Hence, by (F2), we have

$$\frac{\partial H}{\partial \delta}(0, 0) = \int_{\Omega_R} \{f(u_0) - f'(u_0)u_0\}w_0 < 0.$$

Consequently, by the implicit function theorem we conclude that there exists a number  $\varepsilon > 0$  and a smooth function  $\delta = \delta(t) : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$  with  $\delta(0) = 0$  such that (3.9) holds.

Next, we show that  $\delta'(0) = 0$ . To this end, we first note that

$$H(0, t) = t \int_{\Omega_R} \{f(u_0) - f'(u_0)u_0\}w_j + O(t^2)$$

for  $t$  small enough. Accordingly,  $\frac{\partial H}{\partial t}(0, 0) = 0$  due to  $\int_{S^{n-1}} \psi_j(\theta_1, \dots, \theta_{n-1}) = 0$ . Now,  $\delta'(0) = 0$  follows immediately from the identity

$$(3.11) \quad \frac{\partial H}{\partial \delta}(\delta(t), t) \frac{d\delta}{dt} + \frac{\partial H}{\partial t}(\delta(t), t) = 0.$$

Finally, a direct calculation implies that

$$J(u_0 + \delta(t)w_0 + tw_j) = J(u_0) + \frac{1}{2}\delta^2\mu_{0,1} + \frac{1}{2}t^2\mu_{j,1} + o(t^2)$$

for  $t$  small enough. Hence, by Lemma 3.5, we know that  $J(u_0 + \delta(t)w_0 + tw_j) < J(u_0)$  for  $t$  small enough, and this completes the proof of Lemma 3.7.

To prove Theorem 3, we let

$$b_j = \inf\{J(u); u \in V_j\} \quad \text{for } j = 1, \dots, L.$$

Since  $V_j \neq \emptyset$  and the imbeddings  $V_j \hookrightarrow L^{p+1}(\Omega_R)$  are compact, we conclude that for any  $1 \leq j \leq L$  there exists  $u_j \in V_j$  such that  $J(u_j) = b_j$ . Furthermore, by Lemma 3.3 and the symmetric criticality principle, the  $u_j$  are solutions of problem (1.1). Hence, to complete the proof of Theorem 3, we have to prove the following two properties of  $u_j$  with  $1 \leq j \leq k$ :

- (i)  $u_j$  is nonradial.
- (ii)  $u_i \neq u_j$  for  $i \neq j$  and  $1 \leq i, j \leq k$ .

Obviously, (i) follows immediately from Lemma 3.7, since  $b_j < b_0$  by that lemma. To prove (ii), we need the following lemma.

LEMMA 3.8. *For any  $j = 1, 2, \dots$  and  $m = 2, 3, \dots$ , the inequality  $b_{j \cdot m} < b_0$  implies  $b_j < b_{j \cdot m}$ .*

PROOF. Suppose that  $b_{j,m}$  is achieved at  $u_{j,m}$ , that is,

$$b_{j,m} = J(u_{j,m}), \quad u_{j,m} \in V_{j,m}, \quad u_{j,m} \not\equiv 0.$$

For convenience, we set  $u = u_{j,m}$ . According to the regularity theory of elliptic equations,  $u$  is in fact a positive  $C^2$  function.

Let  $(\rho, \theta)$  be the polar coordinates of  $\mathbb{R}^2$  and  $y = (x_3, x_4, \dots, x_n) \in \mathbb{R}^{n-2}$ . We write  $u(x)$  as  $u(\rho, \theta, |y|)$ . Let  $D$  denote the domain  $\Omega_R \cap \mathbb{R}^{n-2}$ . Then

$$\begin{aligned} \int_{\Omega_R} |\nabla u|^2 dx_1 dx_2 dy &= \int_D \int_0^{2\pi} \int_R^1 \left( u_\rho^2 + \frac{1}{\rho^2} u_\theta^2 + |\nabla_y u|^2 \right) \rho d\rho d\theta dy, \\ \int_{\partial\Omega_R} \beta u^2 &= \int_D \int_0^{2\pi} \beta u^2 d\theta dy, \quad \int_{\Omega_R} F(u^+) = \int_D \int_0^{2\pi} \int_R^1 F(u^+) \rho d\rho d\theta dy. \end{aligned}$$

Define  $v \in V_j \setminus \{0\}$  by

$$v(\rho, \theta, |y|) = u\left(\rho, \frac{\theta}{m}, |y|\right), \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\begin{aligned} v_\rho(\rho, \theta, |y|) &= u_\rho\left(\rho, \frac{\theta}{m}, |y|\right), \\ v_\theta(\rho, \theta, |y|) &= \frac{1}{m} u_\theta\left(\rho, \frac{\theta}{m}, |y|\right), \\ \nabla_y v(\rho, \theta, |y|) &= \nabla_y u\left(\rho, \frac{\theta}{m}, |y|\right). \end{aligned}$$

Therefore,

$$\int_{\Omega_R} |\nabla v|^2 dx_1 dx_2 dy = \int_D \int_0^{2\pi} \int_R^1 \left( u_\rho^2 + \frac{1}{\rho^2 m^2} u_\theta^2 + |\nabla_y u|^2 \right) \rho d\rho d\theta dy.$$

Similarly,

$$\int_{\partial\Omega_R} \beta v^2 = \int_{\partial\Omega_R} \beta u^2, \quad \int_{\Omega_R} F(v^+) = \int_{\Omega_R} F(u^+).$$

Since  $u$  does not belong to  $V_0$ , we have

$$\int_D \int_0^{2\pi} \int_R^1 \frac{1}{\rho^2 m^2} u_\theta^2 \rho d\rho d\theta dy > 0.$$

Thus, by the definition of  $J$ ,

$$b_j \leq J(v) < J(u) = b_{j,m}.$$

PROOF OF (ii). Let  $1 \leq i < j \leq k$ , and  $L$  be the least common multiple of  $1, \dots, k$ .

If  $j = i \cdot m$  for some  $m \geq 2$ , then by Lemma 3.8 we have

$$b_i < b_j = b_{i \cdot m},$$

since  $b_{i \cdot m} = b_j < b_0$ . Hence,  $u_j \neq u_i$ .

If  $j \neq i \cdot m$  for any  $m \geq 2$ , then  $j < L$ , and  $V_i \cap V_j = V_L$ . If  $u_i \equiv u_j \equiv \bar{u} \in V_L$ , we have

$$b_L \leq J(\bar{u}) = b_i < b_0.$$

On the other hand, it follows from Lemma 3.8 that  $b_i < b_L$ , because  $L = i \cdot m_L$  for some  $m_L \geq 2$ ; this is a contradiction.

ACKNOWLEDGEMENTS. We are greatly indebted to the referees for their suggestions which helped us to improve the exposition.

This work is supported by NNSF of China (Grant No 10671064).

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Received 25 January 2008,  
and in revised form 30 January 2008.

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