Rend. Lincei Mat. Appl. 19 (2008), 199–204

Mathematical analysis. — *Transversality of equivariant mappings with closed associated differential forms on G-manifolds*, by MARC LESIMPLE and TULLIO VALENT, communicated on 12 June 2008.

ABSTRACT. — It is proved that any equivariant mapping between G-manifolds (G a Lie group), with a closed associated differential form, is transversal to the orbit of any point in its image up to a G-invariant subspace of the tangent space.

KEY WORDS: Perturbation problems; equivariant mappings; transversality.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 22F05, 47J05, 58E99, 57R99, 57S20, 58F30.

INTRODUCTION

Let $\boldsymbol{G}$ be a Lie group, M, N be two Fréchet $\boldsymbol{G}$-manifolds, $A\colon M \to N$ an equivariant differentiable mapping and β an invariant cross section of the tangent bundle over N. Motivated by the problem of existence of solutions for an equation of the type

$$\beta(A(x)) = 0, \tag{0.1}$$

we aim to obtain a transversality criterion on A, as has been done within the context of Banach spaces in [5].

Particular cases of the situation above arise in the framework of perturbation problems in elasticity theory where equation (0.1) is the free part of a perturbation equation and A is an elastostatic operator (see [9] for some examples).

Before going further in the presentation of the motivations, let us point out a question related to some perturbation problems, followed by a remark concerning the finite-dimensional case.

Let $\widetilde{A}$ denote the map $\beta \circ A$ and suppose that a solution $x_0 \in M$ of the equation (0.1) is known. Assume moreover that x_0 is a fixed point for the action of $\boldsymbol{G}$ on M. Let $(\mathcal{V}_{x_0}, \theta)$ be a chart at x_0 such that $\theta(x_0) = 0$ and consider the Taylor development of $\widetilde{A}$ about the origin, $\widetilde{A} = \widetilde{A}^1 + \sum_{n\geq 2} \widetilde{A}^n$. Then $\widetilde{A}^1$ is an intertwining operator for the restriction to A-related vector fields of the linear representations on the tangent spaces $T(M)$ and $T(N)$, induced by the action of $\boldsymbol{G}$ on M and N. A characterization of the space of intertwining operators for the induced representations on tensor spaces over M and N, analogous to the one in [8] for the finite-dimensional case, could simplify the solution of the existence problem for a perturbation equation of the type

$$\beta(A(x)) + \epsilon\beta(B(x)) = 0, \tag{0.2}$$

where B is a differentiable mapping from M into N. Besides, if M and N are of finite dimension and if the isotropy algebra $\mathfrak{g}(x_0)$ is semisimple then the Killing vector fields vanishing at x_0 in M (similarly the Killing vector fields vanishing at $A(x_0)$ in N) can be formally linearized in a neighborhood of x_0 [3].

In the Fréchet case that we are considering, local existence of solutions (in a neighborhood $\mathcal{V}_{x_0}$ of x_0) for the equation (0.1) or (0.2) relies on the Nash implicit function theorem which depends on the transversality of A, at any point $x \in \mathcal{V}_{x_0}$, to the orbit $\boldsymbol{G}_{A(x)}$ of $A(x)$ in N. In [6] existence of local families of solutions for perturbation equations was obtained (in the case where M and N are Banach spaces) as a consequence of the "transversality" of the mapping A (in the sense that $A(x)$ belongs to the polar of the orbit $\boldsymbol{G}_x$ if x is not too far from x_0) provided that A is a submersion at x_0 modulo $\boldsymbol{G}_{A(x_0)}$ (that is to say, $A \pitchfork_{\{x_0\}} \boldsymbol{G}_{A(x_0)}$).

A transversality criterion was obtained for nonlinear actions of a Lie group in [5] extending the results presented for perturbation problems in the presence of linear or affine symmetries [9]. In this paper we shall first obtain a similar transversality criterion (Lemma 2.1) and then show, under some conditions, that for every $x \in M$, the mapping A is transversal to the orbit of $A(x)$ in N up to a $\boldsymbol{G}$-invariant subspace of the tangent space $T_{A(x)}(N)$. This generalizes the previous statement and extends it to the case of infinite-dimensional manifolds and groups.

1. Variational principle

Let M and N be two differentiable manifolds modeled on Fréchet spaces E and F respectively (see e.g. [2]). We suppose that N admits a Riemannian metric $(|)$.

Let $\boldsymbol{G}$ be a Lie group (possibly infinite-dimensional) acting smoothly on M (resp. on N) by $(g, x) \mapsto \varphi_g(x) \in M$ for $g \in \boldsymbol{G}$, $x \in M$ (resp. $(g, y) \mapsto \psi_g(y) \in N$ for $g \in \boldsymbol{G}$, $y \in N$). It is enough to suppose $\boldsymbol{G}$ only to be regular [7, 4], but to simplify we shall assume that the exponential map is defined and gives a local chart in a neighborhood of the identity in $\boldsymbol{G}$.

Let $A\colon M \to N$ be a differentiable mapping that is supposed to be equivariant,

$$A(\varphi_g(x)) = \psi_g(A(x)) \quad \text{for every } g \in \boldsymbol{G} \text{ and } x \in M\,. \tag{1.1}$$

By differentiation we obtain

$$dA_x(X^*_\varphi(x)) = X^*_\psi(A(x))\,, \tag{1.2}$$

where X^*_φ (resp. X^*_ψ) denotes the Killing vector field relative to the Lie transformation group φ (resp. ψ) generated by an element X in $\mathfrak{g}$, the Lie algebra of $\boldsymbol{G}$.

Denote by U (resp. V) the linear representation of $\boldsymbol{G}$ induced by φ (resp. ψ) in the space of vector fields on M (resp. on N) denoted by $T(M)$ (resp. $T(N)$) given by $U_g = d\varphi_g$ (resp. $V_g = d\psi_g$) with $g \in \boldsymbol{G}$. The representation U is differentiable to a representation dU of $\mathfrak{g}$, defined by $dU_X = \frac{d}{dt} U_{\exp tX}|_{t=0}$ (the limit being taken in the topology of E, and dU_X is continuous for the topology induced by $C^\infty(\boldsymbol{G}, E)$ on $T(M)$).

Let there be given a differentiable mapping $\kappa\colon M \to N$ and for every $x \in M$, consider the duality on $T_x(M) \times T_{\kappa(x)}(N)$ defined by $\langle v_x, w_{\kappa(x)}\rangle = (d\kappa_x(v_x)|w_{\kappa(x)})_{\kappa(x)}$, where $v \in T(M)$ and $w \in T(N)$. (Such a mapping κ can be interpreted as a trace operator in the Banach case if A describes a boundary value problem [9].)

Let $\{U_\alpha, \theta_\alpha\}$ be an atlas of N and denote by $\boldsymbol{G}_1$ the group of diffeomorphisms of F given by $d\theta_{\beta\alpha}$, where $\theta_{\beta\alpha} = \theta_\beta \circ \theta_\alpha^{-1}\colon \theta_\alpha(U_\alpha \cap U_\beta) \to \theta_\beta(U_\alpha \cap U_\beta)$. Let $L(N, \boldsymbol{G}_1)$ be the principal fibre bundle with transition functions $d\theta_{\beta\alpha}$ and suppose that there exists a connection on L (that is, a linear connection on N). Then the duality $\langle\, ,\rangle$ induces a duality on $T_x(M) \times T_{A(x)}(N)$ given by $\langle v_x, w_{A(x)}\rangle = \langle v_x, \tau_{A(x)}^{\kappa(x)} w_{A(x)}\rangle$, where $\tau_{A(x)}^{\kappa(x)}$ denotes the parallel displacement along a piecewise differentiable curve τ joining $A(x)$ to $\kappa(x)$ in N.

Let β be an invariant vector field on N. We associate to the mapping $\beta \circ A$ a differential form ω_A on M defined by $\omega_A(x)(v_x) = \langle v_x, \beta(A(x))\rangle$ for $x \in M$ and $v_x \in T_x(M)$.

If the differential form ω_A is exact then the equation (0.1) is equivalent to a variational principle relative to a differentiable function J on M:

$$(dJ_x)(v_x) = 0 \tag{1.3}$$

for every $v_x \in T_x(M)$, where $dJ = \omega_A$.

REMARK 1.1. If the differential form ω_A is only closed, then $d(\beta \circ A)_x$ is symmetric with respect to the bilinear form $\langle\, ,\rangle$. This fact is relevant for some nonlinear variational elliptic problems [1].

2. TRANSVERSALITY CRITERION

We are mainly interested in the case where the differential form ω_A is closed. The closedness condition can be written as

$$\langle v_1(x), d\beta_{A(x)} \circ dA_x(v_2(x))\rangle = \langle v_2(x), d\beta_{A(x)} \circ dA_x(v_1(x))\rangle \tag{2.1}$$

for every $v_1, v_2 \in T(M)$ and $x \in M$. Furthermore, assume that the representation V is contragradient to U; so if $v_x \in T_x(M)$ and $w_{A(x)} \in T_{A(x)}(N)$ we have

$$\langle U_g(v_x), V_g(w_{A(x)})\rangle = \langle v_x, w_{A(x)}\rangle \quad \text{for every } x \in M. \tag{2.2}$$

By differentiation we can write, for every $X \in \mathfrak{g}$,

$$\langle dU_X(v_x), w_{A(x)}\rangle = -\langle v_x, dV_X(w_{A(x)})\rangle. \tag{2.3}$$

LEMMA 2.1. *Under conditions* (1.1), (2.1) *and* (2.2)*, we have the following property: for every* $X \in \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ *and* $v \in T(M)$,

$$\langle dU_X(v_x), d\beta_{A(x)} \circ dA_x(v_x)\rangle = 0$$

at any point $x \in M$.

PROOF. The proof is similar to the one given in [5] for the transversality criterion in the linear case. Let $v_1, v_2 \in T(M)$; since dA_x intertwines dU with dV and β is invariant, we can write

$$\langle v_1(x), d\beta \circ dA_x\, dU_X(v_2(x))\rangle = \langle v_1(x), dV_X\, d\beta \circ dA_x(v_2(x))\rangle$$

and by (2.3),

$$\langle v_1(x), d\beta \circ dA_x\, dU_X(v_2(x))\rangle = -\langle dU_X v_1(x), d\beta \circ dA_x(v_2(x))\rangle.$$

Taking $v_1 = dU_Y(v_2)$ with $Y \in \mathfrak{g}$, we have

$$\langle dU_Y(v_2(x)), (d\beta \circ dA_x) dU_X(v_2)(x)\rangle = -\langle dU_X dU_Y(v_2)(x), d\beta \circ dA_x(v_2(x))\rangle,$$

and $d\beta \circ dA_x$ being symmetric (condition (2.1)) we can also write

$$\langle dU_Y(v_2(x)), d\beta \circ dA_x\, dU_X(v_2(x))\rangle = \langle dU_X(v_2(x)), d\beta \circ dA_x\, dU_Y(v_2(x))\rangle;$$

so $\langle dU_{[X,Y]}(v_2(x)), d\beta \circ dA_x(v_2(x))\rangle = 0.$ □

For each $x \in M$, we denote by $\boldsymbol{G}(x)$ the isotropy subgroup of $\boldsymbol{G}$ at x. We denote by $\boldsymbol{G}_x^\varphi$ and $\boldsymbol{G}_y^\psi$ the orbits of x in M and y in N respectively.

Set $\widetilde{A} = \beta \circ A\colon M \to T(N)$. We shall show that $\operatorname{Im} d\widetilde{A}_x$ belongs to $T_x(\boldsymbol{G}_x^\varphi)^\circ$ (the polar of $T_x(\boldsymbol{G}_x^\varphi)$); that is to say, $\widetilde{A}$ is "transversal" to the orbit $\boldsymbol{G}_{A(x)}^\psi$ at each $x \in M$ (in the sense that there exists a $\boldsymbol{G}$-invariant subspace $W_{A(x)}$ of $T_{A(x)}(N)$ contained in $T_x(\boldsymbol{G}_x^\varphi)^\circ$ such that $\operatorname{Im} d\widetilde{A}_x + T_{A(x)}(\boldsymbol{G}_{A(x)}^\psi) \simeq T_{A(x)}(N)/W_{A(x)}$).

We shall deduce (under some conditions) that A is transversal—up to a $\boldsymbol{G}$-invariant subspace $W_{A(x)}$ of $T_{A(x)}(N)$—to $\boldsymbol{G}_{A(x)}^\psi$ at each $x \in M$, in the sense that $T_{A(x)}(N)/W_{A(x)}$ is $\boldsymbol{G}$-isomorphic to $\operatorname{Im} dA_x \oplus T_{A(x)}(\boldsymbol{G}_{A(x)}^\psi)$.

REMARK 2.1. A point, not developed in this paper, is that the following proposition brings forth (under the same kind of compatibility condition—as in perturbation theory—already considered in [5], namely the closedness condition (2.1)) a reduction scheme for the existence problem of local solutions for the perturbation equation (0.2), in the context of Nash implicit function theorem [2] (in [6] such a reduction scheme is presented in the case of Banach spaces).

PROPOSITION 2.1. *Suppose that $\mathfrak{g}'$ contains all nonvanishing Killing vector fields on M. Assume conditions* (1.1), (2.1) *and* (2.2) *are satisfied. Then for any $x \in M$ we have:*

(i) $\operatorname{Im} d\widetilde{A}_x \subset T_x(\boldsymbol{G}_x^\varphi)^\circ$ *;*
(ii) *if $d\beta_{A(x)|\operatorname{Im} dA_x}$ is injective, then* $\operatorname{Im} dA_x \subset T_x(\boldsymbol{G}_x^\varphi)^\circ$.

Furthermore, if V_g is an isometry relative to the metric $(|)$ on N for each $g \in \boldsymbol{G}$ (that is, if $T(N)$ is a Riemannian $\boldsymbol{G}$-vector bundle) then $A \pitchfork \boldsymbol{G}^{\psi}_{A(x)}$ up to a $\boldsymbol{G}$-invariant subspace of $T_{A(x)}(N)$.

PROOF. Let X^*_φ be a Killing vector field on M. By the preceding lemma we have, dU being the contragradient of dV, $\langle X^*_\varphi(x), d\widetilde{A}_x\, dU_Y(X^*_\varphi(x))\rangle = 0$ for every $Y \in \mathfrak{g}'$. Since it is assumed that $\mathfrak{g}'$ contains all nonvanishing Killing vector fields and that $\boldsymbol{G}^\varphi_x \simeq \boldsymbol{G}/\boldsymbol{G}(x)$, we have $\{dU_Y(X^*_\varphi(x)) : Y \in \mathfrak{g}'\} = T_x(\boldsymbol{G}^\varphi_x)$. Hence $d\widetilde{A}_x(T_x(\boldsymbol{G}^\varphi_x)) \subset T_x(\boldsymbol{G}^\varphi_x)^\circ$. But $d\widetilde{A}_x(T_x(\boldsymbol{G}^\varphi_x)) \subset T_{A(x)}(\boldsymbol{G}^\psi_{A(x)})$; so $T_x(\boldsymbol{G}^\varphi_x) \subset \operatorname{Ker} d\widetilde{A}_x$.

It follows that $\operatorname{Im} d\widetilde{A}_x \cap T_{A(x)}(\boldsymbol{G}^\psi_{A(x)}) = \{0\}$: in fact, let $v_x \in T_x(M)$; if $u = d\widetilde{A}_x(v_x)$ belongs to $T_{A(x)}(\boldsymbol{G}^\psi_{A(x)})$, then the class of v_x in $T_x(M)/\operatorname{Ker} d\widetilde{A}_x$, which is the inverse image of u under the quotient map of $d\widetilde{A}_x$, belongs to $T_x(\boldsymbol{G}^\varphi_x)/\operatorname{Ker} d\widetilde{A}_x$ and so is zero. Thus $v_x \in \operatorname{Ker} d\widetilde{A}_x$ and so $u = 0$. Hence $\operatorname{Im} d\widetilde{A}_x \subset T_x(\boldsymbol{G}^\varphi_x)^\circ$.

If now $d\beta_{A(x)|\operatorname{Im} dA_x}$ is injective, then $\operatorname{Ker} d\widetilde{A}_x \subset \operatorname{Ker} dA_x$ and in the same way as before, we find that $\operatorname{Im} dA_x \cap T_{A(x)}(\boldsymbol{G}^\psi_{A(x)}) = \{0\}$. So $\operatorname{Im} dA_x \subset T_x(\boldsymbol{G}^\varphi_x)^\circ$.

For each $x \in M$, let $W_{A(x)}$ be a complement of $\operatorname{Im} dA_x$ in $T_x(\boldsymbol{G}^\varphi_x)^\circ$ and consider the subbundle W of the $\boldsymbol{G}$-vector bundle $T(N)$ with fibre $W_{A(x)}$ (and $W_y = \{0\}$ if $y \notin A(M)$). Then W is a $\boldsymbol{G}$-invariant subspace of $T(N)$. Indeed, V_g being an isometry, we have $(V_g(w)|X^*_\psi) = (w|V_{g^{-1}}(X^*_\psi)) = 0$ for every $w \in W$ and $X^*_\psi \in T(\boldsymbol{G}^\psi_{A(x)})$, so $V_g(w_{A(x)}) \in T_{\varphi(x)}(\boldsymbol{G}^\varphi_x)^\circ$ and $V_g(w_{A(x)}) \notin \operatorname{Im} dA_{\varphi_g(x)}$ (since $w_{A(x)} \notin \operatorname{Im} dA_x$). Also, the restriction to $\operatorname{Im} dA_x \oplus T_{A(x)}(\boldsymbol{G}^\psi_{A(x)})$ of the projection on $T_{A(x)}(N)/W_{A(x)}$ is a $\boldsymbol{G}$-isomorphism. □

REFERENCES

[1] F. CARDIN, *Global finite generating functions for field theory.* In: Classical and Quantum Integrability, Banach Center Publ. 59, Inst. Math., Polish Acad. Sci., 2003, 133–142.

[2] R. S. HAMILTON, *The inverse function theorem of Nash and Moser*. Bull. Amer. Math. Soc. 7 (1982), 65–222.

[3] R. HERMANN, *The formal linearization of a semisimple Lie algebra of vector fields about a singular point.* Trans. Amer. Math. Soc. 130 (1968), 105–109.

[4] N. KAMRAN - T. ROBART, *On the parametrization problem of Lie pseudogroups of infinite type.* C. R. Acad. Sci. Paris Sér. I 331 (2000), 899–903.

[5] M. LESIMPLE - T. VALENT, *Transversality of covariant mappings admitting a potential.* Rend. Lincei Mat. Appl. 18 (2007), 117–124.

[6] M. LESIMPLE - T. VALENT, *Local existence of solutions for perturbation problems with non linear symmetries.* Boll. Un. Mat. Ital. (8) 10-B (2007), 707–717.

[7] J. MILNOR, *Remarks on infinite-dimensional Lie groups.* In: Relativity, Groups and Topology, II (Les Houches, 1983), B. S. De Witt and R. Stora (eds.), North-Holland, 1984, 1007–1057.

[8] R. S. PALAIS, *Natural operations on differential forms*, Trans. Amer. Math. Soc. 92 (1959), 125–141.

[9] T. VALENT, *Nonlinear symmetries of mappings. Existence theorems for a perturbation problem in the presence of symmetries*. In: Recent Developments in Partial Differential Equations, Quad. Mat. 2, Seconda Univ. Napoli, Caserta, 1998, 211–253.

Received 8 May 2007,
and in revised form 12 March 2008.

M. Lesimple, T. Valent
Dipartimento di Matematica Pura ed Applicata
Università degli Studi di Padova
via Trieste, 63
I-35121 PADOVA, Italy
lesimple@math.unipd.it
valent@math.unipd.it