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Solid mechanics. — *Axial impact on a semi-infinite elastic rod*, by PIERO VILLAGGIO, communicated on 12 June 2008.

ABSTRACT. — An approximate theory for treating the axial collision of a rigid mass against an elastic rod was proposed by Saint-Venant more than 150 years ago. The method works only for short bars, under the assumption that the elastic axial displacement instantaneously propagates from one end to the other after the impact. However, this hypothesis is unrealistic for long rods.

Here we suggest an extension of the method which is able to treat the axial impact on the initial cross-section of a semi-infinite rod.

KEY WORDS: Elastic impact; rod theory; energy method.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 74M20, 74K10.

1. INTRODUCTION

In 1849 H. Cox [2] proposed a simple and elegant method for treating the transversal impact of a heavy mass at the center of an elastic beam of uniform cross section and simply supported at its ends. A few years later Saint-Venant [5] applied the same procedure to the case of a prismatical rod fixed at one end and subject to a longitudinal impact at the other. Both solutions are obtained by a simple balance between the kinetic energy of the striking mass and the strain energy stored in the beam or in the rod. Both theories can also account for the inertia of the struck body after impact. Due to their simplicity and experimental confirmation, they have been widely diffused in structural analysis among engineers.

The method, however, presumes in advance that the impinged body is sufficiently short and stiff so as to become completely deformed as soon as the striking mass touches it at some point. But this happens only if the elastic wave departing from the point of first contact travels with infinite velocity. In general, after a given time, only a part of the structure adjacent to this point is in motion while the remaining part is at rest. After having assumed that displacements propagate instantaneously in the structure which is struck, Saint-Venant's theory adds the hypothesis that this (dynamical) deformation is similar to the statical one, namely it is obtainable by multiplying the elastic displacements due to a concentrated load at the point of impact by an amplification factor which is determined by the equation of energy balance. Once this factor is known, we can also compute the maximum displacement state of the structure when the kinetic energy of the moving mass is completely converted into strain energy. Therefore also the maximal stresses can be estimated and hence the danger of a possible collapse. Several appealing extensions of the method are illustrated in the book of Szabó [7, §24].

But, as mentioned above, this semi-statical theory is inapplicable to long rods or beams. It seems that Saint-Venant himself perceived the inadequacy of the model and tried to establish its limits (cf. Timoshenko [8, §41]). Here we suggest an improvement of the theory considering the longitudinal impact of a semi-infinite rod.

2. THE EQUATION OF THE PROGRESSIVE IMPACT

We consider a semi-infinite rod of uniform cross section A and assume its barycentric fiber coincides with the half-axis $0 \leq x < \infty$ of a Cartesian system of coordinates (Fig. 1). Let E, ρ be the Young's elastic modulus and the density of a material. Hence the ratio $c = \sqrt{E/\rho}$ represents the propagation velocity of sound waves in the medium and, in particular, along the x-axis.

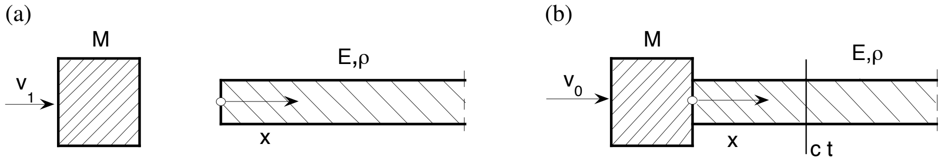

FIG. 1. (a) Before the impact the rod is at rest. (b) At time t after the impact only the portion $0 \leq x < ct$ is in motion.

Assume that a rigid mass M traveling with velocity v_1 along the x-axis hits the initial cross-section of the rod at $x = 0$. At this instant an elastic wave will spread from the origin with a velocity $c = \sqrt{E/\rho}$. At time t the points $0 \leq x < ct$ are in motion while the points $ct \leq x < \infty$ are at rest.

But, before treating this case, suppose for the moment that the rod has a length l and is clamped at the end $x = l$. Saint-Venant made the hypothesis that, given the high value of the velocity c of an elastic wave traveling along a bar of finite length, the whole span l is instantaneously deformed at the first contact with the mass M and the axial displacement $u(x, t)$ has the linear distribution

$$u(x, t) = y(t)\left(1 - \frac{x}{l}\right), \tag{1}$$

where $y(t)$ is the displacement of the initial cross-section. Hence the rod contrasts the axial motion of M with a force $P = -E A\, y(t)/l$, behaving like a spring of compliance $E A/l$. The theory neglects the influence of the strain energy of the rod, but, by imposing the balance of momentum at the instant of impact, it is easy to show (cf. Szabó [7, §24]) that the impinging mass M assumes a velocity v_0 having the value

$$v_0 = \frac{v_1}{1 + \frac{1}{3}\frac{\rho l A}{M}}. \tag{2}$$

It should be noted that this theory excludes the possibility of rebounding of the impinging mass from the end of the rod, but it is possible to treat the case of detachment of M during the period of compression (cf. Szabó [7, §24]).

Let us now return to the case in which the rod is semi-infinite. The motion is governed by the same dynamical equation of a rod with a mass attached at its end, where, however, the compliance of the rod varies with t and so does its mass. We examine three cases.

3. SOLUTIONS FOR THE DYNAMICAL EQUATION

Suppose first that the mass of the rod is negligible with respect to M. Thus $y(t)$, the displacement of the initial cross section, must satisfy the differential equation

$$My''(t) + \frac{EA}{ct}y(t) = 0, \tag{3}$$

with the initial conditions $y(0) = 0$, $y'(0) = v_1$. In order to solve (3) it is convenient to put $b = \frac{EA}{cM}$ and multiply by t^2, so that (3) assumes the form

$$t^2 y''(t) + bty = 0. \tag{4}$$

This is a Bessel differential equation whose solution, recorded by Kamke [4, p. 440], reads

$$y = C_1\sqrt{t}J_1(2\sqrt{bt}) + C_2\sqrt{t}N_1(2\sqrt{bt}), \tag{5}$$

where J_1, N_1 are a Bessel and a Neumann function, respectively, and C_1, C_2 are two arbitrary constants determined by the initial conditions. Since $\sqrt{t}N_1(2\sqrt{bt})$ is infinite at $t = 0$, the constant C_2 must be zero, and hence the solution is

$$y = C_1\sqrt{t}J_1(2\sqrt{bt}), \tag{6}$$

which vanishes for $t = 0$, satisfying in this way the first initial condition $y(0) = 0$. The constant C_1 is determined from the second initial condition $y'(0) = v_1$, and, eventually using some formulae for the derivatives of Bessel functions (cf. Jahnke–Emde–Lösch [3, p. 154]) we obtain the surprisingly simple value $C_1 = v_1/\sqrt{b}$. Therefore the subsequent motion after the impact is governed by the expression

$$y(t) = \frac{v_1}{\sqrt{b}}\sqrt{t}\, J_1(2\sqrt{bt}). \tag{7}$$

A detailed representation of the motion is given by the graph of (7), but let us simply limit ourselves to finding the duration of the first half-period of oscillation of the mass M, namely the time t_1 at which it recovers the position $y(t_1) = 0$. This time is determined by the first zero of the Bessel function $J_1(2\sqrt{bt})$, that is (cf. Jahnke–Emde–Lösch [3, p. 159]), for $2\sqrt{bt_1} \simeq 3.832$, which implies $t_1 = \frac{1}{b}3.6710$.

A second, alternative, situation occurs when the mass M is negligible with respect to the moving part of the rod, which at time t has the value $\frac{1}{3}\rho Act$. In this case the equation of the motion becomes

$$\frac{1}{3}\rho Acty''(t) + \frac{EA}{ct}y(t) = 0, \tag{8}$$

with the initial conditions $y(0) = 0$, $y'(0) = v_0$ since $l = ct = 0$ in formula (2). Dividing by A and recalling that $c^2 = E/\rho$, we reduce (8) to

$$t^2 y'' + 3y = 0, \tag{9}$$

which is again a Bessel differential equation having the general integral (cf. Kamke [4, p. 440])

$$y(t) = C_1 t^{(1+\mu)/2} + C_2 t^{(1-\mu)/2}, \tag{10}$$

where $\mu = \sqrt{1 - 4 \cdot 3} = \pm i\sqrt{11}$, so that, redefining the constants C_1, C_2, we can write (10) in the real form

$$y(t) = C_1\sqrt{t}\cos\left(\frac{\sqrt{11}}{2}\ln t\right) + C_2\sqrt{t}\sin\left(\frac{\sqrt{11}}{2}\ln t\right). \tag{11}$$

At $t = 0$ we have $y(0) = 0$ for any values of the constants C_1, C_2, but $y'(0)$ is singular. Therefore the only possible values of C_1, C_2 are $C_1 = C_2 = 0$, which implies $y(t) \equiv 0$. This result agrees with the exact solution found by Saint-Venant (see, e.g., the books of Timoshenko–Goodier [9, §143] or Stronge [6, Ch. 7.2]), which shows that, in this case, the rod does not undergo any displacement.

We now consider the case in which the two masses, M and $\frac{1}{3}\rho Act$, are of the same order of magnitude. Then the motion after the impact is described by the equation

$$\left(\frac{1}{3}\rho Act + M\right)y'' + \frac{EA}{ct}y = 0, \tag{12}$$

with the initial conditions $y(0) = 0$, $y'(0) = v_1$. Division by $\frac{EA}{ct}$ and setting $b = \frac{EA}{cM}$, $c^2 = \frac{E}{\rho}$ transforms (12) into

$$\left(\frac{1}{3}t^2 + \frac{1}{b}t\right)y''(t) + y(t) = 0. \tag{13}$$

This is a hypergeometric differential equation that, with the change of variables

$$y = \eta(\xi), \qquad t = -\frac{3}{b}\xi,$$

can be converted into the canonical form (cf. Kamke [4, p. 481])

$$\xi(\xi - 1)\eta''(\xi) + 3\eta(\xi) = 0. \tag{14}$$

Equation (14) is a particular case of the general hypergeometric equation

$$\xi(\xi-1)\eta''(\xi)+[(\alpha+\beta+1)\xi-\gamma]\eta'(\xi)+\alpha\beta\eta(\xi)=0,$$

provided that we take $\alpha=\frac{1}{2}(-1+i\sqrt{11})$, $\beta=\frac{1}{2}(-1-i\sqrt{11})$, $\gamma=0$. There are two independent solutions of (14), but only one is bounded at the origin, namely (cf. Kamke [4, p. 467])

$$\eta(\xi)=C_1\xi F(\alpha+1,\beta+1,2,\xi), \tag{15}$$

where C_1 is a constant, and F is a hypergeometric function, whose series expansion is (cf. Abramowitz–Stegun [1, 15])

$$F(\alpha+1,\beta+1,2,\xi)=1+\sum_{n}^{\infty}\frac{(\alpha+1)_n(\beta+1)_n}{(2)_n}\frac{\xi^n}{n!}, \tag{16}$$

where $(x+1)_n=(x+1)(x+2)\dots(x+n)$. Since α and β are complex coniugate the products $(\alpha+1)_n(\beta+1)_n$ are real. Hence $\eta(\xi)$ is real and its explicit expression arrested at the first four terms is

$$\eta(\xi)=C_1\xi\left(1+\frac{3}{2}\xi+\frac{5}{4}\xi^2+\frac{15}{16}\xi^3+\cdots\right). \tag{17}$$

Re-introducing the variable $t=-\frac{3}{b}\xi$ and exploiting the initial condition $y'(0)=v_1$ in order to determine C_1, we obtain

$$y(t)=v_1t\left(1-\frac{1}{2}(bt)+\frac{5}{36}(bt)^2-\frac{5}{144}(bt)^3+\cdots\right). \tag{18}$$

The time t_1 of the end of the impact is given by the first root of the expression inside the parenthesis, namely $t_1\simeq 2.6578/b$. Note that this time is shorter than that calculated in the first case where the inertia of the rod has been neglected.

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