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Partial differential equations. — *Positive solutions of nonlinear Schrödinger–Poisson systems with radial potentials vanishing at infinity*, by CARLO MERCURI, communicated on 12 June 2008.

ABSTRACT. — We deal with a weighted nonlinear Schrödinger–Poisson system, allowing the potentials to vanish at infinity.

KEY WORDS: Nonlinear Schrödinger equations; weighted Sobolev spaces; Pohozaev identity; Palais–Smale condition.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35J50, 34A34.

1. INTRODUCTION AND RESULTS

In this paper we deal with mountain-pass solutions for a system of Schrödinger–Poisson equations of the form

$$\begin{cases} -\Delta u + V(x)u + \phi u = K(x)u^p, & x \in \mathbb{R}^N, \\ -\Delta\phi = u^2. \end{cases} \tag{1}$$

Precisely, we will find solutions having the following properties:

$$u \in H^1(\mathbb{R}^N), \qquad u > 0, \qquad \lim_{|x|\to\infty} u = 0. \tag{2}$$

Here and hereafter $N \in \{3, 4, 5\}$ (see Section 3), $1 < p < (N+2)/(N-2)$ and $V, K : \mathbb{R}^N \to R_+$ are radial and smooth. For (1), existence, non-existence [12] and multiplicity results [4] have been found in the case $V = K = 1$. On the other hand, we do not know any results on (1) in the presence of external potentials. V, K in (1) are assumed to satisfy the same conditions introduced in [1] in the frame of Nonlinear Schrödinger Equations (NLS). Precisely:

$$\frac{a}{1+|x|^\alpha} \le V(x) \le A \tag{3}$$

for some $\alpha \in (0, 2]$, $a, A > 0$, and

$$0 < K(x) \le \frac{b}{1+|x|^\beta} \tag{4}$$

for some $\beta, b > 0$. The purpose of this paper is to extend these existence results to (1). It is convenient to introduce the following quantities:

$$\sigma = \sigma(N, \alpha, \beta) := \begin{cases} \dfrac{N+2}{N-2} - \dfrac{4\beta}{\alpha(N-2)} & \text{if } 0 < \beta < \alpha, \\ 1 & \text{otherwise,} \end{cases} \tag{5}$$

and

$$\alpha^* := \frac{2(N-1)(N-2)}{3N+2}. \tag{6}$$

DEFINITION 1. *Saying that* (u, ϕ) *is a* non-trivial positive solution *of* (1) *we mean that both* u *and* ϕ *are non-trivial, positive and radial. Furthermore,* u *satisfies* (2).

In order to find positive solutions of (1), we will distinguish between $2 < p < 3$ and $p \in [3, 2^* - 1)$. In the latter case we have the following

THEOREM 1. *Let* $\alpha < \alpha^*$ *and* $p \in (\sigma, 2^* - 1) \cap [3, 2^* - 1)$*. If* V *and* K *are radial, smooth, and satisfy* (3) *and* (4)*, then* (1) *has a non-trivial positive classical mountain-pass solution* $(u, \phi) \in H^1(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Moreover, we also have existence of positive classical solutions for p in the interval $(2, 3)$ if we assume that V and K satisfy:

$$\begin{cases} (x, \nabla V) \leq c_V^{(1)} V(x) \quad \text{and} \quad c_V^{(1)} \in (0, 2), \\ (x, \nabla K) \geq c_K^{(1)} K(x) \quad \text{and} \quad c_K^{(1)} \in [2, \infty), \end{cases} \tag{7}$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^N$. We assume that K is such that the following condition holds:

$$\exists \varepsilon \geq 0, q \geq 1 \text{ such that } (x, \nabla K) \in L^q(\mathbb{R}^N) \text{ with } q'(p+1-\varepsilon) \in [2+\alpha/\gamma, 2^*], \tag{8}$$

where

$$\frac{1}{q} + \frac{1}{q'} = 1 \quad \text{for } q \in \mathbb{R}, \qquad q' := 1 \quad \text{for } q = \infty$$

and

$$\gamma := \frac{2(N-1) - \alpha}{4}$$

is a parameter related to inclusions of weighted Sobolev spaces and L^p spaces. Furthermore, assuming V is such that the following condition holds:

$$\exists \varepsilon \geq 0, r \geq 1 \text{ such that } (x, \nabla V) \in L^r(\mathbb{R}^N) \text{ and } r'(2-\varepsilon) \in [2+\alpha/\gamma, 2^*], \tag{9}$$

where r' is defined as for q', we can state the following

THEOREM 2. *Let* $\alpha < \alpha^*$ *and* $p \in (\sigma, 2^* - 1) \cap (2, 3)$. *If V and K are radial, smooth, and satisfy* (3), (4), (7)–(9), *then* (1) *has a non-trivial positive classical solution* $(u, \phi) \in H^1(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$.

If instead of (7), we assume

$$\begin{cases} (x, \nabla V) \geq c_V^{(2)} V(x) \quad \text{and} \quad c_V^{(2)} > 0, \\ (x, \nabla K) \leq c_K^{(2)} K(x) \quad \text{and} \quad c_K^{(2)} \in (0, 2), \end{cases} \tag{10}$$

then, dealing with the case $p \in (2, 3)$, we can state another existence result. Introducing

$$\delta := 2 + c_K^{(2)}/2, \tag{11}$$

we have

THEOREM 3. *Let* $\alpha < \alpha^*$ *and* $p \in (\sigma, 2^* - 1) \cap (\delta, 3)$. *If V and K are radial, smooth, and satisfy* (3), (4), (8)–(10), *then* (1) *has a non-trivial positive classical solution* $(u, \phi) \in H^1(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$.

REMARK 1. We observe that the decaying property in (2) is due to the radiality of the solutions found, while the property $u \in H^1(\mathbb{R}^N)$ is proven in Lemma 6, by adapting an argument from [1].

In the case $p \in (1, 2]$, the previous theorems are completed by some non-existence results in Section 4. In spite of those results, we can also have existence for $p \in (1, 2)$ if we consider the Poisson term as a small perturbation. Indeed, as in [12], we can state the following

PROPOSITION 1. *For* $\alpha < \alpha^*$, $p \in (\sigma, 2^* - 1) \cap (1, 2)$ *and* $\lambda > 0$ *small enough, under the assumptions* (3) *and* (4) *the problem*

$$\begin{cases} -\Delta u + V(x)u + \lambda \phi u = K(x)u^p, \quad x \in \mathbb{R}^N, \\ -\Delta \phi = u^2 \end{cases} \tag{12}$$

has at least two different non-trivial positive classical solutions (u, ϕ), *one of which is a mountain-pass solution.*

Before proving the existence results we focus on giving the variational formulation of (1). So the next two sections deal with some functional preliminaries.

2. NOTATION AND FUNCTIONAL SETTING

Our aim is to use critical point theory, so let us introduce some functional spaces. We denote respectively by $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $H^1(\mathbb{R}^N)$ and $H_V(\mathbb{R}^N)$ the Hilbert spaces defined as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the following norms:

$$\|\phi\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\nabla\phi|^2\,dx,$$
$$\|u\|^2_{H^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\,dx,$$
$$\|u\|^2_{H_V(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)\,dx.$$

In particular, we will work with the closed subspace $H \subset H_V$ defined as its restriction to radial functions:

$$\|u\|^2_H := S_N \int_0^\infty (\varphi'(r)^2 + \tilde{V}(r)\varphi(r))^2 r^{N-1}\,dr,$$

where $\varphi(|x|) = u(x)$, $\tilde{V}(|x|) = V(x)$, and S_N is the Lebesgue surface measure of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Denoting by $L_K^{p+1}(\mathbb{R}^N)$ the weighted L^{p+1} space with norm

$$\|u\|^{p+1}_{L_K^{p+1}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} K(x)\,|u|^{p+1}\;dx, \tag{13}$$

we have

LEMMA 1. *The space $H_V(\mathbb{R}^N)$ is embedded (resp. compactly embedded) in $L_K^{p+1}(\mathbb{R}^N)$ if $\sigma \le p \le (N+2)/(N-2)$ (resp. if $\sigma < p < (N+2)/(N-2)$).*

(For the proof see e.g. [11].) Due to the radiality, we can find that H is compactly embedded in $L^q(\mathbb{R}^N)$ under suitable conditions on q. More precisely, we have the following extension of the Strauss compactness theorem (see [13]) that we give together with its proof for the sake of completness. See also [14] for a more general case.

LEMMA 2. *Let $\gamma := (2(N-1)-\alpha)/4$. The space H is compactly embedded in $L^q(\mathbb{R}^N)$ for any q such that $2+\alpha/\gamma < q < 2N/(N-2)$.*

PROOF. If $N \ge 2$ and $u \in H$, then there exist two positive constants C, $\bar{R}$ such that for a.e. $|x| > \bar{R}$,

$$|u(x)| \le C|x|^{-\gamma}\|u\|_H. \tag{14}$$

By density we can test the inequality on $C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$. Define φ by $\varphi(|x|) = u(x)$. An integration by parts gives

$$\begin{aligned}\varphi(r)^2 &= -2\int_r^\infty \varphi'(s)\varphi(s)\,ds\\ &\le 2\int_r^\infty s^{-(N-1)}\sqrt{\frac{1+s^\alpha}{a}}\sqrt{\frac{a}{1+s^\alpha}}|\varphi'(s)\varphi(s)|s^{N-1}\,ds\\ &\le Cr^{-2\gamma}\|u\|^2_H\end{aligned}$$

for some $C > 0$ and r large enough, where in the last step we have used

$$2\sqrt{\frac{a}{1+s^\alpha}}\,|\varphi'(s)\varphi(s)| \le \varphi'(s)^2 + \varphi(s)^2\frac{a}{1+s^\alpha}$$

and $s^{-(N-1)}\sqrt{1+s^\alpha} \searrow 0$ as $s \to \infty$, because we are focusing on $\alpha \in (0, 2]$.

Let

$$u_n \rightharpoonup 0 \quad \text{in } H.$$

Since on spheres we control the H^1 norm by the H norm, and the Rellich–Kondrashov theorem holds, it is enough to show that, passing to a subsequence, and for R large, the integral

$$\int_{|x|>R} |u_n|^q \; dx$$

can be smaller than an a priori fixed $\varepsilon > 0$ uniformly for $n \ge n_0$ for some $n_0 > 0$. In the following, $c_1, \dots, c_5$ are suitable positive constants. Taking into account that

$$|u_n(x)|^{q-2} \le c_1|x|^{-\gamma(q-2)}\,\|u_n\|_H^{q-2} \le c_2|x|^{-\gamma(q-2)}$$

and $|x|^{\alpha-\gamma(q-2)} \searrow 0$, we have

$$\begin{aligned}\int_{|x|>R} |u_n|^q\,dx &\le c_3\int_{|x|>R} |u_n|^{q-2}|x|^\alpha\frac{a}{1+|x|^\alpha}|u_n|^2\,dx\\ &\le c_4R^{\alpha-\gamma(q-2)}\,\|u_n\|_H^2 \le c_5R^{\alpha-\gamma(q-2)} \searrow 0\end{aligned}$$

as $R \nearrow \infty$. □

REMARK 2. It is worth pointing out that the space H is embedded in L^q for any $q \in [2+\alpha/\gamma, 2^*]$ (see e.g. [14]).

3. VARIATIONAL FORMULATION OF THE PROBLEM

Solutions of (1) are the critical points of the functional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\,dx + \frac{1}{4}\int_{\mathbb{R}^N}\phi_u u^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N}K(x)|u|^{p+1}\,dx$$

(which turns out to be well defined, $C^1(H, \mathbb{R})$ and weakly lower semicontinuous, see below).

This is due to the fact that, given $u \in H$, thanks to the Riesz representation theorem, there exists a unique solution ϕ_u of the problem

$$\int_{\mathbb{R}^N}\nabla\phi\nabla v\,dx = \int_{\mathbb{R}^N}u^2v\,dx, \qquad \forall v \in \mathcal{D}^{1,2}_{\text{rad}}(\mathbb{R}^N).$$

Moreover, since $u^2 \in L^1_{\mathrm{loc}}$, the following representation formula holds for ϕ_u:

$$\phi_u(x) = \omega_N \int_{\mathbb{R}^N} \frac{u(y)^2}{|x-y|^{N-2}}\, dy, \tag{15}$$

where ω_N is the usual normalization factor of the Green function.

Now recall Remark 2 and observe that, because of the embedding of $H_V(\mathbb{R}^N)$ in $L^q(\mathbb{R}^N)$, if $u \in H$, then $u \in L^{4N/(N+2)}$, provided $\alpha \le \alpha^* \Leftrightarrow 4N/(N+2) \ge 2+\alpha/\gamma$. Actually, the strict inequality has been used in order to have the compactness property stated in the following lemma. For the same reason the restriction on N is necessary, because it ensures that $4N/(N+2) < 2^*$.

The Hölder and Sobolev inequalities imply that, given $u \in H$, the operator

$$L_u : v \mapsto \int_{\mathbb{R}^N} u^2 v\, dx \tag{16}$$

is continuous in $\mathcal{D}^{1,2}(\mathbb{R}^N)$:

$$\left| \int_{\mathbb{R}^N} u^2 v\, dx \right| \le \|u^2\|_{L^{2N/(N+2)}} \|v\|_{L^{2N/(N-2)}} = C(u)\|v\|_{\mathcal{D}^{1,2}}.$$

Introducing the notation

$$M(u) := \int_{\mathbb{R}^N} \phi_u(x) u^2\, dx \tag{17}$$

we have

LEMMA 3. *If $\alpha < \alpha^*$, then M is a compact operator on H, i.e., if $u_n \rightharpoonup u$, then, up to a subsequence, $M(u_n) \to M(u)$.*

PROOF. Summing and subtracting $\int_{\mathbb{R}^N} \phi_{u_n} u^2\, dx$, by the Hölder and Sobolev inequalities we have

$$\begin{aligned} |M(u_n) - M(u)| &= \left| \int_{\mathbb{R}^N} [\phi_u(x)u^2 - \phi_{u_n}(x)u_n^2]\, dx \right| \\ &\le \|\phi_{u_n}\|_{L^{2N/(N-2)}} \|u_n^2 - u^2\|_{L^{2N/(N+2)}} + \|\phi_{u_n} - \phi_u\|_{L^{2N/(N-2)}} \|u^2\|_{L^{2N/(N+2)}} \\ &\le \|\phi_{u_n}\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \|u_n^2 - u^2\|_{L^{2N/(N+2)}} + \|\phi_{u_n} - \phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \|u\|^2_{L^{4N/(N+2)}}. \end{aligned}$$

Since

$$\begin{aligned} \|u_n^2 - u^2\|^{2N/(N+2)}_{L^{2N/(N+2)}} &= \int_{\mathbb{R}^N} [|u_n - u|\, |u_n + u|]^{2N/(N+2)}\, dx \\ &\le \|u_n - u\|^{2N/(N+2)}_{L^{4N/(N+2)}} \|u_n + u\|^{2N/(N+2)}_{L^{4N/(N+2)}} \end{aligned}$$

it follows that

$$|M(u_n) - M(u)|$$
$$\leq \|\phi_{u_n}\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\|u_n - u\|_{L^{4N/(N+2)}}\|u_n + u\|_{L^{4N/(N+2)}} + \|\phi_{u_n} - \phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\|u\|^2_{L^{4N/(N+2)}}.$$

Since

$$\alpha < \alpha^* \Leftrightarrow \frac{4N}{N+2} > 2 + \frac{\alpha}{\gamma},$$

Lemma 2 implies $H \hookrightarrow\hookrightarrow L^{4N/(N+2)}(\mathbb{R}^N)$, hence, passing to a subsequence, we obtain $\|u_n - u\|_{L^{4N/(N+2)}} \to 0$ and therefore

$$|M(u_n) - M(u)| \leq \|\phi_{u_n}\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} o(1) + C\|\phi_{u_n} - \phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.$$

In order to estimate $\|\phi_{u_n} - \phi_u\|_{D^{1,2}}$ we argue as follows. One has

$$\|L_{u_n} - L_u\| \leq \sup_{\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}=1} \|u_n^2 - u^2\|_{L^{2N/(N+2)}}\|v\|_{L^{2N/(N-2)}}.$$

Since $\|u_n - u\|_{L^{4N/(N+2)}} \to 0$, passing to a subsequence, we have $u_n \to u$ a.e. and $|u_n|^2 \leq g$ for some $g \in L^{2N/(N+2)}$. Hence, the dominated convergence theorem implies $\|u_n^2 - u^2\|_{L^{2N/(N+2)}} \to 0$, and therefore $L_{u_n} \to L_u$. The Riesz representation theorem implies that $L_u \in \mathcal{D}^{1,2*} \mapsto \phi_u \in \mathcal{D}^{1,2}$ is an isometry, therefore $\phi_{u_n} \to \phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. □

Lemma 3 and the compact embedding of H in L_K^{p+1} imply the weakly lower semicontinuity of I. It is standard to check also that I is a $C^1(H, \mathbb{R})$ functional.

We conclude this section with a Pohozaev-like identity which will be useful later on. For the proof see the Appendix.

LEMMA 4. *Assume that V and K satisfy* (3), (4), (8) *and* (9). *If $u \in H_V(\mathbb{R}^N) \cap H^2_{\text{loc}}(\mathbb{R}^N)$ is a radial solution of the problem* (1), *then u satisfies the following identity:*

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N}V(x)u^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N}(x, \nabla V(x))u^2\,dx + \frac{N+2}{4}\int_{\mathbb{R}^N}\phi_u u^2\,dx = \frac{N}{p+1}\int_{\mathbb{R}^N}K(x)|u|^{p+1}\,dx + \frac{1}{p+1}\int_{\mathbb{R}^N}(x, \nabla K(x))|u|^{p+1}\,dx.$$

4. PROOFS

Because we will use the mountain-pass theorem (see [3], [2]), we will need the following

LEMMA 5. *I has the mountain-pass geometry for $p > 2$.*

PROOF. The continuous embedding of H in L_K^{p+1} gives

$$I(u) = \frac{1}{2}\|u\|_H^2 + o(\|u\|_H^2), \qquad u \to 0, \tag{18}$$

which shows that I has a strict local minimum at the origin. Furthermore, let us show that I attains negative values on the curves $u_t(x) := t^\lambda u(t^\mu x)$ for a suitable choice of $u \in H$, positive λ, μ and large values of t. The case $3 < p < 2^* - 1$ is standard and it can be checked taking any $u \in H \setminus \{0\}$ and putting $\mu = 0$, $\lambda = 1$. The case $p \in (2, 3]$ can be treated as follows. Fix $u \in H \cap L^2 \cap L^{p+1}$. Because of the integrability of u and the boundedness of V and K, the dominated convergence theorem yields the following asymptotics for $t \to \infty$:

$$\|u_t\|_H^2 = t^{2(\lambda+\mu)-\mu N}\|\nabla u\|_{L^2}^2 + t^{2\lambda-\mu N}\int_{\mathbb{R}^N} V(t^{-\mu}x)|u|^2\,dx \approx t^{2(\lambda+\mu)-\mu N}, \tag{19}$$

$$\int_{\mathbb{R}^N} K(x)|u_t|^{p+1}\,dx = t^{\lambda(p+1)-\mu N}\int_{\mathbb{R}^N} K(t^{-\mu}x)|u|^{p+1}\,dx \approx t^{\lambda(p+1)-\mu N}. \tag{20}$$

Moreover, since

$$\phi_{u_t}(x) = \omega_N \int_{\mathbb{R}^N} t^{2\lambda}u^2(t^\mu y)\frac{t^{\mu(N-2)}}{|t^\mu x - t^\mu y|^{N-2}}\,dy = t^{2\lambda+\mu(N-2)-\mu N}\phi_u(t^\mu x),$$

we have

$$\int_{\mathbb{R}^N} \phi_{u_t}(x)u_t(x)^2\,dx = t^{4\lambda+\mu(N-2)-2\mu N}\int_{\mathbb{R}^N} \phi_u(x)u(x)^2\,dx \approx t^{4\lambda-\mu(N+2)}. \tag{21}$$

Summing up (19)–(21) we get

$$I(u_t) \approx t^{2(\lambda+\mu)-\mu N} + t^{4\lambda-\mu(N+2)} - t^{\lambda(p+1)-\mu N}.$$

With the choice $\lambda = 2\mu$ we get (19) $\approx$ (21), and for $p > 2$, we have (20) $\gg$ (21), so $I(u_t) \to -\infty$ as $t \to \infty$, hence the functional has the mountain-pass geometry. □

PROOF OF THEOREM 1.

STEP 1: *For $p \geq 3$, I satisfies the Palais–Smale condition.* Take a sequence such that

$$I(u_n) < C, \qquad I'(u_n) \to 0.$$

We write

$$\begin{aligned}(p+1)I(u_n) - (I'(u_n), u_n) &= \frac{p-1}{2}\|u_n\|_H^2 + \frac{p-3}{4}\int_{\mathbb{R}^N} \phi_{u_n}(x)u_n^2 \\ &\geq \frac{p-1}{2}\|u_n\|_H^2\end{aligned}$$

iff $p \geq 3$. This shows that u_n is bounded in H. Hence, passing to a subsequence, we have

$$u_n \rightharpoonup u \in H \quad \text{and} \quad u_n \to u \quad \text{in } L_K^{p+1}, \quad p \in (\sigma, 2^* - 1).$$

So we write

$$(22) \qquad o(1) = (I'(u_n), (u_n - u)) = \|u_n\|_H^2 - \|u\|_H^2 + o(1) + \int_{\mathbb{R}^N} \phi_{u_n}(x) u_n (u_n - u)\, dx + \int_{\mathbb{R}^N} K(x)|u_n|^p (u_n - u)\, dx.$$

For the Poisson term we have

$$\left| \int_{\mathbb{R}^N} \phi_{u_n}(x) u_n (u_n - u)\, dx \right| \leq \|\phi_{u_n}\|_{D^{1,2}} \|u_n u - u_n^2\|_{L^{2N/(N+2)}}.$$

Now notice that, because of Lemma 3, ϕ_{u_n} is bounded in $\mathcal{D}^{1,2}$. Moreover, because of the compact embedding in $L^{4N/(N+2)}$, passing to a subsequence we have $u_n \to u$ a.e. and $|u_n u - u_n^2| \leq u\sqrt{g} + g \in L^{2N/(N+2)}$ for some $g \in L^{2N/(N+2)}$. Now, using the dominated convergence theorem we infer that $\|u_n u - u_n^2\|_{L^{2N/(N+2)}} \to 0$, and thus

$$\left| \int_{\mathbb{R}^N} \phi_{u_n}(x) u_n (u_n - u)\, dx \right| \to 0.$$

In the same fashion, using Lemma 1, we see that the p-term tends to zero. From this and (22), it follows that $\|u_n\|_H - \|u\|_H \to 0$ and hence $u_n \to u$ strongly in H.

STEP 2: *Conclusion.* Now set $\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}$. The previous steps and Lemma 5 show that the hypotheses of the mountain-pass theorem are satisfied, hence

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

is a critical level of I corresponding to a non-trivial weak solution in H. The bootstrap process can be performed (see the lemma below) and by a maximum principle argument it can be shown that we can actually get a positive classical solution. □

Because we will use Lemma 4, we need to show that H-solutions actually belong to H^2_{loc}. More precisely, we state the following

LEMMA 6. *Let u be a weak solution in H of the problem* (1). *Then $u \in H^2_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, $u \in L^2(\mathbb{R}^N)$, i.e. $u \in H^1(\mathbb{R}^N)$.*

PROOF. For convenience we write the first equation in (1) as $-\Delta u = a(x)u$, with $a(x) := K(x)u^{p-1} - V(x) - \phi(x)$. By standard elliptic regularity theory it is enough to show that $a(x)u \in L^2_{\mathrm{loc}}$. We now claim that $u \in L^q_{\mathrm{loc}}$ for any $2 \leq q < \infty$. In order to prove that, we use the Brezis–Kato result (see e.g. [9, p. 48]), since $a_- u \in L^1_{\mathrm{loc}}$

and $a_+ \in L^{N/2}$. Observe that the former claim is trivial, while dealing with the latter simply observe that $(p-1)N/2 < 2^* \Leftrightarrow p < 2^* - 1$. As a consequence, $\phi \in W^{2,q}_{\mathrm{loc}}$ and by the Morrey embedding theorem, $\phi \in C^{0,\alpha}_{\mathrm{loc}}$. Thanks to the local boundedness of V, K and ϕ, the L^2_{loc} regularity of $a(x)u$ follows, hence the conclusion.

Now we prove that, actually, $u \in H^1$. In order to do that, first observe that ϕ is a positive continuous radial function vanishing at infinity. This is a consequence of the fact that $\phi \in C^{0,\alpha}_{\mathrm{loc}}$ plus the following decay estimate (see [5, p. 340]):

$$|\phi(x)| \le C_N |x|^{(2-N)/2} \|\phi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}, \qquad |x| \ge 1. \tag{23}$$

This observation allows us to define the auxiliary potential $V_u(x) = V(x) + \phi_u(x)$, satisfying the condition

$$\frac{a}{1+|x|^\alpha} \le V_u(x) \le A', \tag{24}$$

which is identical to (3). Observe now that u is a solution of the equation

$$-\Delta u + V_u(x)u = K(x)u^p,$$

which is formally the same as the one studied in [1]. More precisely, it can be shown that

$$\int_{\mathbb{R}^N \setminus B_R(0)} V_u(x)u^2\,dx \approx \exp(-cR^{1-\alpha/2}), \qquad R \gg 1,\ c > 0, \tag{25}$$

where $B_r(y) := \{x \in \mathbb{R}^N : |x-y| < r\}$. Now observe that as a consequence of (24) we have

$$\int_{B_1(y)} u^2\,dx \le c_1 |y|^\alpha \int_{B_1(y)} V_u(x)u^2\,dx. \tag{26}$$

By repeating the same argument in [1, proof of Theorem 16], the equations (25) and (26) yield the existence of a partition $\{B_{r_k}(y_k)\}_{k\ge1}$ of $\mathbb{R}^N \setminus B_2(0)$ such that

$$\int_{\mathbb{R}^N\setminus B_2(0)} u^2\,dx \le \sum_k \int_{B_{r_k}(y_k)} u^2\,dx \le c_2 \sum_k |y_k|^\alpha \exp(-C|y_k|^{1-\alpha/2}) < \infty,$$

completing the proof. □

PROOF OF THEOREM 2. We point out that, for $p \in (2,3)$, the PS condition is not known for I, even in the case $V = K = 1$, although the mountain-pass geometry holds. This is due to the difficulty in proving the boundedness for Palais–Smale sequences. In order to overcome this obstacle, we use a method introduced by Struwe (see e.g. [15] and also [4], [8]).

Let us consider a perturbation of I:

$$I_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\,dx + \frac{1}{4}\int_{\mathbb{R}^N}\phi_u u^2\,dx - \frac{\mu}{p+1}\int_{\mathbb{R}^N}K(x)|u|^{p+1}\,dx \tag{27}$$

for $\mu \in [1/2, 1]$.

Following [4, Proposition 2.3], it is possible to define min-max levels for I_μ, which we denote by c_μ, such that the following properties are satisfied:

(i) $\mu \mapsto c_\mu$ is non-increasing (hence differentiable a.e. in $[1/2, 1]$) and left-continuous.
(ii) Denote by $\mathcal{I}$ the set of μ for which c_μ is differentiable; then for each $\mu \in \mathcal{I}$ there exists a Palais–Smale seqence for I_μ at the level c_μ.
(iii) For almost every $\mu \in [1/2, 1]$, c_μ is a critical level for I_μ.

We remark that thanks to Lemma 5, I has the mountain-pass geometry and we are allowed to use this argument.

We denote by $\mathcal{C}$ the set of values of μ for which c_μ is a critical level for I_μ. Now take a sequence $\mu_n \nearrow 1$ in $\mathcal{C}$ and a sequence $u_n \in H$ of critical points for I_{μ_n}. It is easy to see that, if this sequence is bounded, then Theorem 2 follows. Actually, we can now repeat the same argument carried out in Step 1 above: up to a subsequence, we have $u_n \rightharpoonup u$ in H and

$$u_n \to u \quad \text{in } L_K^{p+1},\ p \in (\sigma, 2^* - 1);$$

hence, from $I'(u_n)(u_n - u) = \|u_n\|_H^2 - \|u\|_H^2 + o(1)$ and $\mu_n \nearrow 1$, we find again that $u_n \to u$ in H and thus $I'(u) = 0$.

To prove that the sequence u_n is bounded we use Lemma 4. First we define the following quantities:

$$\chi_{1,n} := \int_{\mathbb{R}^N}|\nabla u_n|^2, \qquad \chi_{2,n} := \int_{\mathbb{R}^N}V(x)u_n^2,$$
$$\chi_{3,n} := \int_{\mathbb{R}^N}\phi_{u_n}u_n^2, \qquad \chi_{4,n} := \mu_n\int_{\mathbb{R}^N}K(x)|u_n|^{p+1},$$
$$\xi_{V,n} := \int_{\mathbb{R}^N}(x, \nabla V(x))u_n^2, \quad \xi_{K,n} := \mu_n\int_{\mathbb{R}^N}(x, \nabla K(x))|u_n|^{p+1}.$$

Notice that u_n are solutions of the problem $(1)_{\mu_n}$, obtained by replacing K with $\mu_n K$ in (1). Now we can use Lemma 4, having already checked the H^2_{loc} regularity in Lemma 6, to obtain

$$\frac{N-2}{2}\chi_{1,n} + \frac{N}{2}\chi_{2,n} + \frac{N+2}{4}\chi_{3,n} - \frac{N}{p+1}\chi_{4,n} = \frac{1}{p+1}\xi_{K,n} - \frac{1}{2}\xi_{V,n}. \tag{28}$$

By definition, we have

$$\frac{1}{2}\chi_{1,n} + \frac{1}{2}\chi_{2,n} + \frac{1}{4}\chi_{3,n} - \frac{1}{p+1}\chi_{4,n} = c_{\mu_n}. \tag{29}$$

Eliminating $\chi_{3,n}$ in the system (28)–(29) we obtain

$$2\chi_{1,n} + \chi_{2,n} - \frac{1}{2}\xi_{V,n} = (N+2)c_{\mu_n} + \frac{1}{p+1}(2\chi_{4,n} - \xi_{K,n}). \tag{30}$$

Using (7), (30) implies

$$2\chi_{1,n} + \frac{2-c_V^{(1)}}{2}\chi_{2,n} \le (N+2)c_{\mu_n} + \frac{1}{p+1}(2-c_K^{(1)})\chi_{4,n}. \tag{31}$$

Since $2 - c_V^{(1)} > 0$, $2 - c_K^{(1)} \le 0$, and c_{μ_n} is bounded, (31) now implies that $\chi_{1,n}$ and $\chi_{2,n}$ are bounded, so that $\|u_n\|_H \le C$, hence the conclusion. □

PROOF OF THEOREM 3. The proof is the same as the previous one, being reduced to checking the boundedness of u_n. Multiplying the first equation of the problem $(1)_{\mu_n}$ by u and integrating by parts, we find that

$$\chi_{1,n} + \chi_{2,n} + \chi_{3,n} - \chi_{4,n} = 0. \tag{32}$$

Let us solve the system (29)–(32) with respect to the quantities $\chi_{3,n}$ and $\chi_{4,n}$. If $D = (3-p)/[4(p+1)]$ denotes the determinant of the system (since we are considering $p \in (2,3)$, D is positive), we obtain

$$\begin{cases} \chi_{3,n} = \dfrac{1}{D}\left[\dfrac{p-1}{2(p+1)}(\chi_{1,n}+\chi_{2,n}) - c_{\mu_n}\right], \\ \chi_{4,n} = \dfrac{1}{D}\left[\dfrac{1}{4}(\chi_{1,n}+\chi_{2,n}) - c_{\mu_n}\right]. \end{cases} \tag{33}$$

Using (10) in (28), we have

$$\frac{N-2}{2}\chi_{1,n} + \left(\frac{N}{2} + \frac{c_V^{(2)}}{2}\right)\chi_{2,n} + \frac{N+2}{4}\chi_{3,n} - \left(\frac{N}{p+1} + \frac{c_K^{(2)}}{p+1}\right)\chi_{4,n} \le 0. \tag{34}$$

Substituting (33) into (34) we get

$$\left[\frac{N-2}{2} + \frac{N+2}{4D}\cdot\frac{p-1}{2(p+1)} - \frac{1}{4D}\left(\frac{N}{p+1} + \frac{c_K^{(2)}}{p+1}\right)\right]\chi_{1,n}$$
$$+ \left[\frac{N}{2} + \frac{c_V^{(2)}}{2} + \frac{N+2}{4D}\cdot\frac{p-1}{2(p+1)} - \frac{1}{4D}\left(\frac{N}{p+1} + \frac{c_K^{(2)}}{p+1}\right)\right]\chi_{2,n}$$
$$\le \left[\frac{N+2}{4D} - \frac{1}{D}\left(\frac{N}{p+1} + \frac{c_K^{(2)}}{p+1}\right)\right]c_{\mu_n}.$$

It is easy to check that, since $p > \delta := 2+c_K^{(2)}/2$, the coefficient of $\chi_{1,n}$ is positive. For the same reason the coefficient of $\chi_{2,n}$ is also positive. Furthermore, it can be verified that the coefficient of c_{μ_n} is positive for $p > (4c_K^{(2)} + 3N - 2)/(N + 2)$, which is less than δ. Hence we get the boundedness of u_n as above. □

PROOF OF PROPOSITION 1. The proof is based on the mountain-pass theorem and the Ekeland variational principle and it is almost the same as for Theorem 4.3 and Corollary 4.4 in [12]. Precisely, it can be shown that:

(i) $I > -\infty$,
(ii) I satisfies the Palais–Smale condition.

In order to do that, since we work on H, (14) and Lemma 1 must be used instead of the Strauss inequality and the Strauss embedding theorem. The restriction on α is also needed in order to use the continuity property stated in Lemma 3. □

For λ large enough, Proposition 1 does not hold anymore. Indeed, we have the following

PROPOSITION 2. *Assume* $\sigma \in (1, 2]$*,* $p \in [\sigma, 2]$*,* $\alpha \le \alpha^*$ *and suppose* V *and* K *are radial, smooth and satisfy* (3) *and* (4)*. Then:*

(i) *For* $p = 2$*: if* $K(x) \le 1$*, then* (1) *has no non-trivial positive solution* $(u, \phi) \in H \times \mathcal{D}^{1,2}(\mathbb{R}^N)$.
(ii) *For* $p \in [\sigma, 2)$*: if*
$$V(x) \ge (C_p K(x))^{1/(2-p)},$$
where $C_p = (p-1)^{p-1}(2-p)^{2-p}$*, then* (1) *has no non-trivial positive solution* $(u, \phi) \in H \times \mathcal{D}^{1,2}(\mathbb{R}^N)$.

PROOF. Here we follow [10] and [12]. By the assumptions on p and α, H is continuously embedded in L_K^{p+1} and $L^{4N/(N+2)}$, hence all the following integrals are well defined. Now observe that, by the trivial inequality $ab \le a^2 + b^2/4$, it follows that

$$\int_{\mathbb{R}^N} u^3\,dx = \int_{\mathbb{R}^N} \nabla\phi\nabla u\,dx \le \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \frac{1}{4}|\nabla\phi|^2\right) dx. \tag{35}$$

Now we argue by contradiction, assuming that $(u, \phi) \in H \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a non-trivial positive solution. Then we have

$$\begin{aligned} 0 = (I'(u), u) &= \int_{\mathbb{R}^N} (|\nabla u|^2 + \int_{\mathbb{R}^N} (V(x)u^2 + \phi_u u^2\,dx - K(x)|u|^{p+1})\,dx \\ &\ge \int_{\mathbb{R}^N} \left(u^3 - \frac{1}{4}|\nabla\phi|^2\right) dx + \int_{\mathbb{R}^N} (V(x)u^2 + \phi_u u^2 - K(x)|u|^{p+1})\,dx. \end{aligned}$$

Since $\int_{\mathbb{R}^N} \phi u^2\, dx = \int_{R^N} |\nabla\phi|^2\, dx$ we infer that

$$0 \geq \int_{\mathbb{R}^N} u^3\, dx + \int_{\mathbb{R}^N} \left(\frac{3}{4}|\nabla\phi|^2 + V(x)u^2 - K(x)|u|^{p+1}\right) dx \tag{36}$$
$$\geq \int_{\mathbb{R}^N} (u^3 + V(x)u^2 - K(x)u^{p+1})\, dx$$
$$= \int_{\mathbb{R}^N} u^2(u + V(x) - K(x)u^{p-1})\, dx.$$

Now define $f(u) := u + V(x) - K(x)u^{p-1}$. If $p = 2$, then since $K(x) \leq 1$, the function f is strictly increasing, hence strictly positive for $u > 0$. Therefore, (36) implies that $u \equiv 0$ and this is a contradiction. Now consider the case $p \in (1, 2)$. Observe that f has an absolute minimum point $u_m = (K(x)(p-1))^{1/(2-p)}$. Now defining $C_p = (p-1)^{p-1}(2-p)^{2-p}$ and observing that

$$f(u) \geq f(u_m) = V(x) - (C_p K(x))^{1/(2-p)} \geq 0$$

we get a contradiction as above. □

REMARK 3. We remark that the condition $V(x) \geq (C_p K(x))^{1/(2-p)}$ is compatible with the case $\sigma \in (1, 2]$. Therefore, under this condition, we have non-existence although we also have compactness.

As a final remark we also consider

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = K(x)u^p, & x \in \mathbb{R}^N, \\ -\Delta\phi = u^2. \end{cases} \tag{37}$$

For $\lambda \geq 1/4$, by repeating the same proof, it is easy to see that Proposition 2 holds true, extending the result of Theorem 4.1 in [12] to the case of NLSP with radial potentials vanishing at infinity.

5. APPENDIX

PROOF OF LEMMA 4. The proof of this identity follows the standard method in the literature, therefore we only sketch the main steps. Consider $\{\eta^s(x)\}_{s>0} \subset C^\infty_{\mathrm{rad}}(\mathbb{R}^N)$ with the following properties:

$$0 \leq \eta^s(x) \leq 1, \quad |\nabla\eta^s(x)| \leq \frac{C}{s}, \quad \eta^s(x) = \begin{cases} 1, & x \in B(0, s/2), \\ 0, & x \in \mathbb{R}^N \setminus B(0, s), \end{cases}$$

where $B(0, s) := \{x \in \mathbb{R}^N : |x| < s\}$, for some positive constant C. Multiply the first equation in (1) by $x_i \partial_i u(x)\eta^s(x)$, integrate on $B(0, s)$ and sum up over i. Observe that, since $\operatorname{supp}\eta^s$ is contained in $\{x : |x| \leq s\}$, we have $|\nabla\eta^s(x)| \leq C/s \leq C/|x|$.

By the dominated convergence theorem there exists a sequence $s_n \to \infty$ (we simply write $s \to \infty$) (see e.g. [5]–[7], [9, Section 3]) such that

$$-\sum_i \int_{B(0,s)} \Delta u x_i (\partial_i u)\eta^s \, dx = \frac{2-N}{2} \int_{B(0,s)} |\nabla u|^2 \, dx + o(1). \tag{38}$$

In order to perform the calculation for the K-term, integrating by parts, observe that

$$\begin{aligned}\int_{B(0,s)} K(x)u^p x_i (\partial_i u)\eta^s \, dx &= \frac{1}{p+1} \int_{B(0,s)} K(x)(\partial_i u^{p+1}) x_i \eta^s \, dx \\ = -\frac{1}{p+1} \int_{B(0,s)} \eta^s u^{p+1} K(x) \, dx &- \frac{1}{p+1} \int_{B(0,s)} x_i (\partial_i \eta^s) u^{p+1} K(x) \, dx \\ - \frac{1}{p+1} \int_{B(0,s)} \eta^s u^{p+1} x_i \partial_i K(x) \, dx.\end{aligned}$$

In the last step the boundary term has been neglected since $\eta^s(\partial B(0,s)) = 0$. Since $0 \le \eta^s \le 1$ and $\eta^s \to 1$, $|x_i \partial_i \eta^s| \le C$ and $\partial_i \eta^s \to 0$, by the dominated convergence theorem the second integral in the last step tends to zero. Hence

$$\begin{aligned}\int_{B(0,s)} K(x)u^p x_i \partial_i u \eta^s \, dx = &-\frac{1}{p+1} \int_{B(0,s)} \eta^s u^{p+1} K(x) \, dx \\ &- \frac{1}{p+1} \int_{B(0,s)} \eta^s u^{p+1} x_i \partial_i K(x) \, dx + o(1).\end{aligned} \tag{39}$$

We now consider the last integral in (39). Since u is a radial function in $H_V(\mathbb{R}^N)$ the Strauss type inequality (14) holds:

$$|u(x)| \le c|x|^{-\gamma} \|u\|_{H_V} \tag{40}$$

a.e. in $\mathbb{R}^N \setminus B^c(0,s)$ for large s. Since $1 - \eta^s = 0$ on $B(0,s)$ and using (40) we get

$$\begin{aligned}&\left| \int_{\mathbb{R}^N} u^{p+1} x_i \partial_i K(x) \, dx - \int_{\mathbb{R}^N} \eta^s u^{p+1} x_i \partial_i K(x) \, dx \right| \\ &\qquad \le c' s^{-(N-1)\varepsilon/2} \int_{\mathbb{R}^N \setminus B(0,s)} (1-\eta^s) u^{p+1-\varepsilon} |x_i \partial_i K(x)| \, dx.\end{aligned} \tag{41}$$

Notice that, because of (8), since $q'(p+1-\varepsilon) \in [2+\alpha/\gamma, 2^*]$, there exists a constant $C_{p,q',\varepsilon}$ such that $\|u\|_{L^{q'(p+1-\varepsilon)}(\mathbb{R}^N)} \le C_{p,q',\varepsilon} \|u\|_{H_V(\mathbb{R}^N)}$. Therefore, as $0 \le 1 - \eta^s \le 1$, using the Hölder inequality we have

$$\begin{aligned}&\int_{\mathbb{R}^N \setminus B(0,s)} (1-\eta^s) u^{p+1-\varepsilon} |x_i \partial_i K(x)| \, dx \le \int_{\mathbb{R}^N \setminus B(0,s)} u^{p+1-\varepsilon} |x_i \partial_i K(x)| \, dx \\ &\le \|u\|_{L^{q'(p+1-\varepsilon)}(\mathbb{R}^N)} \|(x, \nabla K)\|_{L^q(\mathbb{R}^N)} \le C_{p,q',\varepsilon} \|u\|_{H_V(\mathbb{R}^N)} \|(x, \nabla K)\|_{L^q(\mathbb{R}^N)} < \infty.\end{aligned} \tag{42}$$

Observe that (41) and (42) imply that $\int_{\mathbb{R}^N} u^{p+1}(x,\nabla K)\,dx < \infty$ and

$$\int_{B(0,s)} \eta^s u^{p+1} x_i \partial_i K(x)\,dx = \int_{B(0,s)} u^{p+1} x_i \partial_i K(x)\,dx + o(1). \tag{43}$$

Finally, from (39), (43) and summing up over i we have

$$\sum_i \int_{B(0,s)} K(x) u^p x_i (\partial_i u) \eta^s\,dx = -\frac{N}{p+1}\int_{B(0,s)} K(x) u^{p+1}\,dx - \frac{1}{p+1}\int_{B(0,s)} (x,\nabla K) u^{p+1}\,dx + o(1). \tag{44}$$

In the same fashion as in (44), because of the assumptions on V, we can use the dominated convergence theorem to get

$$\sum_i \int_{B(0,s)} V(x) u^2 x_i (\partial_i u) \eta^s\,dx = -\frac{N}{2}\int_{B(0,s)} V(x) u^2\,dx - \frac{1}{2}\int_{B(0,s)} (x,\nabla V) u^2\,dx + o(1). \tag{45}$$

Moreover, as in (39),

$$\sum_i \int_{B(0,s)} \phi_u u x_i (\partial_i u) \eta^s\,dx = -\frac{N}{2}\int_{B(0,s)} \phi_u u^2\,dx - \frac{1}{2}\int_{B(0,s)} (x,\nabla \phi_u) u^2 \eta^s\,dx + o(1). \tag{46}$$

From the first equation in (1) and (38), (44), (45), (46), we finally have, as $s \to \infty$,

$$\begin{aligned} \frac{2-N}{2}\int_{B(0,s)} |\nabla u|^2\,dx - \frac{N}{2}\int_{B(0,s)} V(x) u^2\,dx - \frac{1}{2}\int_{B(0,s)} (x,\nabla V) u^2\,dx \\ - \frac{N}{2}\int_{B(0,s)} \phi_u u^2\,dx - \frac{1}{2}\int_{B(0,s)} (x,\nabla \phi_u) u^2 \eta^s\,dx + o(1) \\ = -\frac{N}{p+1}\int_{B(0,s)} K(x) u^{p+1}\,dx - \frac{1}{p+1}\int_{B(0,s)} (x,\nabla K) u^{p+1}\,dx. \end{aligned} \tag{47}$$

In the same way as above, we now multiply the second equation in (1) by $(x,\nabla\phi_u)\eta^s$ and integrate on $B(0,s)$, obtaining

$$\frac{2-N}{2}\int_{B(0,s)} |\nabla \phi_u|^2\,dx = \int_{B(0,s)} (x,\nabla\phi_u) u^2 \eta^s\,dx + o(1). \tag{48}$$

Eliminating $\int_{B(0,s)} (x,\nabla\phi_u) u^2 \eta^s\,dx$ from (47) and (48), letting $s \to \infty$ and using $\int_{\mathbb{R}^N} |\nabla\phi_u|^2\,dx = \int_{\mathbb{R}^N} \phi_u u^2\,dx$, we get the conclusion. □

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