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Calculus of variations. — *Boundary regularity of minima*, by JAN KRISTENSEN and GIUSEPPE MINGIONE.

ABSTRACT. — Let $u\colon \Omega \to \mathbb{R}^N$ be any given solution to the Dirichlet variational problem

$$\min_{w} \int_{\Omega} F(x, w, Dw)\, dx, \qquad w \equiv u_0 \quad \text{on } \partial\Omega,$$

where the integrand $F(x, w, Dw)$ is strongly convex in the gradient variable Dw, and suitably Hölder continuous with respect to (x, w). We prove that almost every boundary point, in the sense of the usual surface measure of $\partial\Omega$, is a regular point for u. This means that Du is Hölder continuous in a relative neighbourhood of the point. The existence of even one such regular boundary point was an open problem for the general functionals considered here, and known only under certain very special structure assumptions.

KEY WORDS: Boundary regularity; variational problems; singular sets.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 49N60; Secondary 35J55, 35J60, 49J10.

1. INTRODUCTION

In this paper we describe results recently obtained in [23], concerning the boundary regularity of solutions to Dirichlet variational problems of the type

$$\begin{cases} \min\limits_{w} \mathcal{F}[w], \\ w \equiv u_0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\mathcal{F}[w] := \int_{\Omega} F(x, w, Dw)\, dx. \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded, open, and suitably smooth domain, $n \geq 2$ and the boundary datum u_0 is assumed to be suitably smooth, more specifically,

$$\Omega \text{ is a } C^{1,\alpha} \text{ domain and } u_0 \in C^{1,\alpha}(\Omega, \mathbb{R}^N) \text{ for an } \alpha \in (0, 1). \tag{1.3}$$

The minimization in (1.1) is over all Sobolev maps $w \in u_0 + W^{1,p}_0(\Omega, \mathbb{R}^N)$ and we assume that $p, N \geq 2$. We work under various sets of hypotheses on the integrand $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$. Let us start with the assumptions

$$\begin{cases} \nu|z|^p \leq F(x, y, z) \leq L(1 + |z|^2)^{p/2}, \\ \nu(1 + |z|^2)^{(p-2)/2}|\lambda|^2 \leq \langle F_{zz}(x, y, z)\lambda, \lambda\rangle \leq L(1 + |z|^2)^{(p-2)/2}|\lambda|^2, \\ |F(x_1, y_1, z) - F(x_2, y_2, z)| \leq L\omega_\alpha(|x_1 - x_2| + |y_1 - y_2|)(1 + |z|^2)^{p/2}, \end{cases} \tag{1.4}$$

satisfied for all $x, x_1, x_2 \in \Omega$, $y, y_1, y_2 \in \mathbb{R}^N$ and $z, \lambda \in \mathbb{R}^{Nn}$, where $p \geq 2$, $0 < \nu \leq L$, and the dependence on x and y is Hölder continuous with exponent α in the sense that

$$\omega_\alpha(s) := \min\{s^\alpha, 1\}, \qquad 0 < \alpha \leq 1.$$

Under such assumptions any minimizer of the functional $\mathcal{F}[\cdot]$, that is, any $W^{1,p}$-map u satisfying $\mathcal{F}[u] \leq \mathcal{F}[w]$ whenever $w \in u + W_0^{1,p}(\Omega, \mathbb{R}^N)$, is partially regular. This means that there exists an open subset $\Omega_u \subset \Omega$ such that

$$|\Omega \setminus \Omega_u| = 0, \qquad Du \in C^{1,\tilde{\sigma}}(\Omega_u, \mathbb{R}^N) \tag{1.5}$$

for some $\tilde{\sigma}$ (under (1.4) we have $\tilde{\sigma} = \alpha/2$). We refer to Ω_u as the set of *regular points*. It turns out that it coincides with the largest open subset of Ω where the gradient is continuous:

$$\Omega_u = \{x \in \Omega : Du \text{ is continuous in some neighbourhood } A \text{ of } x\}. \tag{1.6}$$

The complement $\Omega \setminus \Omega_u$ is called the *singular set* of the minimizer u. It is in general non-empty when $n \geq 3$ and $N \geq 2$. Similar partial regularity results also hold for solutions to non-linear elliptic systems, and standard references on the subject include [15, 13, 11, 17, 30, 31]. See also [7, 9, 12, 18, 22, 29].

In the case of solutions to Dirichlet problems like (1.1) it is then natural to ask whether or not partial regularity extends up to the boundary, possibly after replacing (1.5) by

$$\mathcal{H}^{n-1}(\bar{\Omega} \setminus \tilde{\Omega}_u) = 0, \qquad Du \in C^{1,\tilde{\sigma}}(\tilde{\Omega}_u, \mathbb{R}^N), \tag{1.7}$$

where now $\tilde{\Omega}_u = \Omega_u \cup \Omega_u^b$ and $\Omega_u^b \subset \partial\Omega$ is a relatively open subset of $\partial\Omega$, and

$$\Omega_u^b := \{x \in \partial\Omega : Du \in C^{0,\tilde{\sigma}}(\overline{\Omega \cap A}, \mathbb{R}^N) \text{ for some neighbourhood } A \text{ of } x\}. \tag{1.8}$$

The set Ω_u^b is therefore the set of regular boundary points. As far as we know the only result in this direction is due to Jost & Meier [16], where variants of the very special case of functionals

$$w \mapsto \int_\Omega c(x, w)|Dw|^2\, dx \tag{1.9}$$

is considered. Here $c(\cdot)$ is a Hölder continuous and bounded positive function which is uniformly bounded away from zero. They proved that minimizers are Hölder continuous up to the boundary, but did not consider higher order regularity. The main feature of the functional in (1.9) is that the dependence of the energy density on the gradient variable is directly via the quantity $|Dw|$—this is in turn related to the fundamental result of Uhlenbeck [32], and indeed sometimes called "Uhlenbeck structure". This particular structure plays a crucial role in their proof and it allowed them to prove everywhere regularity, known not to hold in general for the multi-dimensional vectorial case. Apart from these very special structures, in the general case of functionals such as

$$w \mapsto \int_\Omega c(x, w) f(Dw)\, dx \quad \text{and} \quad w \mapsto \int_\Omega (c(x) f(Dw) + h(x, w))\, dx, \tag{1.10}$$

the fundamental problem of boundary regularity of minima has remained essentially untouched, and the criteria for boundary regularity available in the literature *do not yield the existence of even one regular boundary point* (see [13, p. 247]). This is a completely unsatisfactory situation, especially compared to the scalar case (i.e. real-valued minima) where solutions are known to be everywhere regular up to the boundary.

Here we shall describe the first general answer to this problem, obtained in [23], and showing that, for large classes of general functionals, including those in (1.10), and therefore not necessarily having an "Uhlenbeck structure" as in (1.9), *almost every boundary point in the sense of the usual surface measure is a regular point.* We prove, in other words, that *partial regularity of minima extends up to the boundary*, that is, (1.7) holds. Let us remark that since under the general assumptions of Hölder continuous coefficients as in (1.10) *the functionals in question do not admit an Euler–Lagrange system, and consequently the regularity theory available for systems does not apply to minimizers*; hence, alternative methods are needed.

2. First results

Before giving the most general results we shall for expository reasons concentrate on certain model functionals, where various phenomena are easily distinguished. We start with the functional

$$\mathcal{F}_1[w] := \int_\Omega (c(x)f(Dw) + d(w)(1+|Dw|^2)^{\gamma/2})\,dx. \tag{2.11}$$

The following standard Hölder continuity and bounds are imposed on the coefficients:

$$0 < \nu \le c(x) \le L, \quad c(\cdot) \in C^{0,\alpha}(\Omega), \tag{2.12}$$

$$0 \le d(y) \le L, \quad d(\cdot) \in C^{0,\beta}(\Omega), \tag{2.13}$$

for all $x \in \Omega$ and $y \in \mathbb{R}^N$. In addition, we assume the following strong convexity condition on $f(\cdot)$:

$$\begin{aligned} &\nu|z|^p \le f(z),\\ &\nu(1+|z|^2)^{(p-2)/2}|\lambda|^2 \le \langle f_{zz}(z)\lambda,\lambda\rangle \le L(1+|z|^2)^{(p-2)/2}|\lambda|^2, \end{aligned} \tag{2.14}$$

for all $x \in \Omega$ and $z, \lambda \in \mathbb{R}^{Nn}$.

THEOREM 2.1. *Under the assumptions* (2.12)–(2.14) *with*

$$\alpha > 1/2, \quad \beta > 2/3, \quad \gamma \le 2p/3, \tag{2.15}$$

let $u \in u_0 + W_0^{1,p}(\Omega,\mathbb{R}^N)$ *be a solution to the Dirichlet problem* (1.1) *with* (1.3) *and* $\mathcal{F} \equiv \mathcal{F}_1$. *Then* $\mathcal{H}^{n-1}$*-almost every boundary point is regular for* u. *Moreover, the global higher differentiability,*

$$Du \in W^{1/p+\varepsilon,p}(\Omega,\mathbb{R}^{Nn}) \quad \text{and} \quad Du|_{\partial\Omega} \in W^{\varepsilon,p}(\partial\Omega,\mathbb{R}^{Nn}), \tag{2.16}$$

holds for some $\varepsilon \equiv \varepsilon(\alpha,\beta) > 0$.

In order to better understand the forthcoming results, let us first discuss the role played by the *main assumption*

$$\alpha > 1/2, \tag{2.17}$$

which also appears as the fundamental assumption to prove the existence of regular boundary points in [10]. This is related to the results obtained in [25, 26] where, by proving that the gradient Du lies in a suitable *fractional Sobolev space*, the author proved that in the case of solutions to elliptic systems the singular set appearing in (1.5) admits the following Hausdorff dimension estimate:

$$\dim_{\mathcal{H}}(\Omega \setminus \Omega_u) \le n - 2\alpha. \tag{2.18}$$

It is then clear that in order to prove the existence of regular boundary points, by an up to the boundary version of estimate (2.18), we have to assume that $\alpha > 1/2$ so that

$$\dim_{\mathcal{H}}(\bar{\Omega} \setminus \tilde{\Omega}_u) \le n - 2\alpha < n - 1 = \dim_{\mathcal{H}}(\partial\Omega). \tag{2.19}$$

The assumption (2.17) also has another natural meaning: it guarantees that Du has a trace $Du|_{\partial\Omega}$ on $\partial\Omega$. The above mentioned partial regularity result is derived in [10] as a consequence of a higher differentiability result, namely that

$$Du \in W^{(2\alpha-\varepsilon)/p,p}(\Omega, \mathbb{R}^{Nn}) \quad \text{for every } \varepsilon > 0. \tag{2.20}$$

The trace theorem then yields

$$Du|_{\partial\Omega} \in W^{(2\alpha-1-\varepsilon)/p,p}(\partial\Omega, \mathbb{R}^{Nn}),$$

provided (2.17) holds.

The above theorem, reported as a first case for expository purposes, is a particular case of results applying to much more general functionals including

$$\mathcal{F}_2[w] := \int_\Omega (f(Dw) + h(x, w, Dw))\, dx, \tag{2.21}$$

where the main point is that the function $h(\cdot)$ grows at a suitably lower rate,

$$0 \le h(x, y, z) \le L(1 + |z|^2)^{\gamma/2}, \qquad \gamma < p\ . \tag{2.22}$$

We refer to Theorem 3.3 below for a precise statement.

The assumption on β in (2.15) is now stronger than that on α, and moreover we are assuming that $\gamma < p$ in (2.22). The reason for this is roughly the following. In order to get singular set estimates we need to bound the oscillations with respect to the x variable, and this is the meaning of (2.17). When considering an energy density depending on u we can rewrite $F(x, u(x), Du) \equiv \tilde{F}(x, Du)$, and in this sense u acts as a measurable coefficient making the significance of (2.17) less obvious. The role of an assumption like $\gamma < p$ emerges at this stage: the perturbation due to the oscillations of $u(x)$ can affect the leading regularizing term only at a lower rate γ. If γ is small enough relative to p, then it turns out that we can efficiently estimate the singular set. A similar role is played by an assumption of more Hölder regularity on the partial function $y \mapsto F(\cdot, y, \cdot)$.

This discussion leads to a more careful examination of the low dimensional cases, i.e. when $n \leq p+2$, already considered in [10, 26] and for functionals in [21]. In low dimensions, the oscillations of $u(x)$ can be better controlled, this being a remote consequence of a suitable Sobolev embedding. A consequence is that in this case no growth restriction of the type $\gamma < p$ is needed. We shall therefore consider the general assumptions

(2.23)
$$\begin{cases} \nu|z|^p \leq F(x,y,z) \leq L(1+|z|^2)^{p/2}, \\ \nu(1+|z|^2)^{(p-2)/2}|\lambda|^2 \leq \langle F_{zz}(x,y,z)\lambda,\lambda\rangle \leq L(1+|z|^2)^{(p-2)/2}|\lambda|^2, \\ |F(x_1,y_1,z)-F(x_2,y_2,z)| \leq L[\omega_\alpha(|x_1-x_2|)+\omega_\beta(|y_1-y_2|)](1+|z|^2)^{p/2}, \\ |F_z(x_1,y_1,z)-F_z(x_2,y_2,z)| \leq L\omega_\alpha(|x_1-x_2|+|y_1-y_2|)(1+|z|^2)^{(p-1)/2}, \end{cases}$$

satisfied for all $x, x_1, x_2 \in \Omega$, $y, y_1, y_2 \in \mathbb{R}^N$ and $z, \lambda \in \mathbb{R}^{Nn}$, where $p \geq 2$, $0 < \nu \leq L$, and the dependence on x and y is Hölder continuous with exponents α and β, respectively:

$$\omega_\alpha(s) := \min\{s^\alpha, 1\}, \qquad \omega_\beta(s) := \min\{s^\beta, 1\}, \qquad 0 < \alpha \leq \beta \leq 1. \tag{2.24}$$

A comparison with (1.4) reveals that the hypotheses (2.23) are essentially the standard ones used for partial interior regularity and boundary regularity criteria, except $(2.23)_4$, which is less common and used sometimes to obtain sharper integrability results [14]. Condition $(2.23)_4$ is anyway mild and automatically satisfied in many cases, including (1.10).

THEOREM 2.2. *Under the assumptions* (2.23) *with*

$$\alpha > 1/2, \qquad \beta > \max\{1-2/n, 2/3\}, \qquad n \leq p+2, \tag{2.25}$$

let $u \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$ *be a solution to the Dirichlet problem* (1.1) *with* (1.3). *Then* $\mathcal{H}^{n-1}$*-almost every boundary point is regular for* u *and* (2.16) *holds.*

REMARK 2.1. For the rest of the paper, we shall confine ourselves to the case

$$n \geq 3, \tag{2.26}$$

since when $n = 2$ solutions are everywhere continuous up to the boundary—see [3, 23, 21].

3. ROUGH COEFFICIENTS AND GENERAL RESULTS

Here we state the main results of [23], that is, Theorems 3.3 and 3.1 below. Our concern here is to relax the assumption (2.17), allowing the maps $x \mapsto F_z(x, \cdot, \cdot)$ and $x \mapsto f(x, \cdot)$ to be Hölder continuous with arbitrarily small exponents $\sigma > 0$.

More precisely, the viewpoint is the following: since the Hölder continuity of the coefficients is used to prove that the gradient lies in a suitable fractional Sobolev space as in (2.20), which in turn implies the singular set estimates, it seems plausible that the same result should hold assuming that the coefficients of the functional are themselves in a fractional Sobolev space. Therefore we shall compensate a very mild Hölder continuity condition on the coefficient $c(\cdot)$ by a fractional Sobolev type differentiability assumption on the map $x \mapsto F_z(x,\cdot,\cdot)$. Furthermore, the results in the previous section will appear as particular cases of the ones we are proposing here.

Let us start with the rough coefficients counterpart of Theorem 2.1, where the assumptions on the coefficient $c(\cdot)$ are considerably weakened. We shall replace the Hölder dependence on x by the following:

$$
\begin{cases} 0 < \nu \le c(x) \le L, \quad c(\cdot) \in W^{\alpha,n_s}(\Omega) \cap C^{0,\sigma}(\Omega) \quad \text{for some } \sigma > 0, \\ n_s < n, \quad n - n_s \text{ depends on } n, N, p, L/\nu, \end{cases} \tag{3.27}
$$

and therefore assuming Hölder continuity with exponent α at the lower L^{n_s} integrability scale, and standard Hölder regularity (L^∞ scale) only with a potentially small exponent σ. Here n_s is an exponent, explicitly computable in terms of $n, N, p, L/\nu$.

In order to formulate assumptions similar to (3.27) for more general functionals of the types (1.2) and (2.21), which do not split as in (2.11), we have to look for an alternative way of stating fractional differentiability. A useful tool is provided by the work of DeVore & Sharpley [8], who noticed that if $c(\cdot) \in W^{s,q}(\mathbb{R}^n)$ then there exists $g(\cdot) \in L^q(\mathbb{R}^n)$ (actually provided by the s-fractional sharp maximal function of $c(\cdot)$) such that, whenever $x_1, x_2 \in \mathbb{R}^n$,

$$
|c(x_1) - c(x_2)| \le (g(x_1) + g(x_2))|x_1 - x_2|^s. \tag{3.28}
$$

We remark that if $c(\cdot) \in W^{1,q}$ one can take $g \approx M(|Dc|)$, i.e., the usual Hardy–Littlewood maximal function of Dc; see also [1]. The authors of [8] thereby define a new function space called $\hat{C}^s_q$, taking (3.28), with $g(\cdot)$ being the corresponding fractional maximal operator, as definition. These are not spaces of Besov type, but are nevertheless comparable to the usual fractional Sobolev spaces in the sense that $W^{\alpha,q}(\mathbb{R}^n) \subset \hat{C}^\alpha_q(\mathbb{R}^n) \subset W^{\alpha-\varepsilon,q}(\mathbb{R}^n)$ for every $\varepsilon \in (0,\alpha)$. Definition (3.28) provides us with the right setting, namely functionals with coefficients in $\hat{C}^\alpha_{n_s}$. Note that $W^{\alpha,n_s} \supset W^{\alpha,\infty} \equiv C^{0,\alpha}$ on bounded domains. We shall use the following set of assumptions:

$$
\begin{cases} \nu|z|^p \le F(x,y,z) \le L(1+|z|^2)^{p/2}, \\ \nu(1+|z|^2)^{(p-2)/2}|\lambda|^2 \le \langle F_{zz}(x,y,z)\lambda,\lambda\rangle \le L(1+|z|^2)^{(p-2)/2}|\lambda|^2, \\ |F(x,y_1,z) - F(x,y_2,z)| \le L\omega_\beta(|y_1-y_2|)(1+|z|^2)^{\gamma/2}, \\ |F_z(x_1,y,z) - F_z(x_2,y,z)| \le (g(x_1)+g(x_2))|x_1-x_2|^\alpha(1+|z|^2)^{(p-1)/2}, \\ |F_z(x,y_1,z) - F_z(x,y_2,z)| \le L\omega_\beta(|y_1-y_2|)(1+|z|^2)^{(\gamma-1)/2}, \end{cases} \tag{3.29}
$$

where $(3.29)_{4,5}$ replace $(2.23)_4$, $\omega_\alpha(\cdot)$, $\omega_\beta(\cdot)$ are as in (2.24), and the rest is as in (2.23) but

$$\gamma \le p \quad \text{and} \quad 0 \le g(\cdot) \in L^{n_s}(\Omega), \quad n_s < n, \tag{3.30}$$

where, as in (3.27), n_s is an exponent which is explicitly computable in terms of $n, N, p, L/\nu$. The function $g(\cdot)$ plays in other words the role of an αth derivative of the function $x \mapsto F_z(x, \cdot, \cdot)$, while $(3.29)_3$ describes the α-Hölder continuity at a weaker rate, since the "Hölder constant $g(\cdot)$" may blow up at certain points. Of course we shall also assume that

$$\begin{cases} |F(x_1, y_1, z) - F(x_2, y_2, z)| \le L\omega_\sigma(|x_1 - x_2| + |y_1 - y_2|)(1 + |z|^2)^{p/2}, \\ \omega_\sigma(s) = \min\{s^\sigma, 1\} \quad \text{for some } \sigma > 0. \end{cases} \tag{3.31}$$

This is essential to prove partial regularity—without which there would be no singular set to estimate. The point is that we are no longer requiring $\sigma > 1/2$. As a matter of fact, with the notation of (1.6)–(1.8), we have $Du \in C^{0,\sigma/2}_{\mathrm{loc}}(\Omega_u \cup \Omega_u^b, \mathbb{R}^{Nn})$.

We start by stating the results in the low dimensional case. In the next theorem we have no restriction on γ except for the natural one $\gamma \le p$.

THEOREM 3.1. *Under the assumptions* (3.29)–(3.31)*, assume that*

$$n \le p + 2, \quad \alpha > 1/2, \quad \beta > \max\{1 - 2/n, 2/3\}. \tag{3.32}$$

Let $u \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$ be a solution to the Dirichlet problem (1.1) *with* (1.3). *Then $\mathcal{H}^{n-1}$-almost every boundary point is regular for u and* (2.16) *holds.*

Gaining a few dimensions more must already be compensated by the assumption $\gamma < p$:

THEOREM 3.2. *Under the assumptions* (3.29)–(3.31) *with $\gamma < p$, assume that*

$$n \le 2p + 2, \quad \alpha > 1/2, \quad \beta > \max\{1 - 2/n, 2/3\}. \tag{3.33}$$

Let $u \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$ be a solution to the Dirichlet problem (1.1) *with* (1.3). *Then $\mathcal{H}^{n-1}$-almost every boundary point is regular for u and* (2.16) *holds.*

This phenomenon becomes more apparent when passing to the general high dimensional case, as also suggested by the discussion after Theorem 2.1.

THEOREM 3.3. *Under the assumptions* (3.29)–(3.31) *with $\gamma < p$, take*

$$\frac{2}{3} \le s \le \frac{p}{p+1} \tag{3.34}$$

and assume that

$$\alpha > 1/2, \quad \beta > s, \quad \gamma \le ps + \frac{2ps}{n-2}. \tag{3.35}$$

Let $u \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$ be a solution to the Dirichlet problem (1.1) *with* (1.3). *Then $\mathcal{H}^{n-1}$-almost every boundary point is regular for u and* (2.16) *holds.*

The above statement should be understood as follows: (α, β, γ) characterize the structure of the integrand $F(\cdot)$ via (3.29), while the s, varying in the range (3.34), parametrize the various results. We note that (3.35) says, in accordance with the discussion after Theorem 2.1, that *the less Hölder regularity we assume on $y \mapsto F(\cdot, y, \cdot)$, the less it can be allowed to grow with respect to z*. The extreme cases of assumptions (3.34)–(3.35) are given by

$$\alpha > 1/2, \quad \beta > 2/3, \quad \gamma \leq \frac{2p}{3} + \frac{4p}{3(n-2)},$$

to be considered if we want to assume less regularity on $y \mapsto F(\cdot, y, \cdot)$, and by

$$\alpha > 1/2, \quad \beta > \frac{p}{p+1}, \quad \gamma \leq \frac{p^2}{p+1} + \frac{2p}{n-2},$$

to be considered if we want to allow a faster growth of $z \mapsto F(\cdot, \cdot, z)$. Note that Theorems 2.1 and 2.2 correspond to particular cases where $g(\cdot) \in L^\infty$ in Theorems 3.3 and 3.1, respectively.

We remark that assumptions (3.29) in particular cover the case (2.11) of splitting integrand.

For certain special energies, we may allow even a discontinuous dependence of the integrand on the variable x. According to the explanations after Theorem 2.1, this happens when x does not enter the regularizing part of the integrand, i.e. the one containing Du. We consider an integrand $f\colon \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}$ satisfying (3.29)–(3.31), obviously recast for the case with no dependence on $u(x)$:

$$\begin{cases} \nu|z|^p \leq f(x,z) \leq L(1+|z|^2)^{p/2}, \\ \nu(1+|z|^2)^{(p-2)/2}|\lambda|^2 \leq \langle f_{zz}(x,z)\lambda, \lambda\rangle \leq L(1+|z|^2)^{(p-2)/2}|\lambda|^2, \\ |f_z(x_1,z) - f_z(x_2,z)| \leq (g(x_1)+g(x_2))|x_1-x_2|^\alpha(1+|z|^2)^{p-1/2}. \end{cases} \tag{3.36}$$

Here $g(\cdot)$ is as in (3.30). Moreover, we shall consider another Carathéodory function $h\colon \Omega \times \mathbb{R}^N \to [0,\infty)$ such that

$$\begin{cases} |h(x,y)| \leq L(1+|y|)^\gamma, \\ |h(x,y_1) - h(x,y_2)| \leq L\omega_\beta(|y_1-y_2|)(1+|y_1|+|y_2|)^\gamma, \end{cases} \quad \gamma < p, \tag{3.37}$$

for every $y, y_1, y_2 \in \mathbb{R}^N$, where $\omega_\beta(\cdot)$ is as in (2.24).

THEOREM 3.4. *Let $u \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$ be a solution to the Dirichlet problem* (1.1) *with* (1.3) *and*

$$\mathcal{F}[w] = \int_\Omega (f(x, Dw) + h(x, w))\, dx. \tag{3.38}$$

Assume that the function $f(\cdot)$ satisfies (3.36) *and* (3.31)*, that $h(\cdot)$ satisfies* (3.37)*, and assume* (3.35)*. Then $\mathcal{H}^{n-1}$-almost every boundary point is regular for u and* (2.16) *holds.*

The significance of the above result lies in the fact that the function $x \mapsto h(x, \cdot)$ is a priori only measurable. It is indeed somehow surprising to obtain regular boundary points even having arbitrarily discontinuous coefficients. A typical model example in this case is given by

$$w \mapsto \int_{\Omega} (c(x) f(Dw) + c_1(x)|u|^{\gamma})\, dx,$$

where $c_1(\cdot)$ is a bounded non-negative measurable function and $\gamma \geq \beta$.

Finally, we want to list a few particular cases of Theorems 3.3 and 3.1 which deserve a separate statement; these involve relevant model cases usually treated in the literature. We consider functionals of the form (2.11) with a more general integrand $f\colon \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ as in (2.23):

$$\begin{cases} \nu|z|^p \leq f(y,z) \leq L(1+|z|^2)^{p/2}, \\ \nu(1+|z|^2)^{(p-2)/2}|\lambda|^2 \leq \langle f_{zz}(y,z)\lambda, \lambda\rangle \leq L(1+|z|^2)^{(p-2)/2}|\lambda|^2, \\ |f(y_1,z) - f(y_2,z)| \leq L\omega_\beta(|y_1-y_2|)(1+|z|^2)^{\gamma/2}, \\ |f_z(y_1,z) - f_z(y_2,z)| \leq L\omega_\beta(|y_1-y_2|)(1+|z|^2)^{(\gamma-1)/2}. \end{cases} \tag{3.39}$$

THEOREM 3.5. *Let $u \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$ be a solution to the Dirichlet problem* (1.1) *with* (1.3) *and*

$$\mathcal{F}[w] = \int_{\Omega} c(x) f(w, Dw)\, dx. \tag{3.40}$$

Assume that the function $f(\cdot)$ satisfies (3.39)*, that $c(\cdot)$ satisfies* (3.27)*, and finally assume* (3.35) *with $\gamma < p$. Then $\mathcal{H}^{n-1}$-almost every boundary point is regular for u and* (2.16) *holds. The conclusion remains valid if we replace* (3.35) *by* (3.33)*, or* (3.35) *by* (2.25) *with $\gamma = p$.*

THEOREM 3.6. *Let $u \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$ be a solution to the Dirichlet problem* (1.1) *with* (1.3) *and*

$$\mathcal{F}[w] = \int_{\Omega} c(x) d(w) f(Dw)\, dx. \tag{3.41}$$

Assume that $f(\cdot)$ satisfies (2.14)*, that $c(\cdot)$ satisfies* (3.27) *with $\alpha > 1/2$, and that $d(\cdot)$ satisfies* (2.13) *with $\beta > p/(p-1)$. If $n \leq p+2$, then $\mathcal{H}^{n-1}$-almost every boundary point is regular for u and* (2.16) *holds.*

4. NEW RESULTS FOR ELLIPTIC SYSTEMS

Here we shall consider Dirichlet problems involving general non-linear homogeneous elliptic systems

$$\begin{cases} \operatorname{div} a(x, u, Du) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{4.42}$$

The approach of the previous section allows us to improve the boundary regularity results of [10] in that we can find partial boundary regularity assuming Hölder continuity of a with an arbitrarily small exponent $\sigma > 0$. We are actually aiming at results as in [10] when a exhibits fractional Sobolev x-dependence. We shall moreover consider more general cases than those of [10]. The assumptions on the Carathéodory vector field $a\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ are now

$$\begin{cases} |a(x,y,z)| + |a_z(x,y,z)|(1+|z|^2)^{1/2} \le L(1+|z|^2)^{(p-1)/2}, \\ \nu(1+|z|^2)^{(p-2)/2}|\lambda|^2 \le \langle a_z(x,y,z)\lambda, \lambda\rangle, \\ |a(x_1,y,z) - a(x_2,y,z)| \le (g(x_1)+g(x_2))|x_1-x_2|^{\alpha}(1+|z|^2)^{(p-1)/2}, \\ |a(x,y_1,z) - a(x,y_2,z)| \le L\omega_{\alpha}(|y_1-y_2|)(1+|z|^2)^{(\gamma-1)/2}, \end{cases} \tag{4.43}$$

with the same meaning as in (3.29). In order to have partial regularity and therefore a singular set to estimate, we shall again assume that

$$\begin{cases} |a(x_1,y_1,z) - a(x_2,y_2,z)| \le L\omega_{\sigma}(|x_1-x_2|+|y_1-y_2|)(1+|z|^2)^{(p-1)/2}, \\ \omega_{\sigma}(s) = \min\{s^{\sigma}, 1\} \quad \text{for some } \sigma > 0. \end{cases} \tag{4.44}$$

A model case for the above assumptions is given by the system

$$\operatorname{div}(c(x)a(Du) + b(u, Du)) = 0, \tag{4.45}$$

where $a(\cdot)$ satisfies $(4.43)_{1,2}$, $c(\cdot)$ satisfies (3.27), and finally $b(\cdot)$ is a differentiable and monotone vector field satisfying $(4.43)_4$.

The first result we obtain extends the one in [10, Theorem 1.1], where the case with no u-dependence is considered.

THEOREM 4.1. *Under the assumptions* (4.43)–(4.44) *with $\gamma < p$, assume that*

$$\alpha > 1/2, \qquad \gamma \le p - \frac{1}{2} + \frac{p}{n-2}, \tag{4.46}$$

and let $u \in u_0 + W^{1,p}_0(\Omega, \mathbb{R}^N)$ be a solution to the Dirichlet problem (4.42) *with* (1.3). *Then $\mathcal{H}^{n-1}$-almost every boundary point is regular for u.*

Recall (2.26). Again in the low dimensional case $n \le p+2$ we have

THEOREM 4.2. *Under the assumptions* (4.43)–(4.44) *with*

$$\alpha > 1/2, \qquad n \le p + 2,$$

let $u \in u_0 + W^{1,p}_0(\Omega, \mathbb{R}^N)$ be a solution to the Dirichlet problem (4.42) *with* (1.3). *Then $\mathcal{H}^{n-1}$-almost every boundary point is regular for u.*

The last result extends [10, Theorem 1.2].

Plan of [23]—technical aspects

The estimates of the Hausdorff dimension of singular sets are essentially consequences of global higher differentiability results. The proofs of these combine several aspects of the regularity theory for vectorial problems. We first remark that in a first attempt to obtain for minimizers results similar to those for weak solutions of elliptic systems from [10], the idea of extending the available results obtained in [21] up to the boundary cannot work. In fact, even in the most favourable cases the interior singular set estimate in the case of minimizers is $\dim_{\mathcal{H}}(\Sigma_u) < n - \alpha$, which is clearly insufficient for the argument in (2.19). Therefore in [23] we first develop a technique for improving the available interior singular set estimates for minima, resulting in estimates of the type $\dim_{\mathcal{H}}(\Sigma_u) < n - 2\alpha$, and variants, in the interior. Next, we extend these bounds up to the boundary, and then we conclude by essentially arguing as in (2.19). The outcome is a theory for minimizers which is comparable to that for weak solutions of elliptic systems developed in [10]. In particular, the existence of regular boundary points still follows from fractional differentiability of the type

$$Du \in W^{\sigma_v/p,p}(\Omega, \mathbb{R}^N). \tag{4.47}$$

This in turn implies (by the trace theorem and potential theory)

$$Du|_{\partial\Omega} \in W^{(\sigma_v-1)/p,p}(\partial\Omega, \mathbb{R}^N) \quad \text{and} \quad \dim_{\mathcal{H}}(\bar{\Omega} \setminus \tilde{\Omega}_u) \le n - \sigma_v.$$

Compare with (1.7) and [25, 28]. At this stage, σ_v is given in terms of n, N, p, L/ν, α, β. The result in (4.47) does not require assumptions such as $\alpha > 1/2$, $\beta > 2/3$, and therefore (4.47) also allows us to improve the interior singular set estimates from [21]. Next, using the size conditions $\alpha > 1/2$, $\beta > 2/3$ yields $\sigma_v > 1$, and consequently we are led to the almost everywhere regularity at the boundary.

The proof of (4.47) relies on various results and techniques. These include two types of regularity results for solutions to basic non-linear elliptic systems. The first are stated in terms of Morrey spaces, and are up-to-the-boundary versions of results due to Campanato [6]. These are crucial in the proofs of our results involving the low dimensional assumption $n \le p + 2$. The second type of results are proper manipulations of certain boundary Caccioppoli estimates. The precise forms of these play a decisive role in subsequent estimates. Another key ingredient is the variational difference-quotient technique introduced in [19, 21]. However, it has to be suitably upgraded using more precise comparison estimates and more careful covering arguments in order to treat the boundary situation. We also invoke the mollification procedure introduced in [10] for systems and now applied on certain cube lattices with small mesh size in combination with suitable boundary estimates. Finally, global higher integrability results based on techniques from [2, 5, 21] are involved at various stages.

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