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Algebra. — *Combinatorics and topology of toric arrangements defined by root systems*, by LUCA MOCI.

A Ilaria, e ai viaggi che ci aspettano

ABSTRACT. — Given the toric (or toral) arrangement defined by a root system Φ, we classify and count its components of each dimension. We show how to reduce to the case of 0-dimensional components, and in this case we give an explicit formula involving the maximal subdiagrams of the affine Dynkin diagram of Φ. Then we compute the Euler characteristic and the Poincaré polynomial of the complement of the arrangement, which is the set of regular points of the torus.

KEY WORDS: Affine Dynkin diagram; Poincaré polynomial; regular points; root system; toric arrangement.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 14N10, 17B10, 20G20.

1. INTRODUCTION

Let $\mathfrak{g}$ be a semisimple Lie algebra of rank n over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra and $\Phi \subset \mathfrak{h}^*$ and $\Phi^\vee \subset \mathfrak{h}$ respectively its root and coroot systems. The equations $\{\alpha(h) = 0\}_{\alpha\in\Phi}$ define in $\mathfrak{h}$ a family $\mathcal{H}$ of intersecting hyperplanes. Let $\langle\Phi^\vee\rangle$ be the lattice spanned by the coroots. Then the quotient $T \doteq \mathfrak{h}/\langle\Phi^\vee\rangle$ is a complex torus of rank n. Each root α takes integer values on $\langle\Phi^\vee\rangle$, so it induces a map $T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ that we denote by e^α. The conditions $\{\alpha(h) \in \mathbb{Z}\}_{\alpha\in\Phi}$ define in $\mathfrak{h}$ a periodic family of hyperplanes, or equivalently the equations $\{e^\alpha(t) = 1\}_{\alpha\in\Phi}$ define in T a finite family $\mathcal{T}$ of hypersurfaces. $\mathcal{H}$ and $\mathcal{T}$ are called respectively the *hyperplane arrangement* and the *toric arrangement* defined by Φ (see for example [8], [10], [23]). We define the *subspaces* of $\mathcal{H}$ to be the intersections of elements of $\mathcal{H}$, and the *components* of $\mathcal{T}$ to be the connected components of the intersections of elements of $\mathcal{T}$. We denote by $\mathcal{S}(\Phi)$ the set of subspaces of $\mathcal{H}$, by $\mathcal{C}(\Phi)$ the set of components of $\mathcal{T}$, and by $\mathcal{S}_d(\Phi)$ and $\mathcal{C}_d(\Phi)$ the sets of d-dimensional subspaces and components. Clearly if $\Phi = \Phi_1 \times \Phi_2$ then $\mathcal{S}(\Phi) = \mathcal{S}(\Phi_1) \times \mathcal{S}(\Phi_2)$ and $\mathcal{C}(\Phi) = \mathcal{C}(\Phi_1) \times \mathcal{C}(\Phi_2)$, so from now on we will suppose Φ to be irreducible.

$\mathcal{H}$ is a classical object, whereas De Concini and Procesi [8] recently showed that $\mathcal{T}$ provides a geometric way to compute the values of the Kostant partition function. This function counts in how many ways an element of the lattice $\langle\Phi\rangle$ can be written as a sum of positive roots, and plays an important role in representation theory, since (by

Kostant's and Steinberg's formulae [18], [25]) it yields efficient computation of weight multiplicities and Littlewood–Richardson coefficients, as shown in [6] using results from [1], [3], [7], [26]. Values of the Kostant partition function can be computed as sums of contributions given by the elements of $\mathcal{C}_0(\Phi)$ (see [6, Th. 3.2]).

Furthermore, let R be the complement in T of the union of all elements of $\mathcal{T}$. Then R is called the set of *regular points* of the torus T and has been intensively studied (see in particular [8], [19], [20]). The cohomology of R is the direct sum of contributions given by the elements of $\mathcal{C}(\Phi)$ (see for example [8]). Then by describing the action of W on $\mathcal{C}(\Phi)$ we implicitly get a W-equivariant decomposition of the cohomology of R, and by counting and classifying the elements of $\mathcal{C}(\Phi)$ we can compute the Poincaré polynomial of R.

We say that a subset Θ of Φ is a *subsystem* if it satisfies the following conditions:

- $\alpha \in \Theta \Rightarrow -\alpha \in \Theta$,
- $\alpha, \beta \in \Theta$ and $\alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Theta$.

For each $t \in T$ let us define the following subsystem of Φ:

$$\Phi(t) \doteq \{\alpha \in \Phi \mid e^{\alpha}(t) = 1\}.$$

The aim of Section 2 is to describe $\mathcal{C}_0(\Phi)$, the set of points $t \in T$ such that $\Phi(t)$ has rank n. Let $\alpha_1, \dots, \alpha_n$ be the simple roots of Φ, α_0 the lowest root (i.e. the opposite of the highest root), and Φ_p the subsystem of Φ generated by $\{\alpha_i\}_{0 \le i \le n,\, i \ne p}$. Let Γ be the affine Dynkin diagram of Φ and $V(\Gamma)$ the set of its vertices (a list of such diagrams can be found for example in [12] or [17]). $V(\Gamma)$ is in bijection with $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$, so we can identify each vertex p with an integer from 0 to n. The diagram Γ_p that we get by removing from Γ the vertex p (and all adjacent edges) is the (genuine) Dynkin diagram of Φ_p. Let W be the Weyl group of Φ and W_p the Weyl group of Φ_p, i.e. the subgroup of W generated by all the reflections $s_{\alpha_0}, \dots, s_{\alpha_n}$ except s_{α_p}. Notice that Γ_0 is the Dynkin diagram of Φ and $W_0 = W$. Since W permutes the roots, its natural action on T restricts to an action on $\mathcal{C}_0(\Phi)$. We denote by $W(t)$ the stabilizer of a point $t \in \mathcal{C}_0(\Phi)$. We prove

THEOREM 1. *There is a bijection between the W-orbits of $\mathcal{C}_0(\Phi)$ and the vertices of Γ, having the property that for every point t in the orbit $\mathcal{O}_p$ corresponding to the vertex p, $\Phi(t)$ is W-conjugate to Φ_p and $W(t)$ is W-conjugate to W_p.*

As a corollary we get the formula

$$|\mathcal{C}_0(\Phi)| = \sum_{p \in V(\Gamma)} \frac{|W|}{|W_p|}. \tag{1}$$

In Section 3 we deal with components of arbitrary dimension. For each component U of $\mathcal{T}$ we consider the subsystem of Φ,

$$\Theta_U \doteq \{\alpha \in \Phi \mid e^{\alpha}(t) = 1 \; \forall t \in U\},$$

and its *completion* $\overline{\Theta_U} \doteq \langle \Theta_U \rangle_{\mathbb{R}} \cap \Phi$.

Let $\mathcal{K}_d$ be the set of subsystems Θ of Φ of rank $n-d$ that are *complete* (i.e. such that $\Theta = \overline{\Theta}$), and let $\mathcal{C}^{\Phi}_{\Theta}$ be the set of components U such that $\overline{\Theta_U} = \Theta$. This gives a partition of the components:

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}^{\Phi}_{\Theta}.$$

Notice that the subsystem of roots vanishing on a subspace of $\mathcal{H}$ is always complete; then $\mathcal{K}_d$ is in bijection with $\mathcal{S}_d$. The elements of $\mathcal{S}_d$ are classified and counted in [22], [23]. Thus the description of the sets $\mathcal{C}^{\Phi}_{\Theta}$ that we give in Theorem 11 yields a classification of the components of $\mathcal{T}$. In particular, we show that $|\mathcal{C}^{\Phi}_{\Theta}| = n_{\Theta}^{-1}|\mathcal{C}_0(\Theta)|$, where n_{Θ} is an integer depending only on the conjugacy class of Θ, and so

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta \in \mathcal{K}_d} n_{\Theta}^{-1}|\mathcal{C}_0(\Theta)|. \tag{2}$$

In Section 4, using results of [8] and [9], we deduce from Theorem 1 that the Euler characteristic of R is equal to $(-1)^n|W|$. Moreover, Corollary 12 yields a formula for the Poincaré polynomial of R:

$$P_{\Phi}(q) = \sum_{d=0}^{n}(-1)^d(q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} n_{\Theta}^{-1}|W^{\Theta}|. \tag{3}$$

This formula allows one to compute $P_{\Phi}(q)$ explicitly.

2. Zero-dimensional components

2.1. *Statements*

For all facts about Lie algebras and root systems we refer to [14]. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$$

be the Cartan decomposition of $\mathfrak{g}$, and let us choose nonzero elements $X_0, X_1, \ldots, X_n$ in the 1-dimensional subalgebras $\mathfrak{g}_{\alpha_0}, \mathfrak{g}_{\alpha_1}, \ldots, \mathfrak{g}_{\alpha_n}$; since $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\alpha'}] = \mathfrak{g}_{\alpha+\alpha'}$ whenever $\alpha, \alpha', \alpha+\alpha' \in \Phi$, we know that $X_0, X_1, \ldots, X_n$ generate $\mathfrak{g}$. Let $a_0 = 1$ and for $p = 1, \ldots, n$ let a_p be the coefficient of α_p in $-\alpha_0$. For each $p = 0, \ldots, n$ we define an automorphism σ_p of $\mathfrak{g}$ by

$$\sigma_p(X_p) \doteq e^{2\pi i a_p^{-1}} X_p, \qquad \sigma_p(X_i) = X_i \ \forall i \neq p.$$

Let G be the semisimple and simply connected algebraic group having root system Φ; $\mathfrak{g}$ and T are respectively the Lie algebra and a maximal torus of G (see for example [13]). G acts on itself by conjugacy, i.e. for each $g \in G$ the map $k \mapsto gkg^{-1}$ is an automorphism of G. Its differential $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}$.

REMARK 2. Let $t \in \mathcal{C}_0(\Phi)$ and let $\mathfrak{g}^{\mathrm{Ad}(t)}$ be the subalgebra consisting of the elements fixed by $\mathrm{Ad}(t)$. For each $\alpha \in \Phi$ and for each $X_\alpha \in \mathfrak{g}_\alpha$ we have $\mathrm{Ad}(t)(X_\alpha) = e^\alpha(t)X_\alpha$, thus

$$\mathfrak{g}^{\mathrm{Ad}(t)} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(t)} \mathfrak{g}_\alpha.$$

Moreover, $\mathfrak{g}^{\sigma_p}$ is generated by the subalgebras $\{\mathfrak{g}_{\alpha_i}\}_{0 \le i \le n,\, i \neq p}$. Then $\mathfrak{g}^{\mathrm{Ad}(t)}$ and $\mathfrak{g}^{\sigma_p}$ are semisimple algebras with root systems respectively $\Phi(t)$ and Φ_p. Our strategy will be to prove that for each $t \in \mathcal{C}_0(\Phi)$, $\mathrm{Ad}(t)$ is conjugate to some σ_p. This implies that $\mathfrak{g}^{\mathrm{Ad}(t)}$ is conjugate to $\mathfrak{g}^{\sigma_p}$ and then $\Phi(t)$ to Φ_p, as claimed in Theorem 1.

Then we want to give a bijection between vertices of Γ and W-orbits of $\mathcal{C}_0(\Phi)$ showing that, for every t in the orbit $\mathcal{O}_p$, $\mathrm{Ad}(t)$ is conjugate to σ_p. However, since some of the σ_p (as well as the corresponding Φ_p) are themselves conjugate, this bijection is not going to be canonical. To make it canonical we should merge the orbits corresponding to conjugate automorphisms; for this we consider the action of a larger group.

Let $\Lambda(\Phi) \subset \mathfrak{h}$ be the lattice of coweights of Φ, i.e.

$$\Lambda(\Phi) \doteq \{h \in \mathfrak{h} \mid \alpha(h) \in \mathbb{Z}\ \forall \alpha \in \Phi\}.$$

The lattice spanned by the coroots $\langle \Phi^\vee \rangle$ is a sublattice of $\Lambda(\Phi)$; set

$$Z(\Phi) \doteq \Lambda(\Phi)/\langle \Phi^\vee \rangle.$$

This finite subgroup of T coincides with $Z(G)$, the *center* of G. It is well known ([13, 13.4]) that

$$\mathrm{Ad}(g) = \mathrm{id}_{\mathfrak{g}} \Leftrightarrow g \in Z(\Phi). \tag{4}$$

Notice that

$$Z(\Phi) = \{t \in T \mid \Phi(t) = \Phi\}$$

thus $Z(\Phi) \subseteq \mathcal{C}_0(\Phi)$. Moreover, for each $z \in Z(\Phi)$, $t \in T$, $\alpha \in \Phi$,

$$e^\alpha(zt) = e^\alpha(z)e^\alpha(t) = e^\alpha(t),$$

and therefore $\Phi(zt) = \Phi(t)$. In particular, $Z(\Phi)$ acts by multiplication on $\mathcal{C}_0(\Phi)$. Clearly this action commutes with that of W and we get an action of $W \times Z(\Phi)$ on $\mathcal{C}_0(\Phi)$.

Let Q be the set of $\mathrm{Aut}(\Gamma)$-orbits of $V(\Gamma)$. If $p, p' \in V(\Gamma)$ are two representatives of $q \in Q$, then $\Gamma_p \simeq \Gamma_{p'}$, thus $W_p \simeq W_{p'}$. Moreover we will see (Corollary 7(ii)) that σ_p is conjugate to $\sigma_{p'}$. Then we can restate Theorem 1 as follows.

THEOREM 3. *There is a canonical bijection between Q and the set of $W \times Z(\Phi)$-orbits in $\mathcal{C}_0(\Phi)$, having the property that if $p \in V(\Gamma)$ is a representative of $q \in Q$, then:*

(i) *every point t in the corresponding orbit $\mathcal{O}_q$ induces an automorphism conjugate to σ_p;*
(ii) *the stabilizer of $t \in \mathcal{O}_q$ is isomorphic to $W_p \times \mathrm{Stab}_{\mathrm{Aut}(\Gamma)}\, p$.*

This theorem immediately implies the formula

$$|\mathcal{C}_0(\Phi)| = \sum_{q \in Q} |q| \frac{|W|}{|W_p|} \tag{5}$$

where p is any representative of q. This is clearly equivalent to formula (1).

REMARK 4. If we view the elements of $\Lambda(\Phi)$ as translations, we can define a group of isometries of $\mathfrak{h}$ by

$$\widetilde{W} \doteq W \ltimes \Lambda(\Phi).$$

$\widetilde{W}$ is called the *extended affine Weyl group* of Φ and contains the affine Weyl group $\widehat{W} \doteq W \ltimes \langle \Phi^\vee \rangle$ (see for example [15], [24]).

The action of $W \times Z(\Phi)$ on $\mathcal{C}_0(\Phi)$ can be lifted to an action of $\widetilde{W}$. Indeed, $\widetilde{W}$ preserves the lattice $\langle \Phi^\vee \rangle$ of $\mathfrak{h}$, and thus acts on $T = \mathfrak{h}/\langle \Phi^\vee \rangle$ and on $\mathcal{C}_0(\Phi) \subset T$. Since the semidirect factor $\langle \Phi^\vee \rangle$ acts trivially, $\widetilde{W}$ acts as its quotient,

$$\widetilde{W}/\langle \Phi^\vee \rangle \simeq W \times Z(\Phi).$$

2.2. *Examples*

In the following examples we denote by $\mathfrak{S}_n$, $\mathfrak{D}_n$, $\mathfrak{C}_n$ respectively the symmetric, dihedral and cyclic group on n letters.

CASE C_n. The roots $2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n$ ($i = 1, \ldots, n$) take integer values at the points $[\alpha_1^\vee/2], \ldots, [\alpha_n^\vee/2] \in \mathfrak{h}/\langle \Phi^\vee \rangle$, and thus at their sums, for a total of 2^n points of $\mathcal{C}_0(\Phi)$. Indeed, let us introduce the following notation. For a fixed basis $h_1^*, \ldots, h_n^*$ of $\mathfrak{h}^*$, the simple roots of C_n can be written as

$$\alpha_i = h_i^* - h_{i+1}^* \quad \text{for } i = 1, \ldots, n-1, \qquad \alpha_n = 2h_n^*. \tag{6}$$

Then $\Phi = \{h_i^* - h_j^*\} \cup \{h_i^* + h_j^*\} \cup \{\pm 2h_i^*\}$ ($i, j = 1, \ldots, n$, $i \neq j$), and writing t_i for $e^{h_i^*}$, we have

$$e^\Phi \doteq \{e^\alpha \mid \alpha \in \Phi\} = \{t_i t_j^{-1}\} \cup \{t_i t_j\} \cup \{t_i^{\pm 2}\}.$$

The system of n independent equations $t_1^2 = 1, \ldots, t_n^2 = 1$ has 2^n solutions: $(\pm 1, \ldots, \pm 1)$, and it is easy to see that the other systems do not have other solutions. The group $W \simeq \mathfrak{S}_n \ltimes (\mathfrak{C}_2)^n$ acts on $T = (\mathbb{C}^*)^n$ by permuting and inverting its coordinates; the second operation is trivial on $\mathcal{C}_0(\Phi)$. Two elements of $\mathcal{C}_0(\Phi)$ are in the same W-orbit if and only if they have the same number of negative coordinates. So we can define the p-th W-orbit $\mathcal{O}_p$ as the set of points with p negative coordinates. (This choice is not canonical: we may choose the set of points with p positive coordinates as well.) Clearly, if $t \in \mathcal{O}_p$ then

$$W(t) \simeq (\mathfrak{S}_p \times \mathfrak{S}_{n-p}) \ltimes (\mathfrak{C}_2)^n.$$

Thus $|\mathcal{O}_p| = \binom{n}{p}$ and we get

$$|\mathcal{C}_0(\Phi)| = \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$

Notice that if $t \in \mathcal{O}_p$ then $-t \in \mathcal{O}_{n-p}$, and $\mathrm{Ad}(t) = \mathrm{Ad}(-t)$ since $Z(\Phi) = \{\pm(1,\dots,1)\}$. In fact, Γ has a symmetry exchanging the vertices p and $n-p$. Finally, notice that $\mathcal{C}_0(\Phi)$ is a subgroup of T isomorphic to $(\mathfrak{C}_2)^n$ and generated by the elements

$$\delta_i \doteq (1,\dots,1,-1,1,\dots,1) \quad \text{(with the } -1 \text{ at the } i\text{-th place).}$$

Then we can return to the original coordinates observing that δ_i is the nontrivial solution of the system $t_i^2 = 1$, $t_j = 1$ for $j \neq i$, and using (6) to get

$$\delta_i \leftrightarrow \Big[\sum_{k=i}^{n} \alpha_k^\vee/2\Big].$$

CASE D_n. We can write $\alpha_n = h_{n-1}^* + h_n^*$ and the other α_i as before, so $e^\Phi = \{t_i t_j^{-1}\} \cup \{t_i t_j\}$. Then each system of n independent equations is W-conjugate to

$$t_1 = t_2, \dots, t_{p-1} = t_p,\ t_{p-1} = t_p^{-1},\ t_{p+1}^{\pm 1} = t_{p+2}, \dots, t_{n-1} = t_n,\ t_{n-1} = t_n^{-1}$$

for some $p \neq 1, n-1$. Thus we get the subset of $(\mathfrak{C}_2)^n$ consisting of the following n-ples: $\{(\pm 1,\dots,\pm 1)\} \setminus \{\pm\delta_i\}_{i=1,\dots,n}$, $2^n - 2n$ in number. However, reasoning as before we see that each such n-ple represents two points in $\mathfrak{h}/\langle\Phi^\vee\rangle$. Namely, the correspondence is given by

$$\left\{\left[\sum_{k=i}^{n-1} \frac{\alpha_k^\vee}{2} \pm \frac{\alpha_{n-1}^\vee - \alpha_n^\vee}{4}\right]\right\} \to \delta_i.$$

From the geometric point of view, the t_is are coordinates of a maximal torus of the orthogonal group, while $T = \mathfrak{h}/\langle\Phi^\vee\rangle$ is a maximal torus of its two-sheet universal covering. Each W-orbit corresponding to the four extremal vertices of Γ is a singleton consisting of one of the four points over $\pm(1,\dots,1)$, all inducing the identity automorphism: indeed, $\mathrm{Aut}(\Gamma)$ acts transitively on these points. The other orbits are defined as in the case C_n.

CASE B_n. This case is very similar to the previous one, but now $\alpha_n = h_n^*$, $e^\Phi = \{t_i t_j^{-1}\} \cup \{t_i t_j\} \cup \{t_i^{\pm 1}\}$, and hence we get the points $\{(\pm 1, \dots, \pm 1)\} \setminus \{\delta_i\}_{i=1,\dots,n}$. Here the projection is

$$\left\{\left[\sum_{k=i}^{n-1} \frac{\alpha_k^\vee}{2} \pm \frac{\alpha_n^\vee}{4}\right]\right\} \to \delta_i$$

so we have $2^n - n$ pairs of points in $\mathcal{C}_0(\Phi)$.

CASE A_n. If we view $\mathfrak{h}^*$ as the subspace of $\langle h_1^*, \dots, h_{n+1}^* \rangle$ of equation $\sum h_i^* = 0$, and T as the subgroup of $(\mathbb{C}^*)^{n+1}$ of equation $\prod t_i = 1$, we can write all the simple roots as $\alpha_i = h_i^* - h_{i+1}^*$; then $e^\Phi = \{t_i t_j^{-1}\}$. In this case Φ has no proper subsystem of the same rank, so all the coordinates must be equal. Therefore

$$\mathcal{C}_0(\Phi) = Z(\Phi) = \{(\zeta, \dots, \zeta) \mid \zeta^{n+1} = 1\} \simeq \mathfrak{C}_{n+1}.$$

Then $W \simeq \mathfrak{S}_{n+1}$ acts on $\mathcal{C}_0(\Phi)$ trivially and $Z(\Phi)$ transitively, as expected since $\mathrm{Aut}(\Gamma) \simeq \mathfrak{D}_{n+1}$ acts transitively on the vertices of Γ. We can write more explicitly $\mathcal{C}_0(\Phi) \subseteq \mathfrak{h}/\langle \Phi^\vee \rangle$ as

$$\mathcal{C}_0(\Phi) = \left\{\left[\frac{k}{n+1} \sum_{i=1}^{n} i\alpha_i^\vee\right] \,\middle|\, k = 0, \dots, n\right\}.$$

2.3. *Proofs*

Motivated by Remark 2, we start by describing the automorphisms of $\mathfrak{g}$ that are induced by the points of $\mathcal{C}_0(\Phi)$.

LEMMA 5. *If $t \in \mathcal{C}_0(\Phi)$, then* $\mathrm{Ad}(t)$ *has finite order.*

PROOF. Let $\beta_1, \dots, \beta_n$ be linearly independent roots such that $e^{\beta_i}(t) = 1$. Then for each root $\alpha \in \Phi$ we have $m\alpha = \sum c_i \beta_i$ for some $m, c_i \in \mathbb{Z}$, and thus

$$e^\alpha(t^m) = e^{m\alpha}(t) = \prod_{i=1}^{n} (e^{\beta_i})^{c_i}(t) = 1.$$

Then $\mathrm{Ad}(t^m)$ is the identity on $\mathfrak{g}$, so by (4), $t^m \in Z(\Phi)$. As $Z(\Phi)$ is a finite group, t^m and t have finite order. □

The previous lemma allows us to apply the following

THEOREM 6 (Kac).

(i) *Each inner automorphism of $\mathfrak{g}$ of finite order m is conjugate to an automorphism σ of the form*

$$\sigma(X_i) = \zeta^{s_i} X_i$$

with ζ a fixed primitive m-th root of unity and $(s_0, \dots, s_n)$ nonnegative integers without common factors such that $m = \sum s_i a_i$.

(ii) *Two such automorphisms are conjugate if and only if there is an automorphism of Γ sending the parameters $(s_0, \dots, s_n)$ of the first to the parameters $(s'_0, \dots, s'_n)$ of the second.*

(iii) *Let $(i_1, \dots, i_r)$ be all the indices for which $s_{i_1} = \cdots = s_{i_r} = 0$. Then $\mathfrak{g}^\sigma$ is the direct sum of an $(n-r)$-dimensional center and a semisimple Lie algebra whose Dynkin diagram is the subdiagram of Γ with vertices $i_1, \dots, i_r$.*

This is a special case of a theorem proved in [16] and more extensively in [12, X.5.15 and 16]. We only need the following

COROLLARY 7.

(i) *Let σ be an inner automorphism of $\mathfrak{g}$ of finite order m such that $\mathfrak{g}^\sigma$ is semisimple. Then there is $p \in V(\Gamma)$ such that σ is conjugate to σ_p. In particular, $m = a_p$ and the Dynkin diagram of $\mathfrak{g}^\sigma$ is Γ_p.*

(ii) *Two automorphisms σ_p, $\sigma_{p'}$ are conjugate if and only if p, p' are in the same* Aut(Γ)*-orbit.*

PROOF. If $\mathfrak{g}^\sigma$ is semisimple, then in Theorem 6(iii) $n = r$, hence all parameters of σ but one are equal to 0, and the nonzero parameter s_p must be equal to 1, otherwise there would be a common factor, contradicting Theorem 6. So we get the first statement. Then the second statement follows from Theorem 6(ii). □

Let $t \in \mathcal{C}_0(\Phi)$; by Remark 2, $\mathfrak{g}^{\mathrm{Ad}(t)}$ is semisimple, so by Corollary 7(i), $\mathrm{Ad}(t)$ is conjugate to some σ_p. Thus there is a canonical map

$$\psi : \mathcal{C}_0(\Phi) \to Q \tag{7}$$

sending t to $\psi(t) = \{p \in V(\Gamma) \mid \sigma_p \text{ is conjugate to } \mathrm{Ad}(t)\}$. Notice that $\psi(t)$ is a well-defined element of Q by Corollary 7(ii).

We now prove the fundamental

LEMMA 8. *Two points in $\mathcal{C}_0(\Phi)$ induce conjugate automorphisms if and only if they are in the same $W \times Z(\Phi)$-orbit.*

PROOF. Let N be the normalizer of T in G. We recall that $W \simeq N/T$ and the action of W on T is induced by the conjugation action of N; it is also well known that two points of T are G-conjugate if and only if they are W-conjugate. Thus W-conjugate points induce conjugate automorphisms. Moreover, by (4),

$$\mathrm{Ad}(t) = \mathrm{Ad}(s) \;\Leftrightarrow\; \mathrm{Ad}(ts^{-1}) = \mathrm{id}_{\mathfrak{g}} \;\Leftrightarrow\; ts^{-1} \in Z(\Phi).$$

Finally, suppose that $t, t' \in \mathcal{C}_0(\Phi)$ induce conjugate automorphisms, i.e.

$$\exists g \in G : \mathrm{Ad}(t') = \mathrm{Ad}(g)\,\mathrm{Ad}(t)\,\mathrm{Ad}(g^{-1}) = \mathrm{Ad}(gtg^{-1}).$$

Then $zt' = gtg^{-1}$ for some $z \in Z(\Phi)$. Thus zt' and t are G-conjugate elements of T, and so they are W-conjugate, proving the claim. □

We can now prove Theorem 3(i). Indeed, by the previous lemma there is a canonical injective map defined on the set of orbits of $\mathcal{C}_0(\Phi)$:

$$\overline{\psi} : \frac{\mathcal{C}_0(\Phi)}{W \times Z(\Phi)} \to Q.$$

We must show that this map is surjective. The system

$$\alpha_i(h) = 1\ (\forall i \neq 0, p), \qquad \alpha_p(h) = a_p^{-1}$$

is composed of n linearly independent equations, so it has a solution $h \in \mathfrak{h}$. Notice that $\alpha_0(h) \in \mathbb{Z}$. Let t be the class of h in T; then $e^{\alpha}(t) = 1 \Leftrightarrow \alpha \in \Phi_p$. Hence by Remark 2, $\mathrm{Ad}(t)$ is conjugate to σ_p, and $\Phi(t)$ to Φ_p.

In order to relate the action of $Z(\Phi)$ to that of $\mathrm{Aut}(\Gamma)$, we introduce the following subset of W. For each $p \neq 0$ such that $a_p = 1$, set $z_p \doteq w_0^p w_0$, where w_0 is the longest element of W and w_0^p is the longest element of the parabolic subgroup of W generated by all the simple reflections $s_{\alpha_1}, \dots, s_{\alpha_n}$ except s_{α_p}. Then we define

$$W_Z \doteq \{1\} \cup \{z_p\}_{p=1,\dots,n,\, a_p=1}.$$

This set has the following properties (see [15, §1.7 and 1.8]):

THEOREM 9 (Iwahori–Matsumoto).

(i) *W_Z is a subgroup of W isomorphic to $Z(\Phi)$.*
(ii) *For each $z_p \in W_Z$, $z_p.\alpha_0 = \alpha_p$. This defines an injective morphism $W_Z \hookrightarrow \mathrm{Aut}(\Gamma)$, and the W_Z-orbits of $V(\Gamma)$ coincide with the $\mathrm{Aut}(\Gamma)$-orbits.*

Therefore Q is the set of W_Z-orbits of $V(\Gamma)$, and the bijection $\overline{\psi}$ between Q and the set of $Z(\Phi)$-orbits of $\mathcal{C}_0(\Phi)/W$ can be lifted to a noncanonical bijection between $V(\Gamma)$ and $\mathcal{C}_0(\Phi)/W$. Thus we just have to consider the action of W on $\mathcal{C}_0(\Phi)$ and prove

LEMMA 10. *If $t \in \mathcal{O}_p$, then $W(t)$ is conjugate to W_p.*

PROOF. Notice that the centralizer $C_N(t)$ of t in N is the normalizer of $T = C_T(t)$ in $C_G(t)$. Thus $W(t) = C_N(t)/T$ is the Weyl group of $C_G(t)$. Since $C_G(t)$ is the subgroup of G of points fixed by the conjugacy by t, its Lie algebra is $\mathfrak{g}^{\mathrm{Ad}(t)}$, conjugate to $\mathfrak{g}^{\sigma_p}$ by Theorem 3(i). Therefore $W(t)$ is conjugate to W_p. □

This completes the proof of Theorem 3 and also of Theorem 1, since by Remark 2 the map ψ defined in (7) can also be seen as the map

$$t \mapsto \psi(t) = \{p \in V(\Gamma) \mid \Phi_p \text{ is conjugate to } \Phi(t)\}.$$

3. Positive-dimensional components

3.1. *From hyperplane arrangements to toric arrangements*

Let S be a d-dimensional subspace of $\mathcal{H}$. The set Θ_S of elements of Φ vanishing on S is a complete subsystem of Φ of rank $n-d$. Hence the map $S \mapsto \Theta_S$ gives a bijection between $\mathcal{S}_d$ and $\mathcal{K}_d$, whose inverse is

$$\Theta \mapsto S(\Theta) \doteq \{h \in \mathfrak{h} \mid \alpha(h) = 0 \ \forall \alpha \in \Theta\}.$$

In [23, 6.4 and C] (following [22] and [5]) the subspaces of $\mathcal{H}$ are classified and counted, and the W-orbits of $\mathcal{S}_d$ are completely described. This is done case-by-case according to the type of Φ. We now show a case-free way to extend this analysis to the components of $\mathcal{T}$.

Given a component U of $\mathcal{T}$, set

$$\Theta_U \doteq \{\alpha \in \Phi \mid e^{\alpha}(t) = 1 \ \forall t \in U\}.$$

In contrast with the case of linear arrangements, Θ_U in general is not complete. For each $\Theta \in \mathcal{K}_d$ define $\mathcal{C}^{\Phi}_{\Theta}$ as the set of components U such that $\overline{\Theta_U} = \Theta$. This is clearly a partition of the set of d-dimensional components of $\mathcal{T}$, i.e.

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}^{\Phi}_{\Theta}. \tag{8}$$

We may think of $S(\Theta)$ as the tangent space at any point of each component of $\mathcal{C}^{\Phi}_{\Theta}$; then by [23] the problem of classifying and counting the components of $\mathcal{T}$ reduces to classifying and counting the components of $\mathcal{T}$ with a given tangent space, i.e. the elements of $\mathcal{C}^{\Phi}_{\Theta}$. We do this in the next section.

3.2. *Theorems*

Let Θ be a complete subsystem of Φ and W^{Θ} its Weyl group. Let $\mathfrak{k}$ and K be respectively the semisimple Lie algebra and the semisimple and simply connected algebraic group with root system Θ, $\mathfrak{d}$ a Cartan subalgebra of $\mathfrak{k}$, $\langle \Theta^{\vee} \rangle$ and $\Lambda(\Theta)$ the coroot and coweight lattices, $Z(\Theta) \doteq \Lambda(\Theta)/\langle \Theta^{\vee} \rangle$ the center of K, D the maximal torus of K defined by $\mathfrak{d}/\langle \Theta^{\vee} \rangle$, $\mathcal{D}$ the toric arrangement defined by Θ on D, and $\mathcal{C}_0(\Theta)$ the set of its 0-dimensional components.

We also consider the *adjoint group* $K_a \doteq K/Z(\Theta)$ and its maximal torus $D_a \doteq D/Z(\Theta) \simeq \mathfrak{d}/\Lambda(\Theta)$. We recall from [13] that K is the universal covering of K_a, and if D' is an algebraic torus with Lie algebra $\mathfrak{d}$, then $D' \simeq \mathfrak{d}/L$ for some lattice $\Lambda(\Theta) \supseteq L \supseteq \langle \Theta^{\vee} \rangle$; so there are natural covering projections $D \twoheadrightarrow D' \twoheadrightarrow D_a$ with kernels respectively $L/\langle \Theta^{\vee} \rangle$ and $\Lambda(\Theta)/L$. Notice that Θ naturally defines an arrangement on each D', and that for $D' = D_a$ the set of 0-dimensional components is $\mathcal{C}_0(\Theta)/Z(\Theta)$. Given a point t of some D' we set

$$\Theta(t) \doteq \{\alpha \in \Theta \mid e^{\alpha}(t) = 1\}.$$

THEOREM 11. *There is a W^Θ-equivariant surjective map*

$$\varphi : \mathcal{C}^\Phi_\Theta \twoheadrightarrow \mathcal{C}_0(\Theta)/Z(\Theta)$$

such that $\ker\varphi \simeq Z(\Phi) \cap Z(\Theta)$ *and* $\Theta_U = \Theta(\varphi(U))$.

PROOF. Let $S(\Theta)$ be the subspace of $\mathfrak{h}$ defined in Section 3.1 and H the corresponding subtorus of T. Then T/H is a torus with Lie algebra $\mathfrak{h}/S(\Theta) \simeq \mathfrak{d}$, so Θ defines an arrangement $\mathcal{D}'$ on $D' \doteq T/H$. The projection $\pi : T \twoheadrightarrow T/H$ induces a bijection between $\mathcal{C}^\Phi_\Theta$ and the set of 0-dimensional components of $\mathcal{D}'$, because $H \in \mathcal{C}^\Phi_\Theta$ and $\Theta_U = \Theta(\pi(U))$ for each $U \in \mathcal{C}^\Phi_\Theta$.

Moreover, the restriction of the projection $d\pi : \mathfrak{h} \twoheadrightarrow \mathfrak{h}/S(\Theta)$ to $\langle\Phi^\vee\rangle$ is simply the map that restricts the coroots of Φ to Θ. Set $R^\Phi(\Theta) \doteq d\pi(\langle\Phi^\vee\rangle)$; then $\Lambda(\Theta) \supseteq R^\Phi(\Theta) \supseteq \langle\Theta^\vee\rangle$ and $D' \simeq \mathfrak{d}/R^\Phi(\Theta)$. Denote by p the projection $\Lambda(\Phi) \twoheadrightarrow \Lambda(\Phi)/\langle\Phi^\vee\rangle$ and embed $\Lambda(\Theta)$ in $\Lambda(\Phi)$ in the natural way. Then the kernel of the covering projection of $D' \twoheadrightarrow D_a$ is isomorphic to

$$\Lambda(\Theta)/R^\Phi(\Theta) \simeq p(\Lambda(\Theta)) \simeq Z(\Phi) \cap Z(\Theta). \qquad \square$$

We set

$$n_\Theta \doteq |Z(\Theta)|/|Z(\Phi) \cap Z(\Theta)|.$$

The following corollary is straightforward from Theorem 11.

COROLLARY 12.

$$|\mathcal{C}^\Phi_\Theta| = n_\Theta^{-1}|\mathcal{C}_0(\Theta)|$$

and then by (8),

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta\in\mathcal{K}_d} n_\Theta^{-1}|\mathcal{C}_0(\Theta)|.$$

Notice that two components U, U' of $\mathcal{T}$ are W-conjugate if and only if the two conditions below are satisfied:

- their tangent spaces are W-conjugate , i.e. there exists $w \in W$ such that $\overline{\Theta_U} = w.\overline{\Theta}_{U'}$;
- U and $w.U'$ are $W^{\overline{\Theta_U}}$-conjugate.

Then the action of W on $\mathcal{C}(\Phi)$ is described by the following remark.

REMARK 13. (i) By Theorem 11, φ induces a surjective map $\overline{\varphi}$ from the set of W^Θ-orbits of $\mathcal{C}^\Phi_\Theta$ to the set of $W^\Theta \times Z(\Theta)$-orbits of $\mathcal{C}_0(\Theta)$, described by Theorem 3.

(ii) In particular, if Θ is irreducible, let Γ^Θ be its affine Dynkin diagram, Q^Θ the set of Aut(Γ)-orbits of its vertices, Γ^Θ_p the diagram that we get from Γ^Θ by removing the vertex p, and Θ_p the associated root system. Then there is a surjective map

$$\widehat{\varphi} : \mathcal{C}^\Phi_\Theta \twoheadrightarrow Q^\Theta$$

such that if $\widehat{\varphi}(U) = q$ and p is a representative of q, then $\Theta_U \simeq \Theta_p$.

3.3. *Examples*

CASE F_4. We have $Z(\Phi) = \{1\}$, thus $n_\Theta = |Z(\Theta)|$. Therefore in this case n_Θ does not depend on the conjugacy class, but only on the isomorphism class of Θ.

We say that a subspace S of $\mathcal{H}$ (respectively a component U of $\mathcal{T}$) is of a given type if the corresponding subsystem Θ_S (respectively Θ_U) is of that type. Then by [23, Tab. C.9] and Corollary 12 there are:

1. one subspace of type A_0, tangent to one component of the same type (the whole spaces);
2. 24 subspaces of type A_1, each tangent to one component of the same type;
3. 72 subspaces of type $\mathsf{A}_1 \times \mathsf{A}_1$, each tangent to one component of the same type;
4. 32 subspaces of type A_2, each tangent to one component of the same type;
5. 18 subspaces of type B_2, each tangent to one component of the same type and one component of type $\mathsf{A}_1 \times \mathsf{A}_1$;
6. 12 subspaces of type C_3, each tangent to one component of the same type and three of type $\mathsf{A}_2 \times \mathsf{A}_1$;
7. 12 subspaces of type B_3, each tangent to one component of the same type, one of type A_3 and three of type $\mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1$;
8. 96 subspaces of type $\mathsf{A}_1 \times \mathsf{A}_2$, each tangent to one component of the same type;
9. one subspace of type F_4 (the origin), tangent to: one component of the same type, 12 of type $\mathsf{A}_1 \times \mathsf{C}_3$, 32 of type $\mathsf{A}_2 \times \mathsf{A}_2$, 24 of type $\mathsf{A}_3 \times \mathsf{A}_1$, and 3 of type C_4.

CASE A_{n-1}. It is easily seen that each subsystem Θ of Φ is complete and is a product of irreducible factors $\Theta_1, \dots, \Theta_k$, with Θ_i of type $\mathsf{A}_{\lambda_i - 1}$ for some positive integers λ_i such that $\lambda_1 + \cdots + \lambda_k = n$ and $n - k$ is the rank of Θ. In other words, as is well known, the W-conjugacy classes of subspaces of $\mathcal{H}$ are in bijection with the partitions λ of n, and if a subspace has dimension d then the corresponding partition has length $|\lambda| \doteq k$ equal to $d + 1$. The number of subspaces of the partition λ is easily seen to be equal to $n!/b_\lambda$, where b_i is the number of λ_j that are equal to i and $b_\lambda \doteq \prod i!^{b_i} b_i!$ (see [23, 6.72]). Now let g_λ be the greatest common divisor of $\lambda_1, \dots, \lambda_k$. By Example 4 in Section 2.2 we know that $|Z(\Theta)| = \lambda_1 \dots \lambda_k = |\mathcal{C}_0(\Theta)|$ and $|Z(\Phi) \cap Z(\Theta)| = g_\lambda$. Then by Corollary 12, $|\mathcal{C}_\Theta^\Phi| = g_\lambda$ and

$$|\mathcal{C}_d(\Phi)| = \sum_{|\lambda| = d+1} \frac{n! g_\lambda}{b_\lambda}.$$

This could also be seen directly as follows. We can view T as the subgroup of $(\mathbb{C}^*)^n$ given by the equation $t_1 \dots t_n - 1 = 0$. Then Θ imposes the equations

$$t_1 = \cdots = t_{\lambda_1}, \ \dots, \ t_{\lambda_1 + \cdots + \lambda_{k-1} + 1} = \cdots = t_n.$$

Thus we have the relation $x_1^{\lambda_1} \dots x_k^{\lambda_k} - 1 = 0$. If $g_\lambda = 1$ this polynomial is irreducible, because the vector $(\lambda_1, \dots, \lambda_k)$ can be completed to a basis of the lattice $\mathbb{Z}^k$. If $g_\lambda > 1$ this polynomial has exactly g_λ irreducible factors over $\mathbb{C}$. Thus in every case it defines an affine variety having exactly g_λ irreducible components, which are precisely the elements of $\mathcal{C}_\Theta^\Phi$.

4. Topological invariants

4.1. *Theorems*

Let R be the complement in T of the union of all the hypersurfaces of the toric arrangement $\mathcal{T}$. In this section we prove that the Euler characteristic of R, denoted by χ_Φ, is equal to $(-1)^n|W|$. This may also be seen as a consequence of [4, Prop. 5.3]. We also give a formula for the Poincaré polynomial of R, denoted by $P_\Phi(q)$.

Let $d_1, \ldots, d_n$ be the *degrees* of W, i.e. the degrees of the generators of the ring of W-invariant regular functions on $\mathfrak{h}$. It is well known that $d_1 \ldots d_n = |W|$. Moreover, by [2], $\mathcal{B}(\Phi) \doteq (d_1 - 1) \ldots (d_n - 1)$ is equal to the leading coefficient of the Poincaré polynomial of the complement of $\mathcal{H}$ in $\mathfrak{h}$, and hence to the number of *unbroken bases* of Φ, because by [21] they give a basis for the n-th cohomology space.

The cohomology of R can be expressed as a direct sum of contributions given by the components of $\mathcal{T}$ (see for example [8, Th. 4.2] or [10, 15.1.5]). In terms of the Poincaré polynomial this expression is:

THEOREM 14.

$$P_\Phi(q) = \sum_{U \in \mathcal{C}(\Phi)} \mathcal{B}(\Theta_U)(q+1)^{d(U)} q^{n-d(U)}$$

where $d(U)$ is the dimension of the component U.

Now we use this expression to compute χ_Φ.

LEMMA 15.

$$\chi_\Phi = (-1)^n \sum_{p=0}^{n} \frac{|W|}{|W_p|} \mathcal{B}(\Phi_p).$$

PROOF. We have

$$\chi_\Phi = P_\Phi(-1) = (-1)^n \sum_{t \in \mathcal{C}_0(\Phi)} \mathcal{B}(\Phi(t)) \tag{9}$$

because the contributions of all components of positive dimension vanish at -1. Obviously isomorphic subsystems have the same degrees, so Theorem 1 yields the statement. □

THEOREM 16.

$$\chi_\Phi = (-1)^n |W|.$$

PROOF. By the previous lemma we must prove that

$$\sum_{p=0}^{n} \frac{\mathcal{B}(\Phi_p)}{|W_p|} = 1.$$

If we write $d_1^p, \dots, d_n^p$ for the degrees of W_p, the previous identity becomes

$$\sum_{p=0}^{n} \frac{(d_1^p - 1)\dots(d_n^p - 1)}{d_1^p \dots d_n^p} = 1.$$

This identity has been proved in [9], and later with different methods in [11]. □

Notice that W acts on R and hence on the cohomology of R. So we can consider the *equivariant Euler characteristic* of R, that is, for each $w \in W$,

$$\widetilde{\chi}_\Phi(w) \doteq \sum_{i=0}^{n} (-1)^i \operatorname{Tr}(w, H^i(R, \mathbb{C})).$$

Let ϱ_W be the character of the regular representation of W. From Theorem 16 we get

COROLLARY 17.

$$\widetilde{\chi}_\Phi = (-1)^n \varrho_W.$$

PROOF. Since W is finite and acts freely on R, it is well known that $\widetilde{\chi}_\Phi = k\varrho_W$ for some $k \in \mathbb{Z}$. Then to compute k we just have to look at $\widetilde{\chi}_\Phi(1_W) = \chi_\Phi$. □

Finally, we give a formula for $P_\Phi(q)$ that, together with the above mentioned results in [23], allows its explicit computation.

THEOREM 18.

$$P_\Phi(q) = \sum_{d=0}^{n} (q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |W^\Theta|.$$

PROOF. By formula (8) we can restate Theorem 14 as

$$P_\Phi(q) = \sum_{d=0}^{n} (q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} \sum_{U \in \mathcal{C}_\Theta^\Phi} \mathcal{B}(\Theta_U).$$

Moreover, by Theorem 11 and Corollary 12 we get

$$\sum_{U \in \mathcal{C}_\Theta^\Phi} \mathcal{B}(\Theta_U) = n_\Theta^{-1} \sum_{t \in \mathcal{C}_0(\Theta)} \mathcal{B}(\Theta(t)).$$

Finally, the claim follows from formula (9) and Theorem 16 applied to Θ:

$$\sum_{t \in \mathcal{C}_0(\Theta)} \mathcal{B}(\Theta(t)) = (-1)^d \chi_\Theta = |W^\Theta|. \qquad \square$$

4.2. *Examples*

CASE F_4. In Section 3.3 we have given the list of all possible types of complete subsystems, together with their multiplicities. So we just have to compute the coefficient $n_\Theta^{-1}|W^\Theta|$ for each type. This is equal to:

- 1 for types 1, 2 and 3,
- 2 for types 4 and 8,
- 4 for type 5,
- 24 for types 6 and 7,
- 1152 for type 9.

Thus

$$P_\Phi(q) = 2153q^4 + 1260q^3 + 286q^2 + 28q + 1.$$

CASE A_{n-1}. By Section 3.3, $n_\Theta^{-1} = g_\lambda/\lambda_1 \dots \lambda_k$ and $|W^\Theta| = \lambda_1! \dots \lambda_k!$. Hence by Theorem 17,

$$P_\Phi(q) = \sum_{d=0}^{n} (q+1)^d q^{n-d} \sum_{|\lambda|=d+1} n! b_\lambda^{-1} g_\lambda (\lambda_1 - 1)! \dots (\lambda_k - 1)!.$$

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