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Partial differential equations. — *On a conjecture of J. Serrin*, by HAÏM BREZIS.

Dedicated to the memory of Guido Stampacchia, a beloved mentor and friend

ABSTRACT. — In 1964 J. Serrin proposed the following conjecture. Let $u \in W^{1,1}_{\text{loc}}(\Omega)$ be a weak solution of the second order elliptic equation (1) below, in divergence form, with Hölder continuous coefficients $a_{ij}(x)$; then u is a "classical" solution. We announce a solution of this conjecture assuming only $u \in BV_{\text{loc}}(\Omega)$ and Dini continuous coefficients.

KEY WORDS: Divergence elliptic equations; very weak solutions.

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Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let $u \in W^{1,1}_{\text{loc}}(\Omega)$ be a weak solution of the equation

$$\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = 0 \quad \text{in } \Omega, \tag{1}$$

where the coefficients $a_{ij}(x)$ are bounded measurable and elliptic, i.e.,

$$\lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad x \in \Omega,\ \xi \in \mathbb{R}^N,$$

with $0 < \lambda \leq \Lambda < \infty$. A weak solution $u \in W^{1,1}_{\text{loc}}(\Omega)$ satisfies, by definition,

$$\sum_{i,j} \int a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \zeta}{\partial x_j} = 0 \quad \forall \zeta \in C^1_c(\Omega) \tag{2}$$

where the subscript c indicates compact support.

A celebrated result of E. De Giorgi [4] asserts that if u is a weak solution of (1) and moreover $u \in H^1_{\text{loc}}(\Omega)$, then u is locally Hölder continuous, and in particular $u \in L^\infty_{\text{loc}}(\Omega)$ (see also [13]). Subsequently J. Serrin produced in [11] a striking example showing that the assumption $u \in H^1_{\text{loc}}(\Omega)$ is essential; more precisely, for every p, $1 < p < 2$, and all $N \geq 2$, he constructed an equation of the form (1) which has a solution $u \in W^{1,p}(\Omega)$ and $u \notin L^\infty_{\text{loc}}(\Omega)$. J. Serrin conjectured in [11] that if the coefficients a_{ij} are locally Hölder continuous, then any weak solution $u \in W^{1,1}_{\text{loc}}(\Omega)$ of

(1) must be a "classical" solution, i.e., $u \in H^1_{\text{loc}}(\Omega)$. Serrin's conjecture was confirmed by R. A. Hager and J. Ross [7] provided u is a weak solution of class $W^{1,p}(\Omega)$ for some p with $1 < p < 2$.

We announce here the solution of Serrin's conjecture in full generality, starting with $u \in W^{1,1}_{\text{loc}}(\Omega)$, or even with $u \in BV_{\text{loc}}(\Omega)$, i.e., $u \in L^1_{\text{loc}}(\Omega)$ and its derivatives (in the sense of distributions) are measures.

Our first result is an improvement of the theorem of Hager and Ross: instead of $a_{ij} \in C^{0,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$, we assume only $a_{ij} \in C^0(\bar{\Omega})$.

THEOREM 1. *Assume $a_{ij} \in C^0(\bar{\Omega})$ and $u \in W^{1,p}(\Omega)$ for some $p > 1$. If u is a weak solution of* (1)*, then $u \in W^{1,q}_{\text{loc}}(\Omega)$ for every $q < \infty$. Moreover,*

$$\|u\|_{W^{1,q}(\omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

for every $\omega \subset\subset \Omega$, where C depends only on N, λ, Λ, p, q, ω, Ω, and the modulus of continuity of a_{ij} on $\bar{\Omega}$.

OPEN PROBLEM. We do not know whether the conclusion of Theorem 1 holds in the two limiting cases: $p = 1$ and/or $q = \infty$. (The answer to both questions is positive if the coefficients a_{ij} are Dini continuous; see Theorem 2 below.)

We now turn to Serrin's conjecture. Here we assume that the coefficients a_{ij} are Dini continuous in $\bar{\Omega}$, i.e., $a_{ij} \in C^0(\bar{\Omega})$ and

$$A(r) = \sum_{i,j} \sup_{x,y\in\Omega,\, |x-y|<r} |a_{ij}(x) - a_{ij}(y)|, \quad r > 0, \tag{3}$$

satisfies

$$\int_0^1 \frac{A(r)}{r}\, dr < \infty. \tag{4}$$

THEOREM 2. *Assume that the coefficients a_{ij} are Dini continuous in $\bar{\Omega}$, and let $u \in BV(\Omega)$ be a weak solution of* (1)*. Then $u \in H^1_{\text{loc}}(\Omega)$. Moreover,*

$$\|u\|_{H^1(\omega)} \leq C\|u\|_{BV(\Omega)} \tag{5}$$

for every $\omega \subset\subset \Omega$, where C depends only on N, λ, Λ, ω, Ω, and the modulus of continuity of a_{ij} on $\bar{\Omega}$.

REMARK 1. Surprisingly, the constant C in (5) depends only on the modulus of continuity of a_{ij} in $\bar{\Omega}$, and *not* on the Dini modulus of continuity of a_{ij} in $\bar{\Omega}$. This suggests that Serrin's conjecture might be true assuming only continuity of a_{ij} in $\bar{\Omega}$ (this corresponds to the open problem mentioned above with $p = 1$).

REMARK 2. Using Lemma 2 below we may assert that, under the assumptions of Theorem 2, $u \in C^1(\Omega)$. If the coefficients a_{ij} belong to $C^{0,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, one can further improve the conclusion of Theorem 2, namely $u \in C^{1,\alpha}(\bar{\omega})$ for every $\omega \subset\subset \Omega$. This is a consequence of the standard Schauder regularity theory for elliptic equations in divergence form with $C^{0,\alpha}$ coefficients (see e.g. [10, Theorem 5.5.3(b)], [6, Theorem 3.7], [5, Theorem 3.5], or [3, Theorem 2.6 in Chapter 9]). All the above results extend to elliptic systems.

The proofs use a duality technique in conjunction with a bootstrap argument and the following tools:

a) Standard L^p-regularity theory for elliptic equations in divergence form:

LEMMA 1 (see e.g. [10, Theorem 5.5.3(a)], or [3, Theorem 2.2 in Chapter 10]). *Assume $a_{ij} \in C^0(\bar{\Omega})$ and $u \in H^1(\Omega)$ is a weak solution of*

$$\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = \sum_j \frac{\partial}{\partial x_j} f_j \tag{6}$$

with $f_j \in L^p(\Omega)$ for all j, and $p \in (2,\infty)$. Then $u \in W^{1,p}_{\text{loc}}(\Omega)$, and for $\omega \subset\subset \Omega$,

$$\|u\|_{W^{1,p}(\omega)} \le C\Big(\|u\|_{H^1(\Omega)} + \sum_j \|f_j\|_{L^p(\Omega)}\Big)$$

where C depends on N, λ, Λ, p, ω, Ω, and the modulus of continuity of a_{ij}.

b) Schauder regularity theory for elliptic equations in divergence form with Dini continuous coefficients:

LEMMA 2. *Assume $a_{ij} \in C^0(\bar{\Omega})$ satisfy* (3), (4), *and let $u \in H^1(\Omega)$ be a weak solution of* (6) *with $f_j \in C^\infty_c(\Omega)$ for all j. Then $u \in C^1(\Omega)$.*

The conclusion of Lemma 2 comes with an estimate of the Dini modulus of continuity of Du involving the Dini modulus of continuity of a_{ij}. However, we do *not* need such an estimate—we use only the qualitative form of Lemma 2; this explains Remark 1. It is not easy to find an early reference for Lemma 2. According to the experts (I am quoting M. Giaquinta), it was common knowledge in Pisa in the late 60's—the proof being based on Campanato's approach to Schauder estimates (as presented in [6] or [3]), combined with a result of S. Spanne (Corollary 1 in [12]). A complete proof may be found e.g. in [9, Theorem 5.1]. Y. Li ([8]) has obtained a similar conclusion (also valid for systems) under weaker assumptions on the coefficients a_{ij}.

The detailed proofs of Theorems 1 and 2 are presented as an Appendix in [1]. Theorem 2 is used in the paper of A. Ancona [1], who solved some open problems raised in [2], and called my attention to Serrin's conjecture.

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REFERENCES

[1] A. ANCONA, *Elliptic operators, conormal derivatives and positive parts of functions*. To appear.

[2] H. BREZIS - A. PONCE, *Kato's inequality up to the boundary*. Comm. Contemp. Math., to appear.

[3] Y.-Z. CHEN - L.-C WU, *Second Order Elliptic Equations and Elliptic Systems*. Transl. Math. Monogr. 174, Amer. Math. Soc., 1998.

[4] E. DE GIORGI, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*. Mem. Accad. Sci. Torino 3 (1957), 25–43.

[5] M. GIAQUINTA, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*. Birkhäuser, 1978.

[6] E. GIUSTI, *Equazioni ellitiche del secondo ordine*. Pitagora Editrice, Bologna, 1978.

[7] R. A. HAGER - J. ROSS, *A regularity theorem for linear second order elliptic divergence equations*. Ann. Scuola Norm. Sup. Pisa 23 (1971), 283–290.

[8] Y. LI. In preparation.

[9] G. LIEBERMAN, *Hölder continuity of the gradient of solutions of uniformly parabolic equations with conormal boundary conditions*. Ann. Mat. Pura Appl. 148 (1987), 77–99.

[10] C. B. MORREY, *Multiple Integrals in the Calculus of Variations*. Springer, 1966.

[11] J. SERRIN, *Pathological solutions of elliptic differential equations*. Ann. Scuola Norm. Sup. Pisa 18 (1964), 385–387.

[12] S. SPANNE, *Some function spaces defined using the mean oscillation over cubes*. Ann. Scuola Norm. Sup. Pisa 19 (1965), 593–608.

[13] G. STAMPACCHIA, *Contributi alla regolarizzazione delle soluzioni dei problemi al contorno per equazioni del secondo ordine ellittiche*. Ann. Scuola Norm. Sup. Pisa 12 (1958), 223–245.

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