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Partial differential equations. — *A remark on the semiclassical Fefferman–Phong inequality for certain systems of PDEs*, by ALBERTO PARMEGGIANI, communicated on 9 January 2009

ABSTRACT. — We give here an extension of the semiclassical version of the Fefferman–Phong inequality to certain $N \times N$ systems of PDEs. In addition, we give an example of system for which the semiclassical Fefferman–Phong inequality cannot hold.

KEY WORDS: Lower bounds; Systems of partial differential operators; Fefferman–Phong inequality; Semiclassical analysis

MATHEMATICS SUBJECT CLASSIFICATION: Primary 35S05; Secondary 35B45; 35A30

Dedicated to Professor Sandro Graffi on the occasion of his 65th birthday

1. INTRODUCTION

The semiclassical version of the Fefferman–Phong inequality for a system of second order partial differential operators can be stated as follows: *given an $N \times N$ second order system of PDEs*

$$p(x,\xi;h) = \sum_{j,k=1}^{n} A_{jk}(x)\xi_j\xi_k + \sum_{j=1}^{n} \xi_j B_j(x;h) + C(x;h) = p(x,\xi;h)^* \geq 0,$$

$(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, $h \in (0,1]$ *(the semiclassical parameter), there exists* $h_0 \in (0,1]$ *and an absolute constant* $C > 0$ *(independent of* h*) such that*

$$(1.1) \qquad (p^{\mathrm{w}}(x,hD;h)u,u) \geq -Ch^2\|u\|_0^2, \quad \forall u \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N), \ \forall h \in (0,h_0].$$

Here $p^{\mathrm{w}}(x,hD;h)$ denotes the semiclassical Weyl-quantization of the symbol $p(x,\xi;h)$ (see, for instance, Dimassi and Sjöstrand [1]), $(\cdot,\cdot)$ stands for the $L^2(\mathbb{R}^n;\mathbb{C}^N)$ inner product, and $\|\cdot\|_0$ for the corresponding norm. The proof of the Fefferman–Phong inequality for systems of PDEs given in Parmeggiani [3] (see also Parmeggiani [4]; see also Sung [6] for a proof in the case $n = 1$) carries over, as mentioned in the final remark of that paper, to the semiclassical case. In this paper, besides making that remark more precise in Section 2 (see Theorem 2.1 below), I shall consider in Section 3 (see Theorem 3.1 below) the validity of inequality (1.1), and its sharpness, in the case of certain systems of PDEs that seem to be natural (e.g. systems related to Dirac operators). In the final Section 4 I shall test the result of Section 3 in the case of "Dirac-squared-type" systems (see (4.4) below), for which the constant appearing in (1.1) may be computed explicitly.

Remark that, in contrast to the scalar case, the Fefferman–Phong inequality may be false for Hermitian nonnegative systems (see Parmeggiani [2] and the reference to Brummelhuis' work therein), and therefore the problem of finding necessary and/or sufficient conditions in order that the Fefferman–Phong inequality for systems hold, both in the semiclassical and usual ($h = 1$) case, is a non-trivial and basic problem.

The method used in Sections 3 and 4 for obtaining the inequality is a natural "completion-of-squares procedure", that is, I will write

$$p^{\mathrm{w}}(x, hD; h) = \sum_{j=1}^{n} L_j^{\mathrm{w}}(x, hD; h)^2 + C_0(x; h),$$

for suitable first-order systems $L_j^{\mathrm{w}}(x, hD; h) = L_j^{\mathrm{w}}(x, hD; h)^*$, so that one gets

$$(p^{\mathrm{w}}(x, hD; h)u, u) = \sum_{j=1}^{n} \|L_j^{\mathrm{w}}(x, hD; h)u\|_0^2 + (C_0(x; h)u, u),$$

and the point is to control from below the term $(C_0 u, u)$, "throwing away" the nonnegative contribution of the terms $\|L_j^{\mathrm{w}}(x, hD; h)u\|_0^2$. I will show that such a "wasteful" procedure is in fact in some cases optimal (for small h), by showing examples in which $\sum_j \|L_j^{\mathrm{w}}(x, hD; h)u_h\|_0^2 = O(h^3)$, while $(C_0(x; h)u_h, u_h) \geq -Ch^2$, for suitable Schwartz functions u_h with $\|u_h\|_0 = 1$.

It is also important to notice that in the completion-of-squares procedure, the resulting term $C_0(x; h)$ is obviously nonnegative when $N = 1$ (i.e. in the scalar case), whereas in the genuinely matrix-valued case ($N \geq 2$) this might no longer be the case. This observation provides an example (see Remark 4.3) of a second order system for which the semiclassical Fefferman–Phong inequality cannot hold.

In the sequel M_N will denote the set of $N \times N$ complex matrices (possibly dependent on the parameter h), and $S^2(\mathbb{R}^n \times \mathbb{R}^n; \mathsf{M}_N) := S^2(\mathbb{R}^n \times \mathbb{R}^n) \otimes \mathsf{M}_N$ will denote the set of matrix-valued usual $S^2_{1,0}$ class of second order symbols. Finally, given $a, b \gtrless 0$, I will write $a \sim b$ when $C^{-1}b \leq a \leq Cb$ for some absolute constant $C > 0$ (independent of the main parameters).

2. A first semiclassical Fefferman–Phong inequality

Following [3] and [4], I consider the $N \times N$ system of second order PDEs in $\mathbb{R}^n$ given by

$$(2.1) \quad p(x, \xi; h) = A(x)|\xi|^2 + \sum_{j=1}^{n} \xi_j B_j(x; h) + C(x; h) = p(x, \xi; h)^* \geq -c_0 h^2 I_N,$$

with $c_0 > 0$, for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $h \in (0, 1]$, where

$$A,\ B_j(\cdot; h),\ C(\cdot; h) \in C^\infty(\mathbb{R}^n; \mathsf{M}_N),\ 1 \le j \le n,\ \forall h \in (0, 1], \tag{2.2}$$

and where for any given $\alpha \in \mathbb{Z}_+^n$ (with $\mathbb{Z}_+ = \{0, 1, \ldots\}$) there exists $C_\alpha > 0$ such that

$$\begin{aligned} \|\partial_x^\alpha A\|_{L^\infty(\mathbb{R}^n;\mathsf{M}_N)} + h^{-1} \sum_{j=1}^n \|\partial_x^\alpha B_j(\cdot; h)\|_{L^\infty(\mathbb{R}^n;\mathsf{M}_N)} \\ + h^{-2}\|\partial_x^\alpha C(\cdot; h)\|_{L^\infty(\mathbb{R}^n;\mathsf{M}_N)} \le C_\alpha, \quad \forall h \in (0, 1]. \end{aligned} \tag{2.3}$$

In other words, conditions (2.1) and (2.3), respectively, are rephrased as

$$h^{-2} p(x, h\xi; h) \ge -c_0 I_N, \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, \quad \forall h \in (0, 1], \tag{2.4}$$

$$h^{-2} p(x, h\xi; h) \in S^2(\mathbb{R}^n \times \mathbb{R}^n; \mathsf{M}_N), \qquad \textit{uniformly in } h \in (0, 1], \tag{2.5}$$

respectively.

One has the following theorem (which makes precise the final remark in [3] and [4]).

Theorem 2.1. *In the above hypotheses the semiclassical Fefferman–Phong inequality holds for p: there exists a constant $C > 0$ such that*

$$(p^{\mathrm{w}}(x, hD; h)u, u) \ge -Ch^2\|u\|_0^2, \quad \forall u \in \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N),\ \forall h \in (0, 1].$$

Proof. The proof follows immediately from the proof given in [3], upon noting that the semiclassical Weyl-quantization $p^{\mathrm{w}}(x, hD; h)$ is nothing but the usual $(h = 1)$ Weyl-quantization of $p(x, h\xi; h)$, that condition (2.5) ensures that the Fefferman–Phong metric introduced in [3] (see also [4])

$$g_{x,\xi} = H(x)^2|dx|^2 + \frac{|d\xi|^2}{M^2}, \qquad |\xi| \sim M,$$
$$H(x)^{-1} := \max\left\{\frac{1}{M}, \sqrt{\operatorname{Tr} A(x)}\right\},$$

is *independent of* $h \in (0, 1]$, and that all the seminorms considered are bounded *uniformly* in h. Thus one may work in the usual $(h = 1)$ Weyl–Hörmander calculus and Theorem 3.1 of [3] yields the existence of an absolute constant $C > 0$ such that

$$h^{-2}(p^{\mathrm{w}}(x, hD; h)u, u) \ge -C\|u\|_0^2, \quad \forall u \in \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N),$$

for all $h \in (0, 1]$, which concludes the proof. □

Remark 2.2. *Notice that, therefore, the proof of Theorem 2.1 is* not *semiclassical, for it is a reduction to the usual Weyl-quantization of a differential symbol whose seminorms of all orders are bounded uniformly in h.*

Notice that if one considers $U_h : \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)$, the isometry of $L^2(\mathbb{R}^n; \mathbb{C}^N)$ (also automorphism of $\mathscr{S}'(\mathbb{R}^n; \mathbb{C}^N)$) such that

$$(U_h u)(x) = h^{-n/4} u(h^{-1/2}x),$$

then

$$U_h^{-1} p^{\mathrm{w}}(x, hD; h) U_h = p^{\mathrm{w}}(h^{1/2}x, h^{1/2}D; h),$$

whence, since

$$(p^{\mathrm{w}}(x, hD; h)u, u) = (p^{\mathrm{w}}(h^{1/2}x, h^{1/2}D; h) U_h^{-1} u, U_h^{-1} u),$$

Theorem 2.1 gives a semiclassical Fefferman–Phong inequality for systems of PDEs belonging to pseudodifferential classes whose associated Hörmander metric (and weights) depend on the semiclassical parameter.

As an example of system for which Theorem 2.1 applies, one may take

$$p(x, \xi; h) = A(x)|\xi|^2 + h \sum_{j=1}^{n} \xi_j \hat{B}_j(x) + h^2 \hat{C}(x) = p(x, \xi; h)^* \geq -c_0 h^2 I_N,$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $h \in (0, 1]$, for a constant $c_0 > 0$ and smooth matrices $A, \hat{B}_j$ $(1 \leq j \leq n)$ and $\hat{C}$ such that for any given $\alpha \in \mathbb{Z}_+^n$ there exists $C_\alpha > 0$ for which

$$\|\partial_x^\alpha A\|_{L^\infty(\mathbb{R}^n; \mathsf{M}_N)} + \sum_{j=1}^{n} \|\partial_x^\alpha \hat{B}_j\|_{L^\infty(\mathbb{R}^n; \mathsf{M}_N)} + \|\partial_x^\alpha \hat{C}\|_{L^\infty(\mathbb{R}^n; \mathsf{M}_N)} \leq C_\alpha.$$

Of course, one would like to consider also other classes of systems of PDEs for which the semiclassical version of the Fefferman–Phong inequality holds. In the next section I will show that when $A(x)$ is uniformly elliptic the semiclassical Fefferman–Phong inequality holds true for classes of second order systems of PDEs for which $h^{-1}\|B_j(\cdot; h)\|_{L^\infty}$ and $h^{-2}\|C(\cdot; h)\|_{L^\infty}$ are not necessarily bounded.

3. The inequality in case the matrix-coefficient A is elliptic

Let us consider the $N \times N$ system of second order PDEs

$$p(x, \xi; h) = A(x)|\xi|^2 + \sum_{j=1}^{n} \xi_j B_j(x; h) + C(x; h) = p(x, \xi; h)^* \geq 0, \tag{3.1}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $h \in (0, 1]$, where

$$A, B_j(\cdot; h), C(\cdot; h) \in C^\infty(\mathbb{R}^n; \mathsf{M}_N), \quad j = 1, \dots, n, \ \forall h \in (0, 1], \tag{3.2}$$

and for any given $\alpha \in \mathbb{Z}_+^n$ there exists $C_\alpha > 0$ such that

$$\|\partial_x^\alpha A\|_{L^\infty(\mathbb{R}^n;\mathsf{M}_N)} + \sum_{j=1}^n \|\partial_x^\alpha B_j(\cdot; h)\|_{L^\infty(\mathbb{R}^n;\mathsf{M}_N)} + \|\partial_x^\alpha C(\cdot; h)\|_{L^\infty(\mathbb{R}^n;\mathsf{M}_N)} \le C_\alpha, \tag{3.3}$$

for all $h \in (0, 1]$. Suppose furthermore that the matrix-coefficient A satisfies the uniform-ellipticity condition: *there exists* $c > 0$ *such that*

$$c^{-1} I_N \le A(x) \le c I_N, \quad \forall x \in \mathbb{R}^n. \tag{3.4}$$

It is readily seen (see, for instance [3]) that (3.1) yields that all the matrices A, B_j and C are Hermitian, and that $A, C \ge 0$.

Remark that hypotheses (3.3) and (3.4) yield that the positive square root

$$A(x)^{1/2} := \frac{1}{2\pi i} \int_\gamma \lambda^{1/2} (\lambda - A(x))^{-1} \, d\lambda, \tag{3.5}$$

$\gamma \subset \{\lambda \in \mathbb{C};\ \operatorname{Re} \lambda > 0\}$ being a counterclockwise-oriented path encircling the interval $[c^{-1}, c]$, is smooth and bounded along with its derivatives of all orders (with bounds depending only on c and the C_α).

One has the following result.

THEOREM 3.1. *Let p be as in (3.1) and satisfy hypotheses (3.2)–(3.4). Define for $j = 1, \dots, n$ the smooth (in x) Hermitian matrices*

$$B_{A,j}(x; h) := A(x)^{-1/2} B_j(x; h) A(x)^{-1/2}, \tag{3.6}$$

and

$$\Lambda_j(x; h) := \frac{N}{4} B_{A,j}(x; h)^2 - \frac{\operatorname{Tr}(B_{A,j}(x; h))}{2} B_{A,j}(x; h) + \frac{\operatorname{Tr}(B_{A,j}(x; h)^2)}{4} I_N. \tag{3.7}$$

Suppose further that there exists $c_1 > 0$ such that (in the sense of Hermitian matrices)

$$\sum_{j=1}^n \Lambda_j(x; h) \le c_1 h^2 I_N, \quad \forall x \in \mathbb{R}^n, \ \forall h \in (0, 1]. \tag{3.8}$$

(Of course, when $N = 1$ the condition is trivially satisfied, for $\Lambda_j(x; h) = 0$ for all x, h, and $j = 1, \dots, n$.) Then there exists $C > 0$, depending only on N, on a finite number of seminorms of $A^{1/2}$, A, the B_j and C, and on the constant c_1 of (3.8), such that

$$(p^{\mathrm{w}}(x, hD; h)u, u) \ge -C h^2 \|u\|_0^2, \quad \forall u \in \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N), \ \forall h \in (0, 1]. \tag{3.9}$$

PROOF. I start by writing

$$p(x,\xi;h) = A(x)^{1/2} p_A(x,\xi;h) A(x)^{1/2}, \tag{3.10}$$

where

$$p_A(x,\xi;h) := |\xi|^2 I_N + \sum_{j=1}^n \xi_j B_{A,j}(x;h) + C_A(x;h),$$

with $C_A(x;h) := A(x)^{-1/2} C(x;h) A(x)^{-1/2}$. It follows from (3.2)–(3.4) that the matrices $B_{A,j}$ and C_A are all smooth (in x) and bounded, along with their x-derivatives to all orders, uniformly in x and h. Remark also, as is clear, that

$$p_A(x,\xi;h) = p_A(x,\xi;h)^* \geq 0, \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \quad \forall h \in (0,1].$$

I now "complete the squares" in p_A and write

$$p_A(x,\xi;h) = \sum_{j=1}^n L_j(x,\xi;h)^2 + \Big(C_A(x;h) - \frac{1}{4} \sum_{j=1}^n B_{A,j}(x;h)^2 \Big) \geq 0, \tag{3.11}$$

where

$$L_j(x,\xi;h) := \xi_j I_N + \frac{1}{2} B_{A,j}(x;h) = L_j(x,\xi;h)^*, \quad j = 1,\dots,n.$$

Hence, in particular, $L_j^{\mathrm{w}}(x,hD;h) = L_j^{\mathrm{w}}(x,hD;h)^*$, $j = 1,\dots,n$. Writing $a \sharp_h b$ for the composition-law of the semiclassical Weyl–Hörmander calculus of symbols a and b, one has in the first place that, since

$$\frac{\partial L_j}{\partial \xi_k} = \delta_{jk} I_N, \quad 1 \leq j,k \leq n, \tag{3.12}$$

$$L_j \sharp_h L_j = L_j^2 - \frac{i}{2} h \{L_j, L_j\} = L_j^2 - \frac{i}{2} h \sum_{k=1}^n \left[\frac{\partial L_j}{\partial \xi_k}, \frac{\partial L_j}{\partial x_k} \right] = L_j^2, \ 1 \leq j \leq n. \tag{3.13}$$

I next compute

$$A^{1/2} \sharp_h p_A \sharp_h A^{1/2} = A^{1/2} p_A A^{1/2} - \frac{i}{2} h (\{A^{1/2}, p_A\} A^{1/2} + \{A^{1/2} p_A, A^{1/2}\}) + h^2 r_0,$$

where (by virtue of the fact that p_A is a second-order *differential* system) $r_0 = r_0(x;h)$ is smooth in x and for all $\alpha \in \mathbb{Z}_+^n$

$$\sup_{h \in (0,1]} \|\partial_x^\alpha r_0(\cdot;h)\|_{L^\infty(\mathbb{R}^n;\mathsf{M}_N)} < +\infty, \tag{3.14}$$

with bounds depending only on the constants appearing in (3.3) and (3.4). One now has

$$\begin{aligned}\{A^{1/2}p_A, A^{1/2}\} &= \sum_{k=1}^{n}\left(\frac{\partial}{\partial\xi_k}(A^{1/2}p_A)\frac{\partial A^{1/2}}{\partial x_k} - \frac{\partial}{\partial x_k}(A^{1/2}p_A)\frac{\partial A^{1/2}}{\partial\xi_k}\right)\\ &= \sum_{k=1}^{n} A^{1/2}\frac{\partial p_A}{\partial\xi_k}\frac{\partial A^{1/2}}{\partial x_k} = \sum_{k=1}^{n} A^{1/2}\sum_{j=1}^{n}\left(\frac{\partial L_j}{\partial\xi_k}L_j + L_j\frac{\partial L_j}{\partial\xi_k}\right)\frac{\partial A^{1/2}}{\partial x_k}\\ &= 2\sum_{k=1}^{n} A^{1/2}L_k\frac{\partial A^{1/2}}{\partial x_k},\end{aligned}$$

and, by the same token,

$$\{A^{1/2}, p_A\} = -2\sum_{k=1}^{n}\frac{\partial A^{1/2}}{\partial x_k}L_k.$$

Hence

$$\begin{aligned}A^{1/2}\sharp_h p_A\sharp_h A^{1/2} &= p - ih\sum_{k=1}^{n}\left(A^{1/2}L_k\frac{\partial A^{1/2}}{\partial x_k} - \frac{\partial A^{1/2}}{\partial x_k}L_k A^{1/2}\right) + h^2 r_0\\ &= p - ih\sum_{k=1}^{n}\left(A^{1/2}\sharp_h L_k\sharp_h\frac{\partial A^{1/2}}{\partial x_k} - \frac{\partial A^{1/2}}{\partial x_k}\sharp_h L_k\sharp_h A^{1/2}\right) + h^2 r_1,\end{aligned}$$

where $r_1 = r_1(x;h)$ satisfies the same properties of r_0. We therefore get, for any given $u \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$,

$$\begin{aligned}(p^{\mathrm{w}}(x,hD;h)u,u) &= (p_A^{\mathrm{w}}(x,hD;h)A^{1/2}u, A^{1/2}u)\\ &\quad + 2h\,\mathrm{Im}\sum_{j=1}^{n}\left(L_j^{\mathrm{w}}(x,hD;h)A^{1/2}u, \frac{\partial A^{1/2}}{\partial x_j}u\right) + h^2 O(\|u\|_0^2)\end{aligned}$$

(in view of (3.11))

$$\begin{aligned}&= \sum_{j=1}^{n}\|L_j^{\mathrm{w}}(x,hD;h)A^{1/2}u\|_0^2 + \left((C_A(x;h) - \frac{1}{4}\sum_{j=1}^{n}B_{A,j}(x;h)^2)A^{1/2}u, A^{1/2}u\right)\\ &\quad + 2h\,\mathrm{Im}\sum_{j=1}^{n}\left(L_j^{\mathrm{w}}(x,hD;h)A^{1/2}u, \frac{\partial A^{1/2}}{\partial x_j}u\right) + h^2 O(\|u\|_0^2)\end{aligned}$$

(by using the Cauchy–Schwarz inequality in the second-to-last term)

$$\begin{aligned}&\geq \frac{1}{2}\sum_{j=1}^{n}\|L_j^{\mathrm{w}}(x,hD;h)A^{1/2}u\|_0^2 - 2h^2\sum_{j=1}^{n}\left\|\frac{\partial A^{1/2}}{\partial x_j}u\right\|_0^2\\ &\quad + (C_0(x;h)A^{1/2}u, A^{1/2}u) + h^2 O(\|u\|_0^2)\\ &\geq -2h^2\sum_{j=1}^{n}\left\|\frac{\partial A^{1/2}}{\partial x_j}\right\|_{L^\infty}^2\|u\|_0^2 + h^2 O(\|u\|_0^2) + (C_0(x;h)A^{1/2}u, A^{1/2}u),\end{aligned}$$

where I have put $C_0(x;h) := C_A(x;h) - \frac{1}{4}\sum_{j=1}^{n} B_{A,j}(x;h)^2$. The problem is now to control from below this latter term, and it is here that we use hypotheses (3.1) and (3.8) as follows.

LEMMA 3.2. *Hypothesis (3.1) yields*

$$C_0(x;h) \geq -\frac{1}{N}\sum_{j=1}^{n} \Lambda_j(x;h), \quad \forall x \in \mathbb{R}^n, \quad \forall h \in (0,1], \tag{3.15}$$

as Hermitian matrices.

PROOF OF THE LEMMA. Fixed any $x \in \mathbb{R}^n$ and $h \in (0,1]$, I write (with repetitions according to multiplicity)

$$\mathrm{Spec}(B_{A,j}(x;h)) := \{\lambda_k^{(j)}(x;h);\ k = 1,\dots,N\} \subset \mathbb{R}, \quad j = 1,\dots,n.$$

I consider next for $k = 1,\dots,N$

$$\xi^{(k)} = \xi^{(k)}(x;h) := \left(-\frac{1}{2}\lambda_k^{(1)}(x;h), -\frac{1}{2}\lambda_k^{(2)}(x;h), \dots, -\frac{1}{2}\lambda_k^{(n)}(x;h)\right),$$

and

$$p_A(x,\xi^{(k)};h) = \sum_{j=1}^{n}\left(\frac{1}{2}B_{A,j}(x;h) - \frac{1}{2}\lambda_k^{(j)}(x;h)I_N\right)^2 + C_0(x;h), \quad k = 1,\dots,n,$$

which is *nonegative* by hypothesis (3.1), that is, in the sense of Hermitian matrices,

$$C_0(x;h) \geq -\sum_{j=1}^{n}\left(\frac{1}{4}B_{A,j}(x;h)^2 - \frac{1}{2}\lambda_k^{(j)}(x;h)B_{A,j}(x;h) + \frac{1}{4}\lambda_k^{(j)}(x;h)^2 I_N\right).$$

Hence averaging over $k = 1,\dots,N$ gives, recalling the definition of the Hermitian matrices $\Lambda_j(x;h)$ given in (3.7),

$$C_0(x;h) \geq -\frac{1}{N}\sum_{j=1}^{n}\Lambda_j(x;h),$$

which concludes the proof of the lemma. □

From the lemma and hypothesis (3.8) it therefore follows that

$$(p^{\mathrm{w}}(x,hD;h)u,u) \geq -2h^2\sum_{j=1}^{n}\left\|\frac{\partial A^{1/2}}{\partial x_j}\right\|_{L^\infty}^2 \|u\|_0^2 + h^2 O(\|u\|_0^2) - \frac{c_1}{N}h^2\|A^{1/2}\|_{L^\infty}^2\|u\|_0^2,$$

for all $u \in \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)$ and all $h \in (0, 1]$, which concludes the proof of the theorem. □

REMARK 3.3. *It is interesting to note that the matrices Λ_j, $1 \leq j \leq n$, have eigenvalues given by*

$$\sum_{k \neq k'; k=1}^{N} \left(\frac{\lambda_{k'}^{(j)} - \lambda_k^{(j)}}{2} \right)^2, \quad k' = 1, \ldots, N.$$

Hence hypothesis (3.8) may be thought of as requiring that the extent to which the matrices $B_{A,j}$ fail to be scalar multiples of the identity I_N be $O(h)$. In particular, $B_{A,j}$ may be given in the form

$$B_{A,j}(x; h) = b_j(x) I_N + h\tilde{B}_{A,j}(x; h),$$

for some smooth real (bounded, along with all the derivatives) functions b_j and some smooth Hermitian matrices $\tilde{B}_{A,j}(\cdot; h)$ $(j = 1, \ldots, n)$ such that $\|\partial_x^\alpha \tilde{B}_{A,j}(\cdot; h)\|_{L^\infty} < +\infty$ for all h and all $\alpha \in \mathbb{Z}_+^n$.

Notice that when $b_j = 0$ for all $j = 1, \ldots, n$, then $B_{A,j} = O(h)$, and the conclusion of the theorem follows at once, for in this case the condition $p_A(x, \xi; h) \geq 0$ immediately yields $C_A(x; h) \geq 0$, so that (in the sense of Hermitian matrices)

$$C_A(x; h) - \frac{1}{4} \sum_{j=1}^{n} B_{A,j}(x; h)^2 \geq -\frac{1}{4} \sum_{j=1}^{n} B_{A,j}(x; h)^2 \geq -ch^2 I_N,$$

for all $x \in \mathbb{R}^n$ and all $h \in (0, 1]$, for some absolute constant $c > 0$.

The method of proof of Theorem 3.1 gives the following slightly more general result.

THEOREM 3.4. *Consider the $N \times N$ systems of second order PDEs*

$$p(x, \xi; h) = \sum_{j=1}^{n} (\xi_j^2 A_j(x) + \xi_j B_j(x; h)) + C(x; h) = p(x, \xi; h)^* \geq 0, \tag{3.16}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $h \in (0, 1]$, where

$$A_j, B_j(\cdot; h), C(\cdot; h) \in C^\infty(\mathbb{R}^n; \mathsf{M}_N), \quad j = 1, \ldots, n, \ \forall h \in (0, 1], \tag{3.17}$$

and for any given $\alpha \in \mathbb{Z}_+^n$ there exists $C_\alpha > 0$ such that

$$\sum_{j=1}^{n} (\|\partial_x^\alpha A_j\|_{L^\infty} + \|\partial_x^\alpha B_j(\cdot; h)\|_{L^\infty}) + \|\partial_x^\alpha C(\cdot; h)\|_{L^\infty} \leq C_\alpha, \ \forall h \in (0, 1]. \tag{3.18}$$

Suppose furthermore that the matrix-coefficients A_j satisfy the uniform-ellipticity condition: there exists $c > 0$ such that

$$c^{-1} I_N \leq A_j(x) \leq c I_N, \quad \forall x \in \mathbb{R}^n, \ j = 1, \ldots, n. \tag{3.19}$$

Let (as in (3.5)) $A_j^{1/2}$ be the smooth positive square root of A_j. Define as in Theorem 3.1 the matrices $B_{A_j,j} := A_j^{-1/2} B_j A_j^{-1/2}$ and Λ_j (see (3.6) and (3.7)), and suppose that there exists $c_1 > 0$ such that

$$\sum_{j=1}^{n} A_j(x)^{1/2} \Lambda_j(x;h) A_j(x)^{1/2} \le c_1 h^2 I_N, \quad \forall x \in \mathbb{R}^n, \ \forall h \in (0,1]. \tag{3.20}$$

Then there exists $C > 0$, depending only on N, on a finite number of seminorms of the $A_j^{1/2}$, A_j, B_j and C, and on the constant c_1 of (3.20), such that

$$(p^{\mathrm{w}}(x,hD;h)u,u) \ge -Ch^2 \|u\|_0^2, \quad \forall u \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N), \ \forall h \in (0,1]. \tag{3.21}$$

Proof. The proof follows the same lines of that of Theorem 3.1. I write in the first place

$$\begin{aligned} p(x,\xi;h) = & \sum_{j=1}^{n} A_j(x)^{1/2} L_j(x,\xi;h)^2 A_j(x)^{1/2} \\ & + C(x;h) - \frac{1}{4}\sum_{j=1}^{n} A_j(x)^{1/2} B_{A_j,j}(x;h)^2 A_j(x)^{1/2}, \end{aligned}$$

where

$$L_j(x,\xi;h) := \xi_j I_N + \frac{1}{2} B_{A_j,j}(x;h) = L_j(x,\xi;h)^*, \quad 1 \le j \le n.$$

Then, as before,

$$L_j \sharp_h L_j = L_j^2, \quad 1 \le j \le n,$$

$$\{A_j^{1/2} L_j^2, A_j^{1/2}\} = 2A_j^{1/2} L_j \frac{\partial A_j^{1/2}}{\partial x_j},$$

and

$$\{A_j^{1/2}, L_j^2\} = -2 \frac{\partial A_j^{1/2}}{\partial x_j} L_j.$$

Therefore

$$\begin{aligned} \sum_{j=1}^{n} A_j^{1/2} \sharp_h L_j^2 \sharp_h A_j^{1/2} = & \sum_{j=1}^{n} A_j^{1/2} L_j^2 A_j^{1/2} \\ & - ih \sum_{j=1}^{n} \left(A_j^{1/2} \sharp_h L_j \sharp \frac{\partial A_j^{1/2}}{\partial x_j} - \frac{\partial A_j^{1/2}}{\partial x_j} \sharp_h L_j \sharp_h A_j^{1/2} \right) + h^2 r, \end{aligned}$$

where $r = r(x;h)$ is smooth in x and for all $\alpha \in \mathbb{Z}_+^n$

$$\sup_{h \in (0,1]} \|\partial_x^\alpha r(\cdot;h)\|_{L^\infty(\mathbb{R}^n;\mathsf{M}_N)} < +\infty,$$

with bounds depending only on the constants appearing in (3.18) and (3.19). Therefore, for any given $u \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$

$$\begin{aligned}(p^{\mathrm{w}}(x,hD;h)u,u) = &\sum_{j=1}^{n} \|L_j^{\mathrm{w}}(x,hD;h)A_j^{1/2}u\|_0^2\\ &+ 2h\,\mathrm{Im}\sum_{j=1}^{n}\left(L_j^{\mathrm{w}}(x,hD;h)A_j^{1/2}u, \frac{\partial A_j^{1/2}}{\partial x_j}u\right)\\ &+ \left((C(x;h) - \frac{1}{4}\sum_{j=1}^{n} A_j(x)^{1/2}B_{A_j,j}(x;h)^2A_j(x)^{1/2})u,u\right)\\ &+ h^2O(\|u\|_0^2)\end{aligned}$$

(by the Cauchy–Schwarz inequality)

$$\begin{aligned}&\geq \frac{1}{2}\sum_{j=1}^{n} \|L_j^{\mathrm{w}}(x,hD;h)A_j^{1/2}u\|_0^2\\ &\quad - 2h^2\sum_{j=1}^{n}\left\|\frac{\partial A_j^{1/2}}{\partial x_j}u\right\|_0^2 + (C_0(x;h)u,u) + h^2O(\|u\|_0^2),\end{aligned}$$

where this time I have put

$$C_0(x;h) := C(x;h) - \frac{1}{4}\sum_{j=1}^{n} A_j(x)^{1/2}B_{A_j,j}(x;h)^2A_j(x)^{1/2}.$$

Now, proceeding in a way similar to that of the proof of Lemma 3.2, let

$$\mathrm{Spec}(B_{A_j,j}(x;h)) := \{\lambda_k^{(j)}(x;h);\ k = 1,\dots,N\} \subset \mathbb{R}, \quad 1 \leq j \leq n,$$

and let for $k = 1,\dots,N$,

$$\xi^{(k)} := \xi^{(k)}(x;h) := \big(-\frac{1}{2}\lambda_k^{(1)}(x;h), -\frac{1}{2}\lambda_k^{(2)}(x;h),\dots,-\frac{1}{2}\lambda_k^{(n)}(x;h)\big).$$

Then by (3.16) one has

$$p(x,\xi^{(k)};h) = \sum_{j=1}^{n} A_j(x)^{1/2}\big(\frac{1}{2}B_{A_j,j}(x;h) - \frac{1}{2}\lambda_k^{(j)}(x;h)I_N\big)^2A_j(x)^{1/2} + C_0(x;h) \geq 0,$$

that is

$$C_0 \geq -\sum_{j=1}^{n} A_j^{1/2}\Big(\frac{1}{4}B_{A_j,j}^2 - \frac{1}{2}\lambda_k^{(j)}B_{A_j,j} + \frac{1}{4}\lambda_k^{(j)^2}I_N\Big)A_j^{1/2},$$

whence averaging over $k = 1, \ldots, N$ gives, as before,

$$C_0(x;h) \geq -\frac{1}{N}\sum_{j=1}^{n} A_j(x)^{1/2}\Lambda_j(x;h)A_j(x)^{1/2}.$$

It thus follows from hypothesis (3.20) that

$$(p^{\mathrm{w}}(x,hD;h)u,u) \geq -2h^2\sum_{j=1}^{n}\left\|\frac{\partial A_j^{1/2}}{\partial x_j}\right\|_{L^\infty}^2 \|u\|_0^2 + h^2 O(\|u\|_0^2) - \frac{c_1}{N}h^2\|u\|_0^2,$$

for all $u \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$ and all $h \in (0,1]$, which concludes the proof of the theorem. □

Of course, a lot of information has been thrown away when disregarding the "sum-of-squares" term appearing in (3.11) in bounding from below the L^2-quadratic form of $p_A^{\mathrm{w}}(x,hD;h)$. In this respect, Theorem 3.1 (and Theorem 3.4) is not yet satisfactory. This will be seen even more clearly in the next section, where I shall test Theorems 2.1 and 3.1 on a Dirac-squared-type system.

However, there are instances in which the result of Theorem 3.1 is sharp for all $h \in (0,h_0]$, for some $h_0 \in (0,1]$ sufficiently small, as the following lemma shows.

Lemma 3.5. *Suppose that*

$$B_j(x;h) = B_{0,j}(h) + h^{\nu_j}B_{1,j}(x;h) = B_j(x;h)^*, \quad j = 1,\ldots,n, \tag{3.22}$$

where $\nu_j \geq 3/2$, $1 \leq j \leq n$, the $B_{0,j}(h)$ are matrices that are constant in x, and the $B_{1,j}(\cdot;h) \in C^\infty(\mathbb{R}^n;\mathsf{M}_N)$ are such that for all $\alpha \in \mathbb{Z}_+^n$ there is $C_\alpha > 0$ with

$$\sum_{j=1}^{n}\|\partial_x^\alpha B_{1,j}(\cdot;h)\|_{L^\infty} \leq C_\alpha, \quad \forall h \in (0,1].$$

Suppose further that there exists $c > 0$ such that for any given $h \in (0,1]$ there are $\xi_j^0(h) \in \mathbb{R}$, $j = 1,\ldots,n$, and a non-zero $v_h \in \mathbb{C}^N$ such that

$$\left|\Big(\xi_j^0(h)I_N + \frac{1}{2}B_{0,j}(h)\Big)v_h\right|_{\mathbb{C}^N} \leq ch^{3/2}|v_h|_{\mathbb{C}^N}, \quad 1 \leq j \leq n. \tag{3.23}$$

For each $h \in (0,1]$ let $u_h \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$ be the function

$$u_h(x) := c_0 h^{n/2} e^{ih^{-1}\langle x,\xi^0(h)\rangle - h^2|x|^2/2} v_h/|v_h|_{\mathbb{C}^N}, \quad x \in \mathbb{R}^n, \tag{3.24}$$

where $\xi^0(h) = (\xi^0_1(h),\ldots,\xi^0_n(h))$, *and* $c_0^{-2} := \int_{\mathbb{R}^n} e^{-|t|^2}dt$. *Consider the first-order symbols* $L_j(x,\xi;h) = \xi_j I_N + \frac{1}{2}B_j(x;h)$, $1 \le j \le n$. *Then*

$$\|L_j^{\mathrm{w}}(x,hD;h)u_h\|_0^2 \le 10h^3\left(c^2 + \frac{C_0^2}{4} + \frac{c_0^2}{c_1^2}\right), \quad \forall h \in (0,1], \quad 1 \le j \le n, \tag{3.25}$$

where $c_1^{-2} := \int_{\mathbb{R}^n} t_1^2 e^{-|t|^2}\,dt$.

Proof. I may suppose that $|v_h|_{\mathbb{C}^N} = 1$ for all $h \in (0,1]$. It is immediate to check that

$$\|u_h\|_0 = 1, \quad \forall h \in (0,1],$$

and that

$$hD_{x_j}u_h(x) = \frac{h}{i}\partial_{x_j}u_h(x) = \xi_j^0(h)u_h(x) + ih^3x_ju_h(x), \ 1 \le j \le n.$$

Now,

$$\|h^3x_ju_h\|_0^2 = c_0^2h^{n+6}\int_{\mathbb{R}^n} x_j^2 e^{-h^2|x|^2}dx = \frac{c_0^2}{c_1^2}h^4, \quad \forall h \in (0,1],$$

and, by hypothesis (3.23),

$$\left\|\left(\xi_j^0(h)I_N + \frac{1}{2}B_{0,j}(h)\right)u_h\right\|_0^2 \le c^2h^3, \quad \forall h \in (0,1].$$

Since

$$L_j^{\mathrm{w}}(x,hD;h)u_h = \left(\xi_j^0(h)I_N + \frac{1}{2}B_{0,j}(h)\right)u_h + h^{\nu_j}\frac{1}{2}B_{1,j}(x;h)u_h + ih^3x_ju_h,$$

I therefore get

$$\|L_j^{\mathrm{w}}(x,hD;h)u_h\|_0^2 \le 10h^3\left(c^2 + \frac{C_0^2}{4} + \frac{c_0^2}{c_1^2}\right),$$

for all $h \in (0,1]$, which concludes the proof. □

REMARK 3.6. *It is easy to give examples for which the hypotheses of the above lemma are fulfilled, as the following two cases show:*

(*i*) *Suppose there is* $C > 0$ *and* $j_0 \in \{1, \ldots, n\}$ *such that*

$$|B_{0,j}(h) - B_{0,j_0}(h)|_{\mathsf{M_N}} \le Ch^{3/2}, \quad \forall h \in (0,1],\ j = 1, \ldots, n;$$

(*ii*) *Suppose that*

$$[B_{0,j}(h), B_{0,j'}(h)] = 0, \quad \forall h \in (0,1],\ j, j' = 1, \ldots, n.$$

In case (*i*) *one just takes for each* $h \in (0,1]$

$$\xi_1^0(h) \in \mathrm{Spec}\Big(-\frac{1}{2}B_{0,1}(h)\Big),$$

a vector

$$0 \neq v_h \in \mathrm{Ker}\Big(\frac{1}{2}B_{0,1}(h) + \xi_1^0(h)I_N\Big),$$

and $\xi^0(h) = (\xi_1^0(h), \xi_1^0(h), \ldots, \xi_1^0(h))$.

In case (*ii*) *one takes for each* $h \in (0,1]$

$$\xi^0(h) = (\xi_1^0(h), \xi_2^0(h), \ldots, \xi_n^0(h)),$$

where $\xi_j^0(h) \in \mathrm{Spec}\big(-\frac{1}{2}B_{0,j}(h)\big)$ *for* $1 \le j \le n$, *and the* $\xi_j^0(h)$ *are chosen in such a way that* $0 \neq v_h$ *is a common eigenvector, i.e.*

$$0 \neq v_h \in \bigcap_{j=1}^{n} \mathrm{Ker}\Big(\frac{1}{2}B_{0,j}(h) + \xi_j^0(h)I_N\Big).$$

Hence Lemma 3.5 yields that when $A = I_N$ and the matrices $B_{A,j} = B_j$ in Theorem 3.1 satisfy in addition the assumptions of the lemma, *then there exists* $h_0 \in (0,1]$ *such that inequality (3.9) is sharp for all* $h \in (0, h_0]$, *in the sense that no better contribution from the terms* $(L_j^{\mathrm{w}}(x, hD; h)^2u, u)$ *may be obtained.*

4. SEMICLASSICAL DIRAC-SQUARED-TYPE SYSTEMS

In this section I test Theorem 3.1 on certain systems that I shall call Dirac-squared-type systems.

Following Salo and Tzou [5], one defines a generalized Dirac-type operator as follows. Let $\ell_{jk} \in \mathbb{C}^n$ and consider the constant-coefficient $N \times N$ system $(N \ge 2)$

$$P_0(\xi) := (\ell_{jk} \cdot \xi)_{1 \le j,k \le N}, \quad \xi \in \mathbb{R}^n \tag{4.1}$$

(where $v \cdot w = \sum_k v_k w_k$, for v, w complex vectors), such that

$$P_0(\xi) = P_0(\xi)^*, \quad \text{and} \quad P_0(\xi)^2 = |\xi|^2 I_N. \tag{4.2}$$

Then it follows that

$$P_0(\zeta)P_0(\zeta') + P_0(\zeta')P_0(\zeta) = (\zeta \cdot \zeta')I_N, \quad \zeta, \zeta' \in \mathbb{C}^n. \tag{4.3}$$

When considering certain problems that involve Carleman estimates (see for instance [5]), one is willing to consider the system given by the principal part, in the semiclassical calculus, of $P_0(hD + i\nabla\varphi)^* P_0(hD + i\nabla\varphi)$, that is

$$p(x, \xi; h) := P_0(\xi)^2 + P_0(\nabla\varphi(x; h))^2 + i[P_0(\xi), P_0(\nabla\varphi(x; h))], \tag{4.4}$$

$(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$, $h \in (0, 1]$, where $\varphi = \varphi(x; h)$ is smooth in x and real-valued, and ∇ denotes the gradient with respect the variable x. Since

$$\langle p(x, \xi; h)v, v\rangle_{\mathbb{C}^N} = |P_0(\xi)v|^2_{\mathbb{C}^N} + |P_0(\nabla\varphi)v|^2_{\mathbb{C}^N} + 2\,\mathrm{Im}\langle P_0(\xi)v, P_0(\nabla\varphi)v\rangle_{\mathbb{C}^N},$$

$v \in \mathbb{C}^N$, by the Cauchy–Schwarz inequality we clearly have that

$$p(x, \xi; h) = p(x, \xi; h)^* \ge 0.$$

I shall call system (4.4), a *Dirac-squared-type system*.

We have the following result, which is a consequence of Theorem 3.1 (it is of course also a consequence of Theorem 2.1 and the final part of Remark 3.3).

PROPOSITION 4.1. *Suppose that $\varphi(x; h) = c_\varphi + h\psi(x; h)$, for a real constant c_φ and a smooth, real-valued ψ such that $\|\partial_x^\alpha \psi(\cdot; h)\|_{L^\infty(\mathbb{R}^n)} < +\infty$ for all $h \in (0, 1]$ and all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \ge 1$. Then there exists $C = C(n) > 0$ such that*

$$(p^{\mathrm{w}}(x, hD; h)u, u) \ge -Ch^2 \|\nabla\psi(\cdot; h)\|^2_{L^\infty} \|u\|_0^2, \quad \forall u \in \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N), \tag{4.5}$$

for all $h \in (0, 1]$. One has from Theorem 3.1 that $C = 2(n-1)$, whereas from its method of proof one has $C = (n-2)$.

PROOF. I write

$$P_0(\xi) = \sum_{j=1}^n \xi_j Q_j.$$

Then from (4.2) one has

$$Q_j = Q_j^*, \quad 1 \le j \le n,$$

and from (4.1) and (4.3)

$$P_0(e_j)^2 = I_N \Longrightarrow Q_j^2 = I_N, \quad 1 \le j \le n,$$

and

$$P_0(e_j)P_0(e_{j'}) + P_0(e_{j'})P_0(e_j) = Q_jQ_{j'} + Q_{j'}Q_j = 0, \quad 1 \le j, j' \le n, \quad j \ne j'. \tag{4.6}$$

By (4.4) I may therefore write

$$p(x, \xi; h) = |\xi|^2 I_N + \sum_{j=1}^{n} \xi_j B_j(x; h) + C(x; h),$$

where

$$\begin{aligned} B_j(x; h) := i[Q_j, P_0(\nabla\varphi(x; h))] &= i \sum_{k=1}^{n} [Q_j, Q_k] \frac{\partial \varphi}{\partial x_k}(x; h) \\ &= ih \sum_{k=1}^{n} [Q_j, Q_k] \frac{\partial \psi}{\partial x_k}(x; h), \end{aligned} \tag{4.7}$$

and

$$C(x; h) = |\nabla\varphi(x; h)|^2 I_N = h^2 |\nabla\psi(x; h)|^2 I_N. \tag{4.8}$$

One may also write, by (4.6),

$$B_j(x; h) = 2i \sum_{k \ne j;\, k=1}^{n} Q_j Q_k \frac{\partial \varphi}{\partial x_k}(x; h) = 2i\big(Q_j P_0(\nabla\varphi(x; h)) - \frac{\partial \varphi}{\partial x_j}(x; h) I_N\big). \tag{4.9}$$

Notice that by (4.7) and (4.8), the final part of Remark 3.3 yields the existence of $C' > 0$ such that

$$(p^{\mathrm{w}}(x, hD; h)u, u) \ge -C'h^2 \|u\|_0^2, \quad \forall u \in \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N), \ \forall h \in (0, 1]. \tag{4.10}$$

Notice also that, again by virtue of (4.7) and (4.8), p satisfies (2.4) and (2.5), whence Theorem 2.1 too gives the existence of $C' > 0$ such that (4.10) holds.

The point here is therefore to have a better control on the constant C', and this is provided by Theorem 3.1 and its proof. This is what I am going to show next. (However, Theorem 2.1 holds for more general matrix-coefficients A which are in fact allowed to be only ≥ 0, i.e. allowed to have a non-trivial kernel; it hence holds, for instance, for the system $A_1(x)^* p(x, \xi; h) A_1(x)$, where p is given by (4.4) and the kernel of $A_1(x)$ is non-trivial for some x.)

One has from (4.7) that $\mathrm{Tr}(B_j(x;h)) = 0$, for all x, h, and $j = 1, \dots, n$. On the other hand (using (4.6))

$$\begin{aligned}
B_j^2 &= \left(ih \sum_{k=1}^{n} [Q_j, Q_k] \frac{\partial\psi}{\partial x_k} \right)^2 = -h^2 \sum_{k=1}^{n} [Q_j, Q_k] \frac{\partial\psi}{\partial x_k} \sum_{k'=1}^{n} [Q_j, Q_{k'}] \frac{\partial\psi}{\partial x_{k'}} \\
&= -h^2 \sum_{k,k' \neq j} \frac{\partial\psi}{\partial x_k} \frac{\partial\psi}{\partial x_{k'}} (Q_j Q_k Q_j Q_{k'} - Q_k Q_j^2 Q_{k'} - Q_j Q_k Q_{k'} Q_j + Q_k Q_j Q_{k'} Q_j) \\
&= -h^2 \sum_{k,k' \neq j} \frac{\partial\psi}{\partial x_k} \frac{\partial\psi}{\partial x_{k'}} (-Q_k Q_j^2 Q_{k'} - Q_k Q_j^2 Q_{k'} - Q_k Q_j^2 Q_{k'} - Q_k Q_j^2 Q_{k'}) \\
&= 4h^2 \sum_{k,k' \neq j} \frac{\partial\psi}{\partial x_k} \frac{\partial\psi}{\partial x_{k'}} Q_k Q_{k'} = 4h^2 \sum_{k=k' \neq j} \frac{\partial\psi}{\partial x_k} \frac{\partial\psi}{\partial x_{k'}} Q_k Q_{k'} \\
&\quad + 4h^2 \sum_{k<k';\, k,k' \neq j} \frac{\partial\psi}{\partial x_k} \frac{\partial\psi}{\partial x_{k'}} Q_k Q_{k'} + 4h^2 \sum_{k'<k;\, k,k' \neq j} \frac{\partial\psi}{\partial x_k} \frac{\partial\psi}{\partial x_{k'}} Q_k Q_{k'} \\
&= 4h^2 \left(\sum_{k \neq j} \left(\frac{\partial\psi}{\partial x_k} \right)^2 \right) I_N + 4h^2 \sum_{k<k';\, k,k' \neq j} \frac{\partial\psi}{\partial x_k} \frac{\partial\psi}{\partial x_{k'}} (Q_k Q_{k'} + Q_{k'} Q_k) \\
&= 4h^2 |\nabla^{(j)} \psi(x;h)|^2 I_N,
\end{aligned}$$

where I have put $\nabla^{(j)}\psi = (\partial\psi/\partial x_k)_{k \neq j}$. Hence

$$B_j(x;h)^2 = 4h^2 |\nabla^{(j)}\psi(x;h)|^2 I_N, \quad \mathrm{Tr}(B_j(x;h)^2) = 4Nh^2 |\nabla^{(j)}\psi(x;h)|^2.$$

It follows that

$$\Lambda_j(x;h) = 2Nh^2 |\nabla^{(j)}\psi(x;h)|^2 I_N,$$

$$\sum_{j=1}^{n} \Lambda_j(x;h) = 2Nh^2 \sum_{j=1}^{n} |\nabla^{(j)}\psi(x;h)|^2 I_N = 2N(n-1)h^2 |\nabla\psi(x;h)|^2 I_N,$$

and that

$$\begin{aligned}
C_0(x;h) &= h^2 |\nabla\psi(x;h)|^2 I_N - \frac{1}{4} \sum_{j=1}^{n} 4h^2 |\nabla^{(j)}\psi(x;h)|^2 I_N \\
&= -(n-2)h^2 |\nabla\psi(x;h)|^2 I_N \geq \text{(by (3.15))} \geq -\frac{1}{N} \sum_{j=1}^{n} \Lambda_j(x;h) \\
&= -2(n-1)h^2 |\nabla\psi(x;h)|^2 I_N,
\end{aligned} \tag{4.11}$$

which concludes the proof. □

I next give an example (which is, in fact a consequence of Lemma 3.5) of Dirac-squared-type system for which the procedure of completing the square is sharp for obtaining the semiclassical Fefferman–Phong inequality.

Lemma 4.2. *Let $n \geq 2$, and for some $\nu \geq 3/2$ let*

$$\varphi(x;h) = b_1(h)x_1 + b_2(h)x_2 + h^{\nu}\phi_1(x;h), \quad x \in \mathbb{R}^n, \quad h \in (0,1], \tag{4.12}$$

with $0 \neq b_1(h), b_2(h) \in \mathbb{R}$, where $\phi_1(\cdot;h) \in C^{\infty}(\mathbb{R}^n;\mathbb{R})$ and for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \geq 1$ there exists $C_\alpha > 0$ such that

$$\|\partial_x^\alpha \phi_1(\cdot;h)\|_{L^\infty} \leq C_\alpha, \quad \forall h \in (0,1].$$

For each $h \in (0,1]$ let $u_h \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$ be the function

$$u_h(x) = c_0 h^{n/2} e^{ih^{-1}\lambda_0(b_2(h)x_1 - b_1(h)x_2) - h^2|x|^2/2} v/|v|_{\mathbb{C}^N}, \quad x \in \mathbb{R}^n,$$

where the real number $\lambda_0 \in \mathrm{Spec}(-iQ_1Q_2)$ and $0 \neq v \in \mathrm{Ker}(iQ_1Q_2 + \lambda_0)$, and where (again) $c_0^{-2} = \int_{\mathbb{R}^n} e^{-|t|^2}dt$. As before, consider the first-order symbols $L_j(x,\xi;h) := \xi_j I_N + \frac{1}{2}B_j(x;h)$, $1 \leq j \leq n$. Then there exists a constant $C > 0$ such that

$$\|L_j^{\mathrm{w}}(x,hD;h)u_h\|_0^2 \leq Ch^3, \quad \forall h \in (0,1], \quad j = 1,\dots,n.$$

Proof. Recalling formula (4.9) and using

$$\frac{\partial\varphi}{\partial x_k} = h^{\nu}\frac{\partial\phi_1}{\partial x_k}, \quad k \geq 3,$$

I write

$$B_j(x;h) = B_{0,j}(h) + h^{\nu}B_{1,j}(x;h), \quad j = 1,\dots,n,$$

where

$$B_{0,1}(h) = 2ib_2(h)Q_1Q_2, \quad B_{0,2}(h) = 2ib_1(h)Q_2Q_1, \quad B_{0,j}(h) = 0, \; j \geq 3,$$

and

$$B_{1,j}(x;h) = 2ih^{\nu}\sum_{k\neq j;\, k=1}^{n} Q_jQ_k\frac{\partial\phi_1}{\partial x_k}(x;h), \quad 1 \leq j \leq n.$$

Therefore (by (4.6))

$$[B_{0,j}(h), B_{0,j'}(h)] = 0, \quad \forall h \in (0,1], \quad \forall j,j' = 1,\dots,n,$$

$$hD_{x_1}u_h + \frac{1}{2}B_{0,1}(h)u_h = b_2(h)\underbrace{(\lambda_0 + iQ_1Q_2)u_h}_{=0} + ih^3x_1u_h,$$

and

$$hD_{x_2}u_h + \frac{1}{2}B_{0,2}(h)u_h = -b_1(h)\underbrace{(\lambda_0 - iQ_2Q_1)u_h}_{=0} + ih^3x_2u_h,$$

so that the proof follows as in Lemma 3.5. □

Lemma 4.2 therefore shows that when, say, $b_1(h) = b_2(h) = hb_0(h)$, where $b_0(h)$ is real and uniformly bounded for all $h \in (0, 1]$, one has the existence of an $h_0 \in (0, 1]$ such that Proposition 4.1 holds *and is sharp for all* $h \in (0, h_0]$, *again in the sense that the terms* $(L_j^{\mathrm{w}}(x, hD; h)^2u, u)$ *do not give a better contribution.*

In the following remark I give an elementary example of second order system (of Dirac-squared-type) for which the semiclassical Fefferman–Phong inequality cannot hold.

REMARK 4.3. *Recall from the proof of Proposition 4.1 that when writing* $p(x, \xi; h) = \sum_{j=1}^{n} L_j(x, \xi; h)^2 + C_0(x; h)$, *one has (see (4.11)) that*

$$C_0(x; h) = -(n-2)|\nabla\varphi(x; h)|^2 I_N.$$

Take thus $n \geq 3$ *and* φ *and* ϕ_1 *as in (4.12), where (say)*

$$b_1(h) = b_2(h) = h^{1/2}b_0, \qquad b_0 \neq 0.$$

Then, for $h_0 \in (0, 1]$ *sufficiently small,*

$$|\nabla\varphi(x; h)|^2 \sim h, \quad \forall h \in (0, h_0], \quad \forall x \in \mathbb{R}^n,$$

and by Lemma 4.2 one has

$$(p^{\mathrm{w}}(x, hD; h)u_h, u_h) \sim O(h^3) - h, \quad \forall h \in (0, h_0],$$

whence (by possibly shrinking h_0*) the semiclassical Fefferman–Phong inequality cannot hold.*

Of course, one may even give "worse" and simpler examples. The simplest one is given just by taking $\varphi(x; h) = x_1$. *In this case, again by Lemma 4.2 (with* $b_1(h) = 1$ *and* $b_2(h) = 0$*), one has*

$$(p^{\mathrm{w}}(x, hD; h)u_h, u_h) \sim O(h^3) - 1, \quad \forall h \in (0, h_0],$$

which shows again (by possibly shrinking h_0*) the failure of the semiclassical Fefferman–Phong inequality.*

It is finally interesting to notice in Lemma 4.2 that the function $b_1(h)x_1 + b_2(h)x_2$ is a harmonic conjugate of $b_2(h)x_1 - b_1(h)x_2$. This leads to the following observation (a localized version of the semiclassical Fefferman–Phong inequality).

Lemma 4.4. *Let $n \geq 3$ and $\nu \geq 3/2$. For each $h \in (0,1]$, let $\phi_0 = \phi_0(x_1, x_2; h)$ be a harmonic polynomial. Consider the Dirac-squared-type system (4.4), with*

$$\varphi(x;h) = \phi_0(x_1, x_2; h) + h^{\nu}\phi_1(x;h), \quad x \in \mathbb{R}^n, \quad h \in (0,1],$$

where $\phi_1(\cdot;h)$ is real and smooth for all $h \in (0,1]$, and such that for any given compact $K \subset \mathbb{R}^n$ and any given $\alpha \in \mathbb{Z}_+^n$ with $1 \leq |\alpha| \leq 2$ there exists $C_{K,\alpha} > 0$ for which

$$\|\partial_x^{\alpha}\phi_1(\cdot;h)\|_{L^{\infty}(K)} \leq C_{K,\alpha}, \quad \forall h \in (0,1].$$

Then for any given compact $K \subset \mathbb{R}^n$ one has

$$(p^{\mathrm{w}}(x, hD; h)u, u) \geq -(n-2)\|\nabla\varphi(\cdot;h)\|_{L^{\infty}(K)}^2 \|u\|_0^2, \quad \forall u \in C_0^{\infty}(K; \mathbb{C}^N),$$

for all $h \in (0,1]$.

Furthermore, one may find a constant $C > 0$ and for each $h \in (0,1]$ a function $u_h \in \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)$ (see (4.14) below) such that

$$\|L_j^{\mathrm{w}}(x, hD; h)u_h\|_0^2 \leq Ch^3, \quad \forall h \in (0,1], \quad j = 1, \dots, n.$$

Proof. I need only prove the second part of the statement. For this purpose I adapt the proof of Lemma 4.2 as follows. Let $\lambda_0 \in \mathrm{Spec}(-iQ_1Q_2)$ and $0 \neq v \in \mathrm{Ker}(iQ_1Q_2 + \lambda_0)$. Then I write

$$B_j(x;h) = B_{0,j}(x;h) + h^{\nu}B_{1,j}(x;h), \quad j = 1, \dots, n,$$

where

$$B_{0,1}(x;h) = 2i\frac{\partial\phi_0}{\partial x_2}(x;h)Q_1Q_2, \; B_{0,2}(x;h) = 2i\frac{\partial\phi_0}{\partial x_1}(x;h)Q_2Q_1,$$

$$B_{0,j}(x;h) = 0, \; j = 3, \dots, n,$$

and

$$B_{1,j}(x;h) = 2ih^{\nu}\sum_{k\neq j;\, k=1}^{n} Q_jQ_k\frac{\partial\phi_1}{\partial x_k}(x;h), \quad 1 \leq j \leq n.$$

Let $\psi_0(x_1, x_2; h)$ be a harmonic function such that $\psi_0 + i\phi_0$ is holomorphic, that is, ϕ_0 is a harmonic conjugate of ψ_0. Then $\psi_0(\cdot;h)$ is a polynomial and

$$\frac{\partial\psi_0}{\partial x_1} = \frac{\partial\phi_0}{\partial x_2}, \quad \frac{\partial\psi_0}{\partial x_2} = -\frac{\partial\phi_0}{\partial x_1}, \quad \forall (x_1, x_2) \in \mathbb{R}^2, \; \forall h \in (0,1]. \tag{4.13}$$

Consider then, for each $h \in (0,1]$, the Schwartz function

$$u_h(x) = c_0 h^{n/2} e^{ih^{-1}\lambda_0\psi_0(x_1,x_2;h) - h^2|x|^2/2} v/|v|_{\mathbb{C}^N}, \quad x \in \mathbb{R}^n, \tag{4.14}$$

where $c_0^{-2} := \int_{\mathbb{R}^n} e^{-|t|^2} dt$. Now, by (4.14) and (4.6) one has

$$\begin{aligned} hD_{x_1} u_h + \frac{1}{2} B_{0,1}(x; h) u_h &= \Big(\lambda_0 \frac{\partial \psi_0}{\partial x_1} + \frac{\partial \phi_0}{\partial x_2} iQ_1 Q_2\Big) u_h + ih^3 x_1 u_h \\ &= \frac{\partial \psi_0}{\partial x_1} \underbrace{(\lambda_0 + iQ_1 Q_2) u_h}_{=0} + ih^3 x_1 u_h, \end{aligned}$$

and

$$\begin{aligned} hD_{x_2} u_h + \frac{1}{2} B_{0,2}(x; h) u_h &= \Big(\lambda_0 \frac{\partial \psi_0}{\partial x_2} + \frac{\partial \phi_0}{\partial x_1} iQ_2 Q_1\Big) u_h + ih^3 x_2 u_h \\ &= \Big(\lambda_0 \frac{\partial \psi_0}{\partial x_2} - \frac{\partial \phi_0}{\partial x_1} iQ_1 Q_2\Big) u_h + ih^3 x_2 u_h \\ &= \frac{\partial \psi_0}{\partial x_2} \underbrace{(\lambda_0 + iQ_1 Q_2) u_h}_{=0} + ih^3 x_2 u_h. \end{aligned}$$

Hence the proof follows as in Lemma 3.5. □

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