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Algebra — *On the number of terms of a power of a polynomial*, by ANDRZEJ SCHINZEL and UMBERTO ZANNIER, communicated on 14 November 2008.

ABSTRACT. — Let $f(x)$ be a polynomial with complex coefficients. Rényi and independently Erdös in 1949 conjectured that a bound for the number of terms of $f(x)^2$ implies a bound for the number of terms of $f(x)$. In 1987 Schinzel found a proof of this conjecture, actually for all powers $f(x)^l$, and he gave some explicit bounds. The aim of this paper is to improve such inequalities in a substantial way.

KEY WORDS: Algebra of Polynomials.

MATHEMATICS SUBJECT CLASSIFICATION: 10M05, 12E05, 12Y05

INTRODUCTION

Let us call the number of non-zero coefficients of a polynomial the number of its terms. In this paper we are concerned with the problem of comparing the number of terms of a polynomial $f(x)$ and the number of terms of a power $f(x)^l$ of it. For given l, it is clear that the latter number may be bounded above depending only on the former, but an opposite comparison is on the contrary rather delicate; already the case $l = 2$ was the object of a qualitative statement conjectured by Rényi and independently by Erdös in 1949 [E], that a bound for the number of terms of $f(x)^2$ implies a bound for the number of terms of $f(x)$. In 1987 Schinzel [S] found a proof of this conjecture, actually for all powers $f(x)^l$, and he gave some explicit bounds; for instance, when $f(x)$ has coefficients in a field of zero characteristic he obtained, letting T (resp. t) be the number of terms of $f(x)$ (resp. $f(x)^l$), an inequality of the shape $t \geq c_l \log\log T$, with a certain explicit $c_l > 0$.

The aim of this paper is to improve these inequalities of [S] by removing one logarithm, roughly speaking; more precisely, we shall prove the following theorem.

THEOREM 1. *Let k be a field, $f \in k[x]$, $l \in \mathbb{N}$, f and f^l have $T \geq 2$ and t terms, respectively. If either* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > l \deg f$, *then*

$$t \geq 2 + \frac{\log(T-1)}{\log 4l}. \tag{1}$$

In positive characteristic we have also the following supplementary result:

THEOREM 2. *Let* char $k > 0$, $f \in k[x]$, $l \in \mathbb{N}$, f *and* f^l *have* $T \geq 2$ *and* t *terms, respectively. If*

$$l^{t-1}(T^2 - T + 2) < \operatorname{char} k,$$

then

$$t \geq 2 + \frac{\log(T-1)}{\log 4l}. \tag{1}$$

Note that even for $l = 2$ the lower bound in Theorem 1 is still very far from the best known upper bound, due to Verdenius [V], which is

$$t \ll T^{\log 8/\log 13}$$

for a sequence of polynomials with the number of terms tending to infinity. (See also [S2], Section 2.6.)

Our proofs follow the approach of [S], introducing however a crucial variation, which occurs through an induction with respect to degrees rather than with respect to t. Another method to deal with similar problems appears in [Z]; this works, more generally, for arbitrary polynomial compositions $g(f(x))$ in place of $f(x)^l$, but leads to weaker bounds compared to [S] and the present method.

PROOFS. For the proof of Theorem 1 we need the following facts established in [S].

LEMMA 1. *If* $f(x) \in k[x]$, $f(0) \neq 0$, $f(x)^l \in k[x^d]$, *then either* $\operatorname{char} k \mid (l, d)$ *or* $f(x) \in k[x^d]$.

PROOF. See [S], Lemma 2.

LEMMA 2. *Assume that* $a_0 + \sum_{j=1}^{t-1} a_j x^{n_j} = f_0(x)^l$, *where* $a_j \neq 0$, $0 < n_1 < \cdots < n_{t-1}$ *and* $f_0 \in k[x]$ *has* T *terms. If* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > l \deg f_0$ *and integers* $p_1, \ldots, p_{t-1}$ *satisfy*

$$0 < p_{t-1} \leq (4l)^{t-2}$$

and

$$\left| \frac{n_j}{n_{t-1}} - \frac{p_j}{p_{t-1}} \right| < \frac{1}{4lp_{t-1}}, \quad j = 1, 2, \ldots, t-2,$$

then

$$0 \leq p_j < p_{t-1}, \quad j = 1, 2, \ldots, t-2,$$

and on taking $r_j = p_{t-1}n_j - n_{t-1}p_j$, $r := \min_{1 \leq j < t} r_j$,

$$F(y, z) := z^{-r}\left(a_0 + \sum_{j=1}^{t-1} a_j y^{p_j} z^{r_j}\right)$$

we have either

(2) $$F = cF_0^l, \quad \textit{where } F_0 \in k[y,z],\ c \in k^*$$

and F_0 *has* $T_0 \geq T$ *terms, or* $T \leq 1 + \frac{(4l)^{t-2}}{l}$.

Proof. See [S], pp. 60–63.

Proof of Theorem 1. Assume that Theorem 1 is false. Take a polynomial f_0 of the least possible degree for which the inequality (1) does not hold, or what comes to the same

(3) $$T > 1 + (4l)^{t-2}.$$

It follows that $l \geq 2$ and $t \geq 2$. Using Dirichlet's approximation theorem we obtain from Lemma 2 the relation (2), where

$$F_0(y,z) = \sum_{\tau=1}^{T_0} b_\tau y^{\alpha_\tau} z^{\beta_\tau},$$

with $\langle \alpha_\tau, \beta_\tau \rangle$ distinct and $b_\tau \neq 0$, $T_0 \geq T$.

Now, $\deg_z F(y,z) = \max r_j - \min r_j < \frac{n_{t-1}}{2l}$, hence

$$\max \beta_\tau = \deg_z F_0(y,z) < \frac{n_{t-1}}{2l^2} \leq \frac{n_{t-1}}{4l}.$$

Now choose the least q satisfying $q > \frac{n_{t-1}}{4l}$, $q \equiv n_{t-1} \pmod{p_{t-1}}$. Clearly $q \leq \frac{n_{t-1}}{4l} + p_{t-1}$.

The polynomial $F_0(x^q, x)$ has T_0 terms, since $q\alpha_\sigma + \beta_\sigma = q\alpha_\tau + \beta_\tau$ implies either $\alpha_\sigma = \alpha_\tau$, $\beta_\sigma = \beta_\tau$ or $\alpha_\sigma \neq \alpha_\tau$ and then

$$q|\alpha_\sigma - \alpha_\tau| \geq q > |\beta_\sigma - \beta_\tau|.$$

Now

$$F(x^q, x) = x^{-r}\Bigl(a_0 + \sum_{j=1}^{t-1} a_j x^{qp_j + r_j}\Bigr).$$

We have $qp_j + r_j \equiv n_{t-1}p_j + r_j \equiv 0 \pmod{p_{t-1}}$. Moreover $qp_j + r_j > 0$, since if $p_j = 0$, then $r_j = p_{t-1}n_j$; if $p_j > 0$, then $qp_j + r_j \geq q + r_j > 0$. Thus $F(x^q, x) = x^{-r} f_1(x^{p_{t-1}})$, where $f_1 \in k[x]$ and f_1 has at most t terms. From (2) we obtain

$$x^{-r} f_1(x^{p_{t-1}}) = cF_0(x^q, x)^l,$$

and since $f_1(0) = a_0 \neq 0$ we have $l \mid r$, $f_1(x^{p_{t-1}}) = c(x^{r/l}F_0(x^q, x))^l$ and then by Lemma 1, $x^{r/l}F_0(x^q, x) = g_1(x^{p_{t-1}})$ where $g_1 \in k[x]$.

Thus $f_1(x) = cg_1(x)^l$, where the number of terms of f_1 is at most t, the number of terms of g_1 is $T_0 \geq T$.

Now, the degree of f_1 is $\frac{1}{p_{t-1}} \deg x^r F(x^q, x) = \frac{1}{p_{t-1}} \max(qp_j + r_j) \le \frac{1}{p_{t-1}}\left(\left(\frac{n_{t-1}}{4l} + p_{t-1}\right)p_{t-1} + \frac{n_{t-1}}{4l}\right) \le \frac{n_{t-1}}{4l} + p_{t-1} + \frac{n_{t-1}}{4lp_{t-1}}$.

Since $1 \le p_{t-1} \le (4l)^{t-2}$ we have

$$p_{t-1} + \frac{n_{t-1}}{4lp_{t-1}} \le \max\left(1 + \frac{n_{t-1}}{4l}, (4l)^{t-2} + \frac{n_{t-1}}{(4l)^{t-1}}\right)$$

and by our assumption on the minimality of the degree n_{t-1} of f_0^l we obtain

$$n_{t-1} \le \frac{n_{t-1}}{4l} + \max\left(1 + \frac{n_{t-1}}{4l}, (4l)^{t-2} + \frac{n_{t-1}}{(4l)^{t-1}}\right),$$

which gives

$$T \le 1 + \deg f_0 = 1 + \frac{n_{t-1}}{l} \le 1 + \frac{1}{l}(4l)^{t-2}\left(1 - \frac{1}{2l}\right)^{-1} < 1 + (4l)^{t-2},$$

contrary to (3).

Proof of Theorem 2. Theorem 2 follows from Theorem 1 in the same way as Theorem 2 from Theorem 1 in [S].

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