



**Mathematical Analysis** — *Systems of nonlinear Schrödinger equations. A survey*,  
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ABSTRACT. — In this paper we survey some recent advances on various kind of systems of nonlinear Schrödinger equations. The arguments rely on critical point theory, the concentration compactness and perturbation methods.

KEY WORDS: Nonlinear Schrödinger Equations and Systems, Variational methods, Perturbation methods.

AMS SUBJECT CLASSIFICATION: 36J60, 35J20, 35Q55.

*Dedicated to Renato Caccioppoli in the occasion of the 50th anniversary of his death.*

## 1. INTRODUCTION

Nonlinear Schrödinger equations (NLS in short) have been broadly investigated, after the pioneering paper by A. Floer and A. Weinstein [14]. We refer, for example, to [4, 13, 21, 22] and to [8] which contains several further references.

More recently, there has been an increasing interest to consider systems of coupled NLS equations, which arise for example in Nonlinear Optics.

Let  $E(x, z)$  denote the complex envelope of an electric field. Planar stationary light beams propagating in the  $z$ -direction in a nonlinear medium are described, up to rescaling, by a NLS equation like

$$iE_z + E_{xx} + k|E|^2E = 0,$$

where  $k$  is a constant which is assumed to be positive, say  $k = 1$ , corresponding to the fact that the medium is *self-focusing*.

If  $E = \phi + \psi$  is the sum of a right-hand polarized wave  $\phi$  and of a left-hand polarized wave  $\psi$ , then the preceding equation gives rise to the following system of NLS equations

$$(1) \quad \begin{cases} i\phi_z + \phi_{xx} + (|\phi|^2 + |\psi|^2)\phi = 0, \\ i\psi_z + \psi_{xx} + (|\phi|^2 + |\psi|^2)\psi = 0. \end{cases}$$

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We look for  $\phi$  and  $\psi$  in the form of standing waves, namely

$$\phi(z, x) = u(x)e^{i\omega_1 z}, \quad \psi(z, x) = v(x)e^{i\omega_2 z},$$

where  $u(x), v(x)$  are real valued functions and  $\omega_i > 0, i = 1, 2$ .

With this notation we get the system

$$(2) \quad \begin{cases} -u'' + \omega_1 u = u^3 + \lambda uv^2, \\ -v'' + \omega_2 v = v^3 + \lambda u^2 v, \end{cases}$$

where the coupling constant  $\lambda > 0$  depends on the anisotropy of the fibers.

Coupled NLS systems also arise from the Hartree–Fock theory for the double Bose–Einstein condensates in two hyperfine states. In such a case one finds a system like

$$(3) \quad \begin{cases} -\varepsilon^2 \Delta u + \omega_1 u = \mu_1 u^3 + \lambda uv^2, \\ -\varepsilon^2 \Delta v + \omega_2 v = \mu_2 v^3 + \lambda u^2 v, \end{cases}$$

on a bounded domain  $\Omega \subset \mathbb{R}^3$ , with Dirichlet boundary conditions. Here  $u$  and  $v$  represent the condensate amplitudes, and  $\varepsilon^2 \sim \hbar^2$ ,  $\hbar$  being the Planck constant.

Furthermore, the propagation of optical pulses in nonlinear dual-core fiber can be described by two linearly coupled NLS equations like

$$(4) \quad \begin{cases} -u'' + u = u^3 + \lambda v, \\ -v'' + v = v^3 + \lambda u. \end{cases}$$

## 2. THE VARIATIONAL SETTING

Systems (2) and (4) are in the form

$$(S) \quad \begin{cases} -\Delta u + u = u^3 + \lambda F_u(u, v), \\ -\Delta v + v = v^3 + \lambda F_v(u, v). \end{cases}$$

If  $F(u, v) = uv$ , (S) becomes the linearly coupled system (4), while if  $F(u, v) = \frac{1}{2}u^2v^2$  we find the nonlinearly coupled system (2).

System (S) has a variational structure. Precisely, setting  $H := W^{1,2}(\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$ ,  $n = 1, 2, 3$ , solutions of (S) are the critical points  $(u, v) \in H$  of the functional

$$I_\lambda(u, v) = I(u) + I(v) - \lambda \int F(u, v) dx,$$

where

$$I(u) = \frac{1}{2} \int [|\nabla u|^2 + u^2] dx - \frac{1}{4} \int u^4 dx.$$

A solution  $(u, v) \in H$  of (S) is called a *Bound State*. The set of non-trivial, i.e.  $(u, v) \neq (0, 0)$ , bound states will be denoted by  $\mathfrak{B}_\lambda$ .

A solution  $(\tilde{u}, \tilde{v}) \in \mathfrak{B}_\lambda$  is called a *Ground State* if

$$I_\lambda(\tilde{u}, \tilde{v}) = \min\{I_\lambda(u, v) : (u, v) \in \mathfrak{B}_\lambda\}.$$

The Ground States are important because they are the natural candidates to possess some stability property for the  $z$ -dependent system (1). These Ground States can be found as minima of  $I_\lambda$  on a suitable manifold (usually named Nehari manifold):

$$M_\lambda = \{(u, v) \in H \setminus \{0, 0\} : \langle \nabla I_\lambda(u, v) | (u, v) \rangle = 0\},$$

where  $\langle \cdot | \cdot \rangle$  denotes the standard scalar product on  $H$ . Actually, it is possible to show that non-trivial stationary points of  $I_\lambda$  are the critical points of  $I_\lambda$  *constrained* on  $M_\lambda$ .

### 3. BOUND AND GROUND STATES OF (2)

The nonlinearly coupled system (2) possesses explicit solutions which have one trivial component. We will call these solutions *semi-trivial* solutions.

Precisely,  $(u, 0)$  is a semi-trivial solution of (2) provided  $u$  verifies

$$-\Delta u + \omega_1 u = \mu_1 u^3.$$

Hence  $u := U_1(x) = \sqrt{\omega_1/\mu_1} U(\sqrt{\omega_1}x)$ , where  $U$  is the radial positive soliton like solution satisfying

$$-\Delta U + U = U^3.$$

Similarly,  $(0, v)$  is a semi-trivial solution of (2) provided  $v$  verifies

$$v := U_2(x) = \sqrt{\omega_2/\mu_2} U(\sqrt{\omega_2}x).$$

Using Morse theoretical arguments, it is possible to classify these semi-trivial solutions. Actually one can prove the following lemma.

LEMMA 3.1. *There exist  $\Lambda, \Lambda' > 0$  such that:*

- (i) *For all  $\lambda < \Lambda$  the semi-trivial solutions  $(U_1, 0), (0, U_2)$  are strict local minima of  $I_\lambda$  on  $M_\lambda$ .*
- (ii) *For all  $\lambda > \Lambda'$  the semi-trivial solutions  $(U_1, 0), (0, U_2)$  are saddle points of  $I_\lambda$  on  $M_\lambda$ .*

More precisely, it is possible to show that  $\Lambda, \Lambda'$  are given by

$$\Lambda = \min\{\gamma_1^2, \gamma_2^2\}, \quad \Lambda' = \max\{\gamma_1^2, \gamma_2^2\},$$

where

$$\gamma_1^2 = \inf_{\phi \in W_r^{1,2}(\mathbb{R}^n)} \frac{\int |\nabla \phi|^2 + \omega_2 \phi^2}{\int U_1^2 \phi^2},$$

$$\gamma_2^2 = \inf_{\phi \in W_r^{1,2}(\mathbb{R}^n)} \frac{\int |\nabla \phi|^2 + \omega_1 \phi^2}{\int U_2^2 \phi^2},$$

and  $W_r^{1,2}(\mathbb{R}^n)$  denotes the space of radial functions in  $W^{1,2}(\mathbb{R}^n)$ .

Furthermore, more explicit estimates in terms of  $\omega_i$ ,  $\mu_i$  can also be given. Letting  $\kappa := \mu_2/\mu_1$ , there holds:

$$\gamma_1^2 \geq \omega_1 \kappa^{1-n/4}, \quad \gamma_2^2 \geq \omega_2 \left(\frac{1}{\kappa}\right)^{1-n/4}.$$

Moreover,

$$\gamma_1^2 \leq \max\{\omega_1 \kappa, \omega_1 \kappa^{1-n/2}\}, \quad \gamma_2^2 \leq \max\left\{\omega_2 \frac{1}{\kappa}, \omega_2 \left(\frac{1}{\kappa}\right)^{1-n/2}\right\}.$$

Using Lemma 3.1 and working on  $H_r := W_r^{1,2}(\mathbb{R}^n) \times W_r^{1,2}(\mathbb{R}^n)$ , it is possible to use critical point theory to find a minimum, respectively a mountain-pass critical point, for  $I_\lambda$  on  $M_\lambda$  provided  $\lambda > \Lambda'$ , respectively  $\lambda < \Lambda$ . Then an additional symmetry argument and the maximum principle allow us to prove the following result.

**THEOREM 3.2** [6]. (i) *If  $\lambda > \Lambda'$ , then (2) has a radial ground state  $(u, v) \in H_r$ , with  $u > 0$ ,  $v > 0$ .*

(ii) *If  $\lambda < \Lambda$ , then (2) has a radial bound state  $(u, v)$  different from  $(U_1, 0)$  and  $(0, U_2)$ . Furthermore, if  $\lambda > 0$  then  $u > 0$ ,  $v > 0$ .*

Similar results with slightly different estimates on  $\Lambda$ ,  $\Lambda'$  have been found in [16, 19] and [12], where the case indimension  $n = 1$  is also considered. For some related results dealing with (3) we refer to [20].

#### 4. BOUND AND GROUND STATES OF (4)

In this section we deal with the linearly coupled system (4). The specific feature of these systems is that it possesses a very rich set of solutions with various different behavior. The section is divided into several subsections.

##### 4.1. Explicit Solutions

First of all, there are two families of explicit solutions of (4): the Symmetric States in which  $v = u$  and the Antisymmetric States in which  $v = -u$ . The former ones solve the single equation

$$-\Delta u + (1 - \lambda)u = u^3,$$

the latter ones the equation

$$-\Delta u + (1 + \lambda)u = u^3$$

Therefore, letting  $U_\omega(x) = \sqrt{\omega}U(\sqrt{\omega}x)$  denote the solution of minimal energy of  $-\Delta u + \omega u = u^3$ ,  $\omega > 0$ , the Symmetric States exist for  $\lambda \in [0, 1)$  and are given by the pairs

$$(U_{1-\lambda}, U_{1-\lambda}), \quad (0 \leq \lambda < 1),$$

while the Antisymmetric States exist for all  $\lambda \geq 0$  and are given by the pairs

$$(U_{1+\lambda}, -U_{1+\lambda}), \quad (\lambda \geq 0).$$

#### 4.2. Secondary Bifurcations

In [1] it has been proved in dimension  $n = 1$  that for  $\lambda = 3/5$  there is a secondary bifurcation from the symmetric states. Moreover, using the Implicit Function theorem it is easy to check that a branch of solutions emanates at  $\lambda = 0$  from  $(U, 0)$ . We suspect that such a branch can be continued into the bifurcation point  $\lambda = 3/5$ . In addition, there is a numerical evidence that for  $\lambda = 1$  there is a secondary bifurcation from the Antisymmetric States, see [1]. A rigorous proof of this result has not jet given. A bifurcation diagram of these solutions is reported in Fig. 1 below.

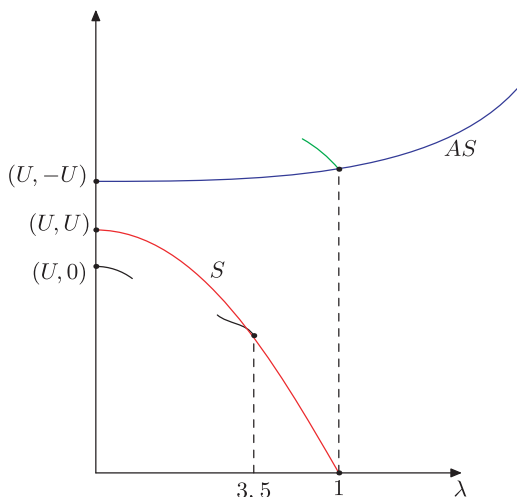


Figure 1.  $S$  denotes the family of the symmetric states,  $AS$  the antisymmetric ones.

The remarkable fact is that the solutions emanating from  $(U, -U)$  at  $\lambda = 1$  do not behave like a soliton but have, in dimension  $n = 1$ , two bumps whose peaks move to infinity as  $\lambda \rightarrow 0$ .

#### 4.3. Existence of Multi-bump Solutions for $\lambda \sim 0$

Using perturbation arguments it is possible to give a rigorous proof of the existence of a family of multi-bump solutions of (4) for small  $\lambda$ . Referring to [7] for more details, one can show:

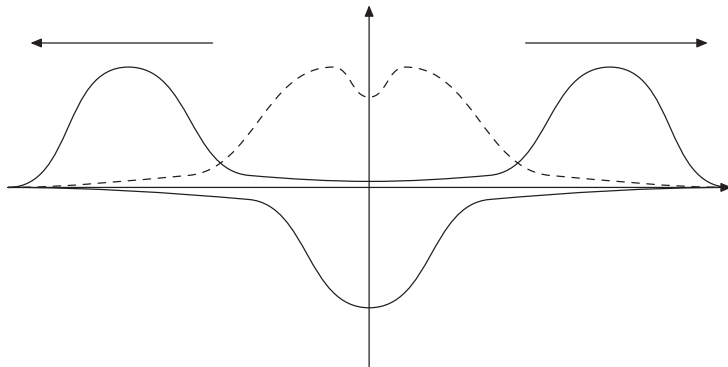


Figure 2. Multi-bump solutions in dimension  $n = 1$ .

**THEOREM 4.1.** *Let  $n = 1$ . For  $\lambda = \varepsilon \sim 0$ , (4) has a solution  $(u_\varepsilon, v_\varepsilon) \in W^{1,2}(\mathbb{R}) \times W^{1,2}(\mathbb{R})$  of the type  $u_\varepsilon \sim U(x + \xi_\varepsilon) + U(x - \xi_\varepsilon)$ ,  $v_\varepsilon \sim -U(x)$  as  $\varepsilon \sim 0$ , with*

$$-\frac{\log \varepsilon}{1 + \delta} < \xi_\varepsilon < -\log \varepsilon.$$

The case  $n = 2, 3$  requires some further notation. Let  $\mathcal{P}$  be a regular polygons in  $\mathbb{R}^2$  or a Platonic solid in  $\mathbb{R}^3$ , centered at  $x = 0$ . Let  $\{p_1, \dots, p_m\}$  be the vertices of  $\mathcal{P}$ ,  $s$  the sides and  $r$  the rays, we will suppose that

$$(P) \quad s = \min\{|p_i - p_j| : i \neq j\} > r = |p_1|.$$

In  $\mathbb{R}^2$  assumption (P) is satisfied by the regular polygons with less than 6 sides while in  $\mathbb{R}^3$  by all the platonic solids but the dodecahedron, in which

$$s < r = \frac{s\sqrt{3}}{4}(1 + \sqrt{5}).$$

In the case of dimension  $n = 1$  we understand that  $\mathcal{P}$  contains the symmetric intervals  $[-p, p]$ .

**THEOREM 4.2.** *Let  $n = 2, 3$ . If (P) holds, then for every  $\lambda = \varepsilon \sim 0$ , (4) has a solution  $(u_{1,\varepsilon}, u_{2,\varepsilon}) \in W^{1,2}(\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$  such that  $u_{2,\varepsilon} \sim -U(x)$ , while  $u_{1,\varepsilon}$  has maxima near  $\xi_\varepsilon p_i$ , where  $\xi_\varepsilon$  satisfies:*

$$\xi_\varepsilon \sim \frac{\log(1/\varepsilon)}{s-r}.$$

**REMARK 4.3.** In a single NLS equation the existence of multi-bump solutions is usually due to the presence of a suitable external potential depending upon  $x$ . See e.g. [11, 15, 17] and references therein. The new feature of the previous theorems is that system (4) is autonomous. Roughly, one can solve the second equation with respect to  $v$ . If  $v = K(x, u)$  denotes such a solution and we substitute such a  $v$  in the first equation of the system, we obtain a NLS equation with a non-local term, depending on  $x$ , which plays the role of the external potential.  $\square$

As anticipated before, the proof of these theorems relies on perturbation methods. One looks for  $\xi \in \mathbb{R}^n$  and  $(w_1, w_2) \in H$ , in such a way that the pair  $(u, v) \in H$  of the form

$$u = U(x + \xi) + U(x - \xi) + w_1, \quad v = -U(x) + w_2,$$

solves (4). Using a Lyapunov-Schmidt reduction, in a variational setting, one first finds  $w_1, w_2$  on the orthogonal to the manifold  $\{U(x + \xi) + U(x - \xi), -U(x) : \xi \in \mathbb{R}^n\}$ . Substituting  $w_1, w_2$  in the bifurcation equation, one is lead to study a finite dimensional functional, depending on  $\xi$ , whose critical points give rise to the solutions we are looking for. It is worth pointing out that these critical points can be found because we are working close to the Antisymmetric States, otherwise the arguments does not work. Actually, since the leading part of the second component is  $-U(x)$  and not  $U(x)$ , the finite dimensional functional contains two competing parts which balance each other and give rise to the existence of a critical point.

#### 4.4. Properties of the Ground States of (4)

For the results we will discuss in later on it is convenient to state a Lemma with the properties of the Ground States of (4). The proof can be found in [5].

If  $(u_\lambda, v_\lambda)$  denotes a Ground State of (4) we set

$$m_\lambda = I_\lambda(u_\lambda, v_\lambda).$$

**LEMMA 4.4.** (i) *For any  $\lambda \in [0, 1)$  there exist a ground state of (4) which, up to translation, is Steiner-symmetric. Furthermore,  $u_\lambda \cdot v_\lambda > 0$ .*

(ii) *The map  $[0, 1) \ni \lambda \mapsto m_\lambda$  is continuous and strictly decreasing and there holds*

$$(5) \quad \lim_{\lambda \rightarrow 0} m_\lambda = m_0 = I_0(U, 0) = I_0(0, U).$$

Since  $m_0 < I_0(U, U) = \lim_{\lambda \rightarrow 0} I_\lambda(U_{1-\lambda}, U_{1-\lambda})$ , (5) implies there exist  $\delta > 0$  such that

$$m_\lambda < I_\lambda(U_{1-\lambda}, U_{1-\lambda}), \quad \forall 0 < \lambda < \delta.$$

- Therefore for  $\lambda \in (0, \delta)$  the ground states of (4) are not the Symmetric States but they belong to the branch bifurcating from  $(U, 0)$ . On the other hand:
- (iii) There exist  $\delta' > 0$  such that for  $\lambda \in (1 - \delta', 1)$  the ground states of (4) are the Symmetric States  $(U_{1-\lambda}, U_{1-\lambda})$ . In particular,

$$\lim_{\lambda \rightarrow 1} m_\lambda = 0.$$

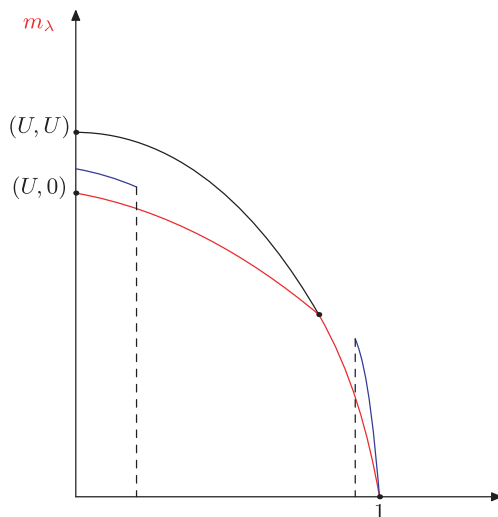


Figure 3. For  $\lambda \sim 0$  and  $\lambda \sim 1$  the bold line is the branch of the Ground States.

## 5. LINEARLY COUPLED NON-AUTONOMOUS SYSTEMS

In this section we report the results of [5] dealing with the following class of linearly coupled NLS systems like

$$(6) \quad \begin{cases} -\Delta u + u = (1 + a(x))u^3 + \lambda v, \\ -\Delta v + v = (1 + b(x))v^3 + \lambda u, \end{cases}$$

whose solutions are the critical points of

$$J_\lambda(u, v) = I_\lambda(u, v) - \frac{1}{4} \int (a(x)u^4 + b(x)v^4) dx.$$

In the sequel we will always assume that



$$(7) \quad a, b \in L^\infty(\mathbb{R}^n), \quad \lim_{|x| \rightarrow \infty} a(x) = \lim_{|x| \rightarrow \infty} b(x) = 0,$$

and

$$(8) \quad \inf_{\mathbb{R}^n} \{1 + a(x)\} > 0, \quad \inf_{\mathbb{R}^n} \{1 + b(x)\} > 0.$$

As usual, since we cannot anymore work in the space of symmetric functions, the main difficulty is the lack of compactness. One can show:

LEMMA 5.1. *The Palais-Smale, (PS) in short, compactness condition is verified at every level  $c < m_\lambda$ , namely at every level smaller than the ground state of the limit functional  $I_\lambda(u, v)$ .*

Our first result deals with the case in which  $a + b \geq 0$ . Let  $c_\lambda$  be the Mountain-Pass level of  $J_\lambda$ .

LEMMA 5.2. *If  $a(x) + b(x) \geq 0$  then  $c_\lambda < m_\lambda$ .*

From Lemmas 5.1 and 5.2 it follows immediately that that  $c_\lambda$  is a critical level, whence

THEOREM 5.3. *Suppose that (7) and (8) hold. If  $a(x) + b(x) \geq 0$ , then  $\forall 0 < \lambda < 1$ , (6) has a positive Ground State.*

Theorem 5.3 is the counterpart of a similar result which holds for the single NLS

$$(9) \quad -\Delta u + u = (1 + a(x))u^3.$$

Our second result is different. The new feature is that we assume only that  $a \geq 0$ , while  $b$  can be arbitrary.

THEOREM 5.4. *Suppose that (7) and (8) hold. If  $a(x) \geq 0$ ,  $a \not\equiv 0$ , then  $\exists \lambda^* \in (0, 1)$  depending only on  $a$ , such that (6) has a positive Ground State for every  $\lambda \in (0, \lambda^*]$ .*

To prove this theorem we let  $m_a$  denote the Ground State level of (9). If  $a \geq 0$ ,  $a \not\equiv 0$  then  $\gamma_a$  is achieved at a Ground State  $z_a$  and there holds

$$\gamma_a = I(z_a) < I(U) = m_{\lambda=0}.$$

In addition, still using the fact that  $a \geq 0$ ,  $a \not\equiv 0$ , one readily shows that

$$c_\lambda < m_a.$$

In conclusion we can state the following lemma.

LEMMA 5.5. *If  $a(x) \geq 0$ ,  $a \not\equiv 0$ , then  $c_\lambda < m_a$ . Moreover,  $m_a < m_{\lambda=0}$  and hence  $c_\lambda < m_\lambda$  for  $\lambda \sim 0$ .*

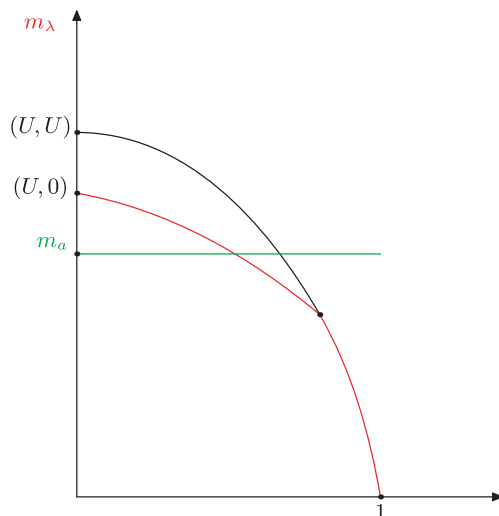


Figure 4. The bold line is the branch of the Ground States.

It is clear that from Lemma 5.5 and Lemma 5.1 we can conclude that  $I_\lambda$  has a critical point for  $\lambda \in (0, \lambda^*]$  which give rise to a positive Ground State of (6).

Our last Theorem deals with the case in which both  $a$  and  $b$  are smaller or equal than zero and is the counterpart for a single NLS of the results proved in [9]. In this case the situation is more difficult because  $c_\lambda = m_\lambda$  and  $c_\lambda$  is not a critical level of  $J_\lambda$ . Actually the solutions we find are Bound States and a different min-max procedure is in order. For this, we need to assume that

(B1)  $m_\lambda$  is an isolated critical level of  $I_\lambda$ .

LEMMA 5.6. (B1) holds for  $\lambda \sim 0$  as well as for  $\lambda \sim 1$ .

PROOF. (Sketch) As  $\lambda \sim 0$ , (ii) of Lemma 4.4 implies that the Ground States belong to the branch bifurcating from  $(U, 0)$ , which satisfies a non-degeneracy condition. This readily yields (B1). Similarly, as  $\lambda \sim 1$ , (iii) of Lemma 4.4 implies that  $m_\lambda$  is achieved at the Symmetric States  $(U_{1-\lambda}, U_{1-\lambda})$ . It is possible to show that also this branch is non-degenerate and therefore (B1) holds.  $\square$

The fact that  $m_\lambda$  is isolated provides us with a range of values greater than  $m_\lambda$  such that the (PS) condition holds:

LEMMA 5.7. There exists  $\delta > 0$  such that  $I_\lambda$  satisfies the (PS) condition at any level  $d \in (m_\lambda, m_\lambda + \delta)$ .

Finally, if  $a \leq 0$ ,  $b \leq 0$  and  $\max\{|a|_\infty, |b|_\infty\} \ll 1$ , we can use the definition of “barycenters”, see [10], to define a min-max level  $\tilde{c}_\lambda$  such that  $m_\lambda < \tilde{c}_\lambda < m_\lambda + \delta$ .

**THEOREM 5.8.** *Suppose that (7) and (8) hold. If  $a \leq 0$ ,  $b \leq 0$ ,  $a + b \neq 0$ , there exist  $0 < \lambda_1 \leq \lambda_2 < 1$  such that (6) has a positive Bound State for every  $\lambda \in (0, \lambda_1) \cup (\lambda_2, 1)$ , provided  $\max\{|a|_\infty, |b|_\infty\}$  is sufficiently small.*

The preceding results can be improved in dimension  $n = 1$  dealing with

$$(10) \quad \begin{cases} -u'' + u = (1 + \varepsilon a(x))u^3 + \lambda v, \\ -v'' + v = (1 + \varepsilon b(x))v^3 + \lambda u. \end{cases}$$

Here

$$J_\lambda(u, v) = I_\lambda(u, v) - \varepsilon \frac{1}{4} \int (au^4 + bv^4) dx$$

and the lack of compactness can be bypassed by using perturbation methods, cfr. [8]. Precisely, there holds

**THEOREM 5.9** [3]. *Suppose that*

$$a, b \in L^\infty(\mathbb{R}), \quad \lim_{|x| \rightarrow \infty} a(x) = \lim_{|x| \rightarrow \infty} b(x) = 0.$$

*Then,  $\forall \lambda \in (0, 1)$ ,  $\lambda \neq 3/5$ , (10) has a solution close to  $(U_{1-\lambda}, U_{1-\lambda})$ , for  $\varepsilon$  sufficiently small.*

**PROOF.** (Sketch) The Symmetric states  $(U_{1-\lambda}, U_{1-\lambda})$  are, for every fixed  $\lambda \in (0, 1)$ , critical points of  $I_\lambda$ . Since  $I_\lambda$  is translation invariant, also  $(U_{1-\lambda}(x + \xi), U_{1-\lambda}(x + \xi))$  are solutions for all  $\xi \in \mathbb{R}$ . Here, taking advantage of the fact that we are working in dimension  $n = 1$ , it is possible to sharpen the non-degeneracy arguments by proving that the Symmetric States are non-degenerate not only for  $\lambda \sim 0$ , but for every  $\lambda \neq 3/5$ . Then we can use, as in the proof of Theorems 4.1 and 4.2, a Lyapunov-Schmidt reduction in order to show that there exist solutions of (10), for  $\varepsilon \sim 0$ , and the result follows.  $\square$

**REMARK 5.10.** Taking advantage to be in dimension  $n = 1$  it is possible to give a precise description of the Ground State levels of  $I_\lambda$ , namely:

- (i) If  $0 < \lambda < \frac{3}{5}$ , then  $m_\lambda < I_\lambda(U_{1-\lambda}, U_{1-\lambda})$ .
- (ii) If  $\frac{3}{5} \leq \lambda < 1$ , then  $m_\lambda = I_\lambda(U_{1-\lambda}, U_{1-\lambda})$ .

Moreover:

- (iii) (B1) holds for all  $\lambda \in (0, 1)$  but  $\lambda = \frac{3}{5}$ .

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