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Partial Differential Equations — *Existence and multiplicity results for a weighted p-Laplace equation involving Hardy potentials and critical nonlinearities*, by ROBERTA MUSINA, communicated by Carlo Sbordone on 13 February 2009.

ABSTRACT. — We study a class of elliptic problems involving weighted p-Laplace operators, critical growths and Hardy potentials. The main motivation lies in some Hardy-Sobolev type inequalities that were proved by Caffarelli-Kohn-Nirenberg in 1984.

KEY WORDS: Variational methods; critical growth; weighted L^p-Laplace operator; Hardy inequalities; Caffarelli-Kohn-Nirenberg inequalities; breaking symmetry.

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Dedicated to Renato Caccioppoli on the 50th anniversary of his death.

1. INTRODUCTION

This paper deals with a class of variational problems involving weights that are powers of the distance from the origin. More precisely, we look for nonnegative weak solutions to

$$\begin{cases} -\mathrm{div}(|x|^a |\nabla u|^{p-2} \nabla u) = \lambda |x|^{a-p} u^{p-1} + |x|^{-b_q} u^{q-1} \quad \text{on } \mathbb{R}^N \\ \int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, dx < \infty, \end{cases} \tag{1.1}$$

where

$$\begin{gathered} 1 < p < q, \quad a > p - N, \quad b_q = N - q \frac{N - p + a}{p}, \\ \lambda < \left(\frac{N - p + a}{p} \right)^p, \quad q \le p^* := \frac{Np}{N - p} \quad \text{if } p < N. \end{gathered} \tag{1.2}$$

Much interest has been payed to problems of the form (1.1). Assume for instance that $a = 0$. Then $-\mathrm{div}(|\nabla u|^{p-2} \nabla u) = -\Delta_p u$ is the standard p-Laplace operator. Let $p < N$, and notice that $b_{p^*} = N - p^*(N - p)/p = 0$. Positive solutions to

$$\begin{cases} -\Delta_p U = U^{p^*-1} \qquad \text{on } \mathbb{R}^N \\ \int_{\mathbb{R}^N} |\nabla U|^p \, dx < \infty \end{cases} \tag{1.3}$$

are explicitly known since the celebrated papers [1] by Aubin and [27] by Talenti. In particular, it turns out that $U > 0$ solves (1.3) if and only if U is an extremal for the Sobolev constant S_p.

A large number of papers deal with (1.1) and with similar variational problems. We quote for example [2], [5], [7]–[23], [25], [26], [28], [29] and references there-in. At our knowledge, all the available results for (1.1) require $p = 2$, or $\lambda = 0$, or $a = 0$.

The purpose of the present paper is twofold. We survey some of the results from [11] and [13] about the semilinear elliptic case $p = 2$, from [5], [15], [26], where $\lambda = 0$ is assumed, and from the appendix of the paper [14], that deals with a non-compact problem for the p-Laplace operator. In addition we prove new existence and multiplicity results by suitably adapting the arguments of [15] and [16].

The remaining of the present paper is organized as follows.

In Section 2 we point out the main features of problem (1.1).

In Section 3 we focus our attention on the existence of a ground state (see Section 2 for the definition) and of a radially symmetric nontrivial solution. The main results in this section are Theorems 3.1 and 3.4.

In Section 4 we compare the ground state and the radially symmetric solution. We report on the breaking symmetry results from [13], where $p = 2$, and from [5], [26], where $\lambda = 0$ is assumed. Then we take $2 \le p < q < p^*$ and we use the arguments in [16] to find out a region of parameters a, q and λ where breaking symmetry occurs. The main result in this section is Theorem 4.4.

Notation

We denote by c any constant $c \in \mathbb{R}_+ := (0, \infty)$ that depends only on fixed parameters.

Let $p > 1$ and let $N \ge 1$ be an integer. We set $p^* := \frac{Np}{N-p}$ if $p < N$ and $p^* = \infty$ if $p \ge N$. If $a \in \mathbb{R}$ we put

$$\lambda_{p,a} := \Big(\frac{N - p + a}{p}\Big)^p.$$

We denote by B_R the N-dimensional ball of radius R centered at the origin. The surface measure of $\mathbb{S}^{N-1} = \partial B_1$ is $\omega_N = |\partial B_1|$.

Let $X = (X, \|\cdot\|)$ be a Banach space. Then X' is its topological dual space. For any sequence g_h in X, we write $g_h \rightharpoonup g$ if g_h converges to $g \in X$ weakly, and $g_h \to g$ if $\|g_h - g\| \to 0$.

Let $q \in (1, +\infty)$, $\alpha \in \mathbb{R}$ and let Ω be a domain in $\mathbb{R}^N$. We denote by $L^q(\Omega; |x|^\alpha\, dx)$ the space of measurable functions u, such that $|x|^{\alpha/q} u \in L^q(\Omega)$.

For any exponent $p \in (1, N)$, the space $\mathcal{D}^{1,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to the L^p norm of $|\nabla u|$. It is well known that $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is continuously embedded into $L^{p^*}(\mathbb{R}^N)$. The explicit value of the Sobolev constant

$$S_p := \inf_{\substack{U \in \mathscr{D}^{1,p}(\mathbb{R}^N) \\ U \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla U|^p \, dx}{(\int_{\mathbb{R}^N} |U|^{p^*} \, dx)^{p/p^*}}$$

and of its minimizers were given in [1] and in [27].

2. Preliminaries

In this preliminary section we recall some well known integral inequalities and we describe the mean features of problem (1.1).

2.1. Hardy and Caffarelli-Kohn-Nirenberg Inequalities

Let $p > 1$ and $a > p - N$. The Hardy inequality states that

$$\lambda_{p,a} \int_{\mathbb{R}^N} |x|^{a-p} |u|^p \, dx < \int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, dx \tag{2.1}$$

for any $u \in C_c^\infty(\mathbb{R}^N)$. Thanks to (2.1), we can define the reflexive Banach space $\mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a \, dx)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the L^p-norm of $|x|^{a/p}|\nabla u|$. Notice that $\mathscr{D}^{1,p}(\mathbb{R}^N; dx) = \mathscr{D}^{1,p}(\mathbb{R}^N)$ if $p < N$ and $a = 0$.

We will deal also with the Caffarelli-Kohn-Nirenberg inequalities. Let $1 < p < q$ and assume $q \le p^*$ if $p < N$. Set $b_q := N - q(N - p + a)/p$. In [6] it is proved that there exists a constant $c = c(N, p, a, q) > 0$ such that

$$c\Big(\int_{\mathbb{R}^N} |x|^{-b_q} |u|^q \, dx\Big)^{p/q} \le \int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, dx \tag{2.2}$$

for any $u \in \mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a \, dx)$. If $\lambda < \lambda_{p,a}$, then inequalities (2.1) and (2.2) plainly imply that the infimum

$$S_p(a, \lambda, q) := \inf_{\substack{u \in \mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a \, dx) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, dx - \lambda \int_{\mathbb{R}^N} |x|^{a-p} |u|^p \, dx}{(\int_{\mathbb{R}^N} |x|^{-b_q} |u|^q \, dx)^{p/q}} \tag{2.3}$$

is positive. Notice that $S_p(a, 0, p) = \lambda_{p,a}$ and $S_p(0, 0, p^*) = S_p$ if $p < N$.

Assume that $S_p(a, \lambda, q)$ is attained by a function $u \in \mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a \, dx)$. Then u is nonnegative weak solution to (1.1), up to a multiplicative constant. The argument is nowadays standard and it will be omitted. Any solution to (1.1) which achieves $S_p(a, \lambda, q)$ is called *ground state*.

2.2. Lack of Compactness

In this section we describe some lack of compactness phenomena that may be observed in studying the minimization problem (2.3). In order to simplify notations we put

(2.4) $$n(u) := \int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, dx - \lambda \int_{\mathbb{R}^N} |x|^{a-p} |u|^p \, dx.$$

Notice that $n(u)^{1/p}$ is bounded from below and from above by the norm of u in $\mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a \, dx)$, by Hardy inequality. In particular, if $p = 2$ or if $\lambda = 0$ then $n(\cdot)^{1/p}$ is an (equivalent) norm in $\mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a \, dx)$.

The invariances of the functional $n(\cdot)$ and of the norm in $L^q(\mathbb{R}^N; |x|^{-b_q} \, dx)$ generate noncompact minimizing sequences. Assume for instance that $S_p(a, \lambda, q)$ is attained by a function $u \in \mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a \, dx)$. Take any sequence $t_h > 0$ and set

$$u_h(x) := t_h^{(N-p+a)/p} u(t_h x).$$

Since

$$n(u_h) = n(u), \quad \int_{\mathbb{R}^N} |x|^{-b_q} |u_h|^q \, dx = \int_{\mathbb{R}^N} |x|^{-b_q} |u|^q \, dx,$$

then u_h achieves $S_p(a, \lambda, q)$ for any h. Now we can easily exhibit noncompact minimizing sequences. Take for instance $t_h \to \infty$. Then the functions u_h *concentrate* at 0, that is, $|x|^a |\nabla u_h|^p \to 0$ in $L^1(\{|x| > R\})$ for any $R > 0$. Also *vanishing* may be produced: if $t_h \to 0$ then $|x|^a |\nabla u_h|^p \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$.

In the limiting case $p < N$ and $q = p^*$, the group of translations is responsible of additional and worst lack of compactness phenomena. For any $\varepsilon > 0$ choose a map $U_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ such that

$$S_p \le \frac{\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^p \, dx}{(\int_{\mathbb{R}^N} |U_\varepsilon|^{p^*} \, dx)^{p/p^*}} < S_p + \varepsilon.$$

Fix a point $x_0 \neq 0$ and put

$$U_{\varepsilon,h}(x) := h^{(N-p)/p} U_\varepsilon(h(x - x_0)).$$

Notice that

$$\lim_{\varepsilon \to 0} \lim_{h \to \infty} \frac{n(U_{\varepsilon,h})}{(\int_{\mathbb{R}^N} |x|^{Na/(N-p)} |U_{\varepsilon,h}|^{p^*} \, dx)^{p/p^*}} = S_p.$$

Test $S_p(a, \lambda, p^*)$ with $U_{\varepsilon,h}$ and pass to the limit to get

(2.5) $$S_p(a, \lambda, p^*) \le S_p.$$

This is a crucial inequality. Assume that $S_p(a, \lambda, p^*) = S_p$. If $\varepsilon_h \to 0$ is a suitably chosen sequence, then $U_{\varepsilon_h,h}$ approaches $S_p(a, \lambda, p^*)$. Notice that $U_{\varepsilon_h,h}$ concentrates at $x_0 \neq 0$ as $h \to \infty$. In addition, it can be proved that it blows-up an extremal for the Sobolev constant S_p. Actually, the infimum $S_p(a, \lambda, p^*)$ might be not achieved if equality holds in (2.5). This happens, for instance, when $a = 0$, $\lambda < 0$ and when $p = 2$, $a > 0$ (see Propositions 3.5 and 3.6). On the other hand, in the next section we will show that the strict inequality $S_p(a, \lambda, p^*) < S_p$ guarantees enough compactness and the existence of a minimizer.

3. Existence

The first result in this section provides sufficient conditions for the existence of a ground state.

THEOREM 3.1. *Let* $1 < p < q$, $a > p - N$ *and assume that* (1.2) *is satisfied.*

i) *If* $p \geq N$ *or if* $q < p^*$, *then* $S_p(a, \lambda, q)$ *is achieved.*
ii) *If* $p < N$, *then* $S_p(a, \lambda, p^*)$ *is achieved provided that* $S_p(a, \lambda, p^*) < S_p$.

Theorem 3.1 was already known for some special values of the parameters involved. We quote [11] for $p = 2$, [15] for $\lambda = 0$ and finally [14] for $a = 0$.

The proof in [11] is based on a helpful functional change that does not behaves nicely when $p \neq 2$. The argument adopted in [14] to handle the case $a = 0$ is based on a hard adaptation of the Concentration-Compactness Lemmata by P. L. Lions (see also [2], [28] for a noncompact problem with cylindrical weights). We notice also that in case $a < 0$ and $\lambda = 0$, a change of the x-variable reduces the problem to the case $a = 0$, where Schwarz symmetrization gives the existence of a ground state that is radially symmetric (see [18]). In general, when $a \geq 0$ or $\lambda \neq 0$ one can not look forward to a radially symmetric ground state (compare with the results in Section 4).

To prove Theorem 3.1 we follow the main ideas of the papers [22] and [15], that deal with a class of variational problems with spherical and cylindrical weights. In particular, the paper [22] is concerned with the semilinear case $p = 2$, while $\lambda = 0$ is assumed in [15]. The proofs in [22], [15] were inspired by arguments that have been developed by Sacks and Uhlenbeck in their seminal paper [24] on minimal spheres in a Riemannian manifold (see also [22] and [23] for a similar variational problem and [3], [4], [9] for the H-surfaces problem).

The strategy consists in selecting a "good" minimizing sequence via Ekeland's variational principle and rescaling argument. The proof of Theorem 3.1 turns out to be direct, self-contained and flexible.

The next Lemma is the main step in the proof.

LEMMA 3.2. *"ε-compactness lemma". Let* $1 < p < q$, $a > p - N$, $\lambda \in \mathbb{R}$ *and let* Ω *be a domain in* $\mathbb{R}^N$. *Assume that* (1.2) *is satisfied. Let* $u_h \rightharpoonup 0$ *be a sequence in* $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\, dx)$ *such that*

$$\limsup_{h \to \infty} \int_\Omega |x|^{-b_q} |u_h|^q \, dx < S_p(a, \lambda, q)^{q/(q-p)}, \tag{3.1}$$

$$-\mathrm{div}(|x|^a |\nabla u_h|^{p-2} \nabla u_h) - \lambda |x|^{a-p} |u_h|^{p-2} u_h = |x|^{-b_q} |u_h|^{q-2} u_h + f_h \quad \text{on } \Omega, \tag{3.2}$$

where $f_h \to 0$ *in* $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\, dx)'$. *Then* $u_h \to 0$ *in* $L^q_{\mathrm{loc}}(\Omega; |x|^{-b_q}\, dx)$.

PROOF. Let Ω' be any domain compactly contained in Ω. Take any nonnegative function $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \equiv 1$ on Ω'. Use Lemmata 1.1 and 1.2 in [15] to check that $\varphi^p u_h$ is an admissible test function for (3.2) and that

$$\int_{\mathbb{R}^N} |x|^a |\nabla u_h|^{p-2} \nabla u_h \cdot \nabla(\varphi^p u_h)\, dx = \int_{\mathbb{R}^N} |x|^a |\nabla(\varphi u_h)|^p\, dx + o(1).$$

Therefore

$$n(\varphi u_h) = \int_{\mathbb{R}^N} |x|^{-b_q} |u_h|^{q-p} |\varphi u_h|^p\, dx + o(1), \tag{3.3}$$

where $n(\cdot)$ is defined in (2.4). We notice that

$$n(\varphi u_h) \ge S_p(a,\lambda,q)\Big(\int_{\mathbb{R}^N} |x|^{-b_q} |\varphi u_h|^q\, dx\Big)^{p/q}$$

by (2.3). Then we use Hölder inequality to estimate the right-hand side of (3.3). In this way we get

$$\begin{aligned} S_p(a,\lambda,q)&\Big(\int_{\mathbb{R}^N} |x|^{-b_q} |\varphi u_h|^q\, dx\Big)^{p/q} \\ &\le \Big(\int_{\Omega} |x|^{-b_q} |u_h|^q\, dx\Big)^{(q-p)/q}\Big(\int_{\mathbb{R}^N} |x|^{-b_q} |\varphi u_h|^q\, dx\Big)^{p/q} + o(1). \end{aligned}$$

Therefore from (3.1) we infer

$$o(1) = \int_{\mathbb{R}^N} |x|^{-b_q} |\varphi u_h|^q\, dx \ge \int_{\Omega'} |x|^{-b_q} |u_h|^q\, dx,$$

as $\varphi \equiv 1$ on Ω'. Since Ω' was arbitrarily chosen, this proves that $u_h \to 0$ strongly in $L^q_{\mathrm{loc}}(\Omega; |x|^{-b_q}\, dx)$. □

PROOF OF THEOREM 3.1. Fix a small $\varepsilon_0 < S_p(a,\lambda,q)^{q/(q-p)}$ and let $n(\cdot)$ be the functional defined in (2.4). The proof will be carried out in two steps.

STEP 1. We claim that there exists a weakly convergent sequence u_h, such that

$$S_p(a,\lambda,q)^{q/(q-p)} = \int_{\mathbb{R}^N} |x|^{-b_q} |u_h|^q\, dx = n(u_h) + o(1), \tag{3.4}$$

$$\lim_{h\to\infty} \int_{B_2} |x|^{-b_q} |u_h|^q\, dx = \varepsilon_0, \tag{3.5}$$

$$-\mathrm{div}(|x|^a |\nabla u_h|^{p-2} \nabla u_h) - \lambda |u_h|^{p-2} u_h = |x|^{-b_q} |u_h|^{q-2} u_h + f_h \quad \text{on } \mathbb{R}^N, \tag{3.6}$$

where $f_h \to 0$ in $\mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a\, dx)'$. For, it suffices to use Ekeland's variational principle and to notice that the ratio

$$\frac{n(u_h)}{(\int_{\mathbb{R}^N} |x|^{-b_q} |u_h|^q \, dx)^{p/q}}$$

is homogeneous and invariant under rescaling. The sequence u_h is bounded in $\mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a \, dx)$ by (3.4) and by Hardy's inequality. Thus we can assume that $u_h \rightharpoonup u$ in $\mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a \, dx)$, for some $u \in \mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a \, dx)$.

STEP 2. We claim that $u \neq 0$ and that u achieves $S_p(a, \lambda, q)$.

Assume by contradiction that $u_h \rightharpoonup 0$. Then Lemma 3.2 and (3.5) imply that $u_h \to 0$ in $L^q_{\text{loc}}(B_2; |x|^{-b_q} \, dx)$. If $q < p^*$ then $u_h \to 0$ in $L^q(B_2; |x|^{-b_q} \, dx)$ by Rellich Theorem. This conclusion clearly contradicts (3.5). Hence we assume $p < N$ and $q = p^*$. Since $u_h \to 0$ in $L^{p^*}_{\text{loc}}(B_2; |x|^{Na/(N-p)} \, dx)$, then $\int_{B_1} |x|^{Na/(N-p)} |u_h|^{p^*} \, dx \to 0$. From (3.5) we infer

$$\lim_{h\to\infty} \int_K |x|^{Na/(N-p)} |u_h|^{p^*} \, dx = \varepsilon_0 > 0, \tag{3.7}$$

where $K = \{x \in \mathbb{R}^N \mid 1 < |x| < 2\}$. Choose a nonnegative smooth function $\varphi \in C^\infty_c(\mathbb{R}^N)$ such that $\varphi \equiv 0$ in a neighbourhood of 0 and $\varphi \equiv 1$ on K. As in the proof of Lemma 3.2 we can use $\varphi^p u_h$ as test function in (3.6) to get

$$n(\varphi u_h) \le S_p(a, \lambda, p^*) \Big(\int_{\mathbb{R}^N} |x|^{Na/(N-p)} |\varphi u_h|^{p^*} \, dx \Big)^{p/p^*} + o(1). \tag{3.8}$$

Notice that

$$|x|^{(a-p)/p} |\varphi u_h| \to 0, \quad |x|^{a/p} |\nabla(\varphi u_h)| - |\nabla(|x|^{a/p} \varphi u_h)| \to 0 \quad \text{in } L^p(\mathbb{R}^N)$$

by Rellich Theorem, as φ has compact support in $\mathbb{R}^N \backslash \{0\}$. Thus

$$n(\varphi u_h) = \int_{\mathbb{R}^N} |\nabla(|x|^{a/p} \varphi u_h)|^p \, dx + o(1) \ge S_p \Big(\int_{\mathbb{R}^N} |x|^{Na/(N-p)} |\varphi u_h|^{p^*} \, dx \Big)^{p/p^*} + o(1)$$

by the Sobolev inequality. In particular, from (3.8) it follows that

$$\begin{aligned} S_p &\Big(\int_{\mathbb{R}^N} |x|^{Na/(N-p)} |\varphi u_h|^{p^*} \, dx \Big)^{p/p^*} \\ &\le S_p(a, \lambda, p^*) \Big(\int_{\mathbb{R}^N} |x|^{Na/(N-p)} |\varphi u_h|^{p^*} \, dx \Big)^{p/p^*} + o(1). \end{aligned}$$

Since $S_p(a, \lambda, p^*) < S_p$ and since $\varphi \equiv 1$ on K, we infer

$$\int_K |x|^{Na/(N-p)} |u_h|^{p^*} \, dx \le \int_{\mathbb{R}^N} |x|^{Na/(N-p)} |\varphi u_h|^{p^*} \, dx = o(1),$$

which contradicts (3.7). Thus, $u_h \rightharpoonup u \neq 0$ in $\mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a \, dx)$.

Next, from (3.6) we get that u solves

$$-\mathrm{div}(|x|^a|\nabla u|^{p-2}\nabla u) - \lambda|x|^{a-p}|u|^{p-2}u = |x|^{-b_q}|u|^{q-2}u \quad \text{on } \mathbb{R}^N.$$

In particular

$$S_p(a,\lambda,q)\Big(\int_{\mathbb{R}^N}|x|^{-b_q}|u|^q\,dx\Big)^{p/q} \le n(u) = \int_{\mathbb{R}^N}|x|^{-b_q}|u|^q\,dx$$

by (3.2). Since $u \neq 0$, using also (3.4) we get

$$n(u) = \int_{\mathbb{R}^N}|x|^{-b_q}|u|^q\,dx \ge S_p(a,\lambda,q)^{q/(q-p)} = \int_{\mathbb{R}^N}|x|^{-b_q}|u_h|^q\,dx.$$

But then the lower semicontinuity of the norm in $L^q(\mathbb{R}^N;|x|^{-b_q}\,dx)$ implies

$$n(u) = \int_{\mathbb{R}^N}|x|^{-b_q}|u|^q\,dx = S_p(a,\lambda,q)^{q/(q-p)}.$$

Therefore u achieves $S_p(a,\lambda,q)$. The proof is complete. □

Remark 3.3. Assume $N = 1 < p < q$, $a > p-1$, $\lambda < |(1-p+a)/p|^p$. If $\underline{u}$ is a minimizer for $S_p(a,\lambda,q)$, then $\underline{u}$ vanishes on a half-line. For, argue as in the proof of Lemma A.7 in [15]. As a corollary to Theorem 3.1 we get the existence of a positive solution $u \in \mathscr{D}^{1,p}(\mathbb{R}_+;s^a\,ds)$ to the ODE problem

$$\begin{cases} -(s^a|u'|^{p-2}u')' = \lambda s^{a-p}u^{p-1} + s^{-b_q}u^{q-1} & \text{on } \mathbb{R}_+ \\ u(0) = 0. \end{cases}$$

The arguments in Section 4 of [15] can be used in order to prove existence for any $a \neq 1-p$.

In the next existence result we take advantage of the invariance of problem (1.1) with respect to rotations. Notice that no upper bounds on q are needed.

Theorem 3.4. *Let $N \ge 2$, $1 < p < q$ and $a > p-N$. Then problem* (1.1) *has a nonnegative radially symmetric solution $\underline{u} \neq 0$ for any $\lambda < \lambda_{p,a}$.*

Proof. We introduce the space $\mathscr{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N;|x|^a\,dx)$ of radially symmetric maps in $\mathscr{D}^{1,p}(\mathbb{R}^N;|x|^a\,dx)$. Use (2.1) to check that $C^\infty_c(\mathbb{R}^N\backslash\{0\}) \cap \mathscr{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N;|x|^a\,dx)$ is dense in $\mathscr{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N;|x|^a\,dx)$ (see for example [22]). Fix any function $u = u(|x|) \in C^\infty_c(\mathbb{R}^N\backslash\{0\})$. Since

$$\int_{\mathbb{R}^N}|x|^a|\nabla u|^p\,dx = \omega_N\int_0^\infty r^{a+N-1}|u'|^p\,dr,$$

then the Caffarelli-Kohn-Nirenberg inequalities (2.2) and the Hardy inequality (2.1) (with $N = 1$ and a replaced by $a+N-1 > p-1$) imply that the infimum

$$S_{p,\mathrm{rad}}(a,\lambda,q) := \inf_{\substack{u\in\mathscr{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N;|x|^a\,dx)\\ u\neq 0}} \frac{\int_{\mathbb{R}^N}|x|^a|\nabla u|^p\,dx - \lambda\int_{\mathbb{R}^N}|x|^{a-p}|u|^p\,dx}{\left(\int_{\mathbb{R}^N}|x|^{-b_q}|u|^q\,dx\right)^{p/q}}$$

is positive. We claim that $S_{p,\mathrm{rad}}(a,\lambda,q)$ is achieved on $\mathscr{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N;|x|^a\,dx)$. The proof goes as for Theorem 3.1. Only minor modifications are needed. In particular, one has to take radially symmetric cut-off functions φ. The conclusion follows via standard arguments. □

3.1. *On the Inequality $S_p(a,\lambda,p^*) < S_p$*

Let $p\in(1,N)$, $a>p-N$, $\lambda<\lambda_{p,a}$ and take $q=p^*$. If in addition $S_p(a,\lambda,p^*)<S_p$, then problem (1.1) has a ground state solution, by Theorem 3.1.

In this section we collect a few sufficient conditions for $S_p(a,\lambda,p^*)<S_p$. We start by recalling some known facts.

PROPOSITION 3.5. *Let $a=0$. Then $S_p(0,\lambda,p^*)<S_p$ if and only if $0<\lambda<\lambda_{p,0}$. If $\lambda<0$ then $S_p(0,\lambda,p^*)$ is not achieved.*

The results in Proposition 3.5 were already noticed in [29] for $p=2$ and in Appendix A of [14] for general exponents $p\in(1,N)$. Let us check it for completeness. By (2.5) it turns out that $S_p(0,\lambda,p^*)\le S_p$ for any λ. Thus equality holds if $\lambda\le 0$. Notice that $S_p(0,\lambda,p^*)$ can not be achieved if $\lambda<0$, as S_p is achieved. Next, assume $\lambda>0$ and test $S_p(0,\lambda,p^*)$ with the radially symmetric function

$$U(x) = (1+|x|^{p/(p-1)})^{-(N-p)/p}. \tag{3.9}$$

By the results in [1], [27], it turns out that U achieves the Sobolev constant S_p. The strict inequality $S_p(0,\lambda,p^*)<S_p$ immediately follows.

When $p=2$, Proposition 3.5 easily implies the next result.

PROPOSITION 3.6 ([11]). *Let $p=2$. Then $S_2(a,\lambda,2^*)<S_2$ if and only if*

$$\lambda_{2,a}-\lambda_{2,0}\le\lambda<\lambda_{2,a}. \tag{3.10}$$

If $\lambda<\lambda_{2,a}-\lambda_{2,0}$ then $S_2(a,\lambda,2^*)$ is not achieved.

To prove Proposition 3.6 it is convenient to use the functional transform

$$u\in\mathscr{D}^{1,2}(\mathbb{R}^N;|x|^a\,dx)\to|x|^{a/2}u\in\mathscr{D}^{1,2}(\mathbb{R}^N).$$

It turns out that the minimization problems for $S_2(a,\lambda,2^*)$ and for $S_2(0,\mu,2^*)$ are equivalent, provided that λ and μ satisfy the identity $\lambda_{2,a}-\lambda=\lambda_{2,0}-\mu$. In particular (3.10) holds if and only if $0<\mu<\lambda_{2,0}$. The conclusion readily follows from Proposition 3.5.

As far as we know, if $p \neq 2$ no similar transform is available and very few is known. In the next result we take $p \in (1, N)$, $a < 0$ and we improve Theorem 0.2 in [15].

Proposition 3.7. *Let $p - N < a < 0$. Then there exists $\lambda^* < \lambda_{p,a}\frac{N-1}{N-p}\frac{ap}{N+a}$ such that $S_p(a, \lambda, p^*) < S$ if $\lambda_* < \lambda < \lambda_{p,a}$.*

Proof. Set

$$\lambda_0 := \lambda_{p,a}\frac{N-1}{N-p}\frac{ap}{N+a},$$

and let U be the Aubin-Talenti function defined in (3.9). Notice that

$$\int_{\mathbb{R}^N} |x|^a|\nabla U|^p\,dx - \lambda_0\int_{\mathbb{R}^N} |x|^{a-p}|U|^p\,dx < \Big(1 - \frac{N-1}{N-p}\frac{ap}{N+a}\Big)\int_{\mathbb{R}^N} |x|^a|\nabla U|^p\,dx$$

by Hardy inequality. Hence

$$S_p(a, \lambda_0, p^*) < \frac{N}{N-p}\,\frac{N-p-a(p-1)}{N+a}\,\frac{\int_{\mathbb{R}^N} |x|^a|\nabla U|^p\,dx}{(\int_{\mathbb{R}^N} |x|^{Na/(N-p)}|U|^{p^*}\,dx)^{p/p^*}}.$$

On the other hand, it has been shown in [15], proof of Theorem 0.2, that

$$\frac{\int_{\mathbb{R}^N} |x|^a|\nabla U|^p\,dx}{(\int_{\mathbb{R}^N} |x|^{Na/(N-p)}|U|^{p^*}dx)^{p/p^*}} \le S_p\frac{N-p}{N}\,\frac{N+a}{N-p-a(p-1)}.$$

Therefore $S_p(a, \lambda_0, p^*) < S_p$. To conclude, notice that $S_p(a, \lambda_0 - \varepsilon, p^*) < S_p$ if $\varepsilon > 0$ is small enough, and that the map $\lambda \to S_p(a, \lambda, p^*)$ is non-increasing. □

When a is positive an additional restriction is needed.

Proposition 3.8. *Let $0 < a < \frac{N-p}{p-1}$. Then there exists $\lambda^* < a\big(\frac{N-p}{p-1}\big)^{p-1}$ such that $S_p(a, \lambda, p^*) < S$ if $\lambda_* < \lambda < \lambda_{p,a}$.*

Proof. Set

$$c_N := \Big(\frac{N-p}{p-1}\Big)^{p-1},$$

and notice that $ac_N < \lambda_{p,a}$ for any $a < \frac{N-p}{p-1}$. As in Proposition 3.7 it suffices to prove that $S_p(a, ac_N, p^*) < S_p$. The strategy again consists in testing $S_p(a, \lambda, p^*)$ with the Aubin-Talenti function U defined in (3.9). In order to simplify notations we set

$$\Phi(x) = 1 + |x|^{p/(p-1)},$$

in such a way that $U = \Phi^{-(N-p)/p}$ and $U^{p^*} = \Phi^{-N}$. Notice that

$$-\Delta_p U = Nc_N U^{p^*-1} \quad \text{on } \mathbb{R}^N,$$
$$\int_{\mathbb{R}^N} |\nabla U|^p\, dx = Nc_N \int_{\mathbb{R}^N} |U|^{p^*}\, dx = Nc_N \int_{\mathbb{R}^N} \Phi^{-N}\, dx = (Nc_N)^{1-N/p} S_p^{N/p}.$$

In particular

$$\int_{\mathbb{R}^N} \Phi^{-N}\, dx = (Nc_N)^{-N/p} S_p^{N/p}. \tag{3.11}$$

For $x \neq 0$ we compute

$$\begin{aligned} -\mathrm{div}(|x|^a |\nabla U|^{p-2} \nabla U) &= (-\Delta_p U + ac_N U^{p^*-1}\Phi)|x|^a \\ &= Nc_N(1 + aN^{-1}\Phi)|x|^a U^{p^*-1}. \end{aligned}$$

Therefore $U \in \mathscr{D}^{1,p}(\mathbb{R}^N; |x|^a\, dx)$ since $a < (N-p)/(p-1)$, and

$$\int_{\mathbb{R}^N} |x|^a |\nabla U|^p\, dx = Nc_N \int_{\mathbb{R}^N} |x|^a \Phi^{-N}(1 + aN^{-1}\Phi)\, dx. \tag{3.12}$$

We notice also that

$$\int_{\mathbb{R}^N} |x|^{a-p} |U|^p\, dx = \int_{\mathbb{R}^N} |x|^a \Phi^{-N}(|x|^{-1}\Phi)^p\, dx. \tag{3.13}$$

Next, use Hölder inequality to estimate

$$\int_{\mathbb{R}^N} |x|^a \Phi^{-N}\, dx \le \Big(\int_{\mathbb{R}^N} |x|^{Na/(N-p)} \Phi^{-N}\, dx\Big)^{p/p^*} \Big(\int_{\mathbb{R}^N} \Phi^{-N}\, dx\Big)^{p/N}.$$

Thus from (3.11) we infer

$$\Big(\int_{\mathbb{R}^N} |x|^{Na/(N-p)} |U|^{p^*}\, dx\Big)^{-p/p^*} \le S_p \Big(Nc_N \int_{\mathbb{R}^N} |x|^a \Phi^{-N}\, dx\Big)^{-1}. \tag{3.14}$$

From (3.12), (3.13) and (3.14) we finally get

$$S_p(a, ac_N, p^*) \le S_p \frac{\int_{\mathbb{R}^N} |x|^a \Phi^{-N}(1 - G(x))\, dx}{\int_{\mathbb{R}^N} |x|^a \Phi^{-N}\, dx},$$

where

$$G(x) = aN^{-1}(|x|^{-p}\Phi^{p-1} - 1)\Phi = aN^{-1}[(1 + |x|^{-p/(p-1)})^{p-1} - 1]\Phi > 0.$$

The conclusion readily follows. □

Remark 3.9. The estimate for λ^* in Propositions 3.7 and 3.8 are not sharp, at least when $p = 2$ (compare with Proposition 3.6). Also the restriction $a < \frac{N-p}{p-1}$ in Proposition 3.8 is purely technical.

4. Breaking symmetry and multiplicity

Assume $p < q < p^*$, $a > p - N$ and $\lambda < \lambda_{p,a}$. By the results in Section 3, the best constants $S_p(a, \lambda, q)$ and $S_{p,\mathrm{rad}}(a, \lambda, q)$ are both (well defined and) achieved. In general it turns out that

$$S_p(a, \lambda, q) \le S_{p,\mathrm{rad}}(a, \lambda, q).$$

When $S_p(a, \lambda, q) < S_{p,\mathrm{rad}}(a, \lambda, q)$ a "breaking symmetry" phenomenon appears. In this case problem (1.1) has a radial solution and a ground state solution which is not radially symmetric.

In this last part of the paper we collect some sufficient conditions to have $S_p(a, \lambda, q) < S_{p,\mathrm{rad}}(a, \lambda, q)$. The goal is to get the existence of multiple non-negative solutions to (1.1).

Breaking symmetry was already observed by Catrina and Wang in [11], in case $p = 2$. In [13] Felli and Schneider gave a sharper description of the region in which braking symmetry occurs.

Theorem 4.1 ([13]). *Let $N > 2 = p$, $a > 2 - N$ and $2 < q < 2^*$. If*

$$\lambda < \lambda_{2,a} - 4\frac{N-1}{q^2 - 4}$$

then $S_2(\lambda, a, q) < S_{2,\mathrm{rad}}(a, \lambda, q)$ and problem (1.1) *has at least two distinct nontrivial positive solutions.*

Theorem 4.1 is equivalent to Corollary 1.2 in [13]. For, use the functional transform $u \to |x|^{\bar{a}/2}u(x)$, where $\bar{a} = 2 - N + 2\sqrt{\lambda_{2,a} - \lambda}$.

Breaking symmetry was investigated also in [5] and [26] in case $p > 1$ and $\lambda = 0$.

Theorem 4.2 ([5], [26]). *Let $N \ge 2$ and $1 < p < q < p^*$. Then there exists $a_0(N, p, q)$ such that for $a > a_0(N, p, q)$ no minimizer for $S_p(a, 0, q)$ is radial.*

Theorem 4.2 and Theorems 3.1, 3.4 in Section 3.1 immediately imply the next multiplicity result.

Corollary 4.3. *Let $N \ge 2$, $1 < p < q < p^*$ and $a > p - N$. Then for every a large enough problem*

$$\begin{cases} -\mathrm{div}(|x|^a |\nabla u|^{p-2}\nabla u) = |x|^{-b_q} u^{q-1} & \text{on } \mathbb{R}^N \\ \displaystyle\int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, dx < \infty \end{cases}$$

has at least two distinct nonnegative and nontrivial solutions.

The proofs of Theorem 4.2 in [5] and in [26] are based on the analysis of the asymptotic behaviour of the best constants $S_p(a, 0, q)$ and $S_{p,\mathrm{rad}}(a, 0, q)$ as $a \to \infty$.

In [26], Smets and Willem showed that some partial symmetry is preserved in the region where $S_p(a,0,q) < S_{p,\mathrm{rad}}(a,0,q)$.

Next we state a new breaking symmetry result. Notice that the additional restriction $p \geq 2$ is needed.

THEOREM 4.4. *Let $N \geq 2$, $2 \leq p < q$ and $a > p - N$. Assume $q \leq p^*$ if $p < N$. Then there exists $\lambda_{sb} = \lambda(N,p,a,q)$ such that for any $\lambda < \lambda_{sb}$ no minimizer for $S_p(a,\lambda,q)$ is radial.*

Theorems 3.1 and 3.4 imply the following corollary to Theorem 4.4.

COROLLARY 4.5. *Let $N \geq 2$, $2 \leq p < q < p^*$ and $a > p - N$. Then there exists $\lambda_{sb} = \lambda(N,p,a,q)$ such that for any $\lambda < \lambda_{sb}$ problem* (1.1) *has at least two distinct nonnegative and nontrivial solutions.*

To prove Theorem 4.4 we show that the Morse index of nontrivial radially symmetric solutions increases as $\lambda \ll \lambda_{p,a}$. The responsible are the eigenfunctions of the (strongly elliptic) Laplace-Beltrami operator on the sphere $\mathbb{S}^{N-1}$. In (4.5) we give an explicit sufficient condition to have $S_p(a,\lambda,q) < S_{p,\mathrm{rad}}(a,\lambda,q)$. For instance, breaking symmetry occurs in the following cases:

- *when $\lambda = 0$ and $a \geq p - N + p\sqrt{\frac{(N-1)(p-1)}{q-p}}$;*
- *when $a = 0$, $p > 2$ and $\lambda < -\frac{2}{p-2}\left(\frac{(N-1)(p-1)(p-2)}{(q-p)p}\right)^{p/2}$;*
- *when a is large enough and*

$$\lambda \leq \left(\frac{N-p+a}{p}\right)^p - \left(\frac{N-p+a}{p}\right)^{p-2}\frac{(N-1)(p-1)}{q-p}.$$

The proof of Theorem 4.4 needs a preliminary Lemma. In order to simplify notations, for $u \neq 0$ we set

$$n(u) = \int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, dx - \lambda \int_{\mathbb{R}^N} |x|^{a-p} |u|^p \, dx$$

as in Section 2.2, and

$$d(u) = \left(\int_{\mathbb{R}^N} |x|^{-b_q} |u|^q \, dx\right)^{p/q}, \quad Q(u) = \frac{n(u)}{d(u)}.$$

LEMMA 4.6. *Let $p > 1$, $a > p - N$, $\lambda \in \mathbb{R}$, $q > p$ and assume $q \leq p^*$ if $p < N$. If $\underline{u}$ is a radially symmetric local minimum for Q on $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,dx)\backslash\{0\}$, then*

$$Q(\underline{u}) \leq \frac{(N-1)(p-1)}{q-p}\left(\int_{\mathbb{R}^N} |x|^a |\nabla \underline{u}|^p \, dx\right)^{(p-2)/p}\left(\int_{\mathbb{R}^N} |x|^{a-p} |\underline{u}|^p \, dx\right)^{2/p}.$$

PROOF. By homogeneity we can assume that $d(\underline{u}) = 1$, so that $Q(\underline{u}) = n(\underline{u})$. Compute the partial derivative of Q at $\underline{u}$, along any direction

$h \in \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,dx)$. Since $\underline{u}$ is a minimum point for Q, then $Q'(\underline{u}) \cdot h = 0$ and $Q''(\underline{u}) \cdot h \cdot h \geq 0$. Hence $n'(\underline{u}) \cdot h = Q(\underline{u})d'(\underline{u}) \cdot h$ and

$$Q(\underline{u})d''(\underline{u}) \cdot h \cdot h \leq n''(\underline{u}) \cdot h \cdot h. \tag{4.1}$$

Now we choose the direction h. Let $f_1 \in H^1(\mathbb{S}^{N-1})$ be an eigenfunction of the Laplace operator on $\mathbb{S}^{N-1}$ relatively to the smaller positive eigenvalue, that is,

$$\int_{\mathbb{S}^{N-1}} f_1\,d\sigma = 0, \quad \int_{\mathbb{S}^{N-1}} |f_1|^2\,d\sigma = 1, \quad \int_{\mathbb{S}^{N-1}} |\nabla_\sigma f_1|^2\,d\sigma = N-1. \tag{4.2}$$

We are allowed to take $h(x) = u(|x|)f_1(x/|x|)$, as $h \in \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a\,dx)$. It turns out that

$$d''(\underline{u}) \cdot h \cdot h = p\left[\frac{p-q}{q}\Big(\int_{\mathbb{R}^N} |x|^{-b_q}|\underline{u}|^{q-2}uh\Big)^2 + (q-1)\int_{\mathbb{R}^N} |x|^{-b_q}|\underline{u}|^{q-2}|h|^2\right]$$

that is

$$d''(\underline{u}) \cdot h \cdot h = p(q-1) \tag{4.3}$$

by (4.2). Now notice that $|\nabla h|^2 = |\nabla \underline{u}|^2|f_1|^2 + |x|^{-2}|\underline{u}|^2|\nabla_\sigma f_1|^2$ and

$$n''(\underline{u}) \cdot h \cdot h = p(p-1)\left[\int_{\mathbb{R}^N} |x|^a|\nabla \underline{u}|^{p-2}|\nabla h|^2 - \lambda\int_{\mathbb{R}^N} |x|^{a-p}|\underline{u}|^{p-2}|h|^2\right].$$

Since $p \geq 2$, from (4.2) we infer

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^a|\nabla \underline{u}|^{p-2}|\nabla h|^2 &= \int_{\mathbb{R}^N} |x|^a|\nabla \underline{u}|^p + (N-1)\int_{\mathbb{R}^N} |x|^{a-2}|\nabla \underline{u}|^{p-2}|\underline{u}|^2\\ &\leq \int_{\mathbb{R}^N} |x|^a|\nabla \underline{u}|^p + (N-1)\\ &\quad\times \Big(\int_{\mathbb{R}^N} |x|^a|\nabla \underline{u}|^p\Big)^{(p-2)/p}\Big(\int_{\mathbb{R}^N} |x|^{a-p}|\underline{u}|^p\Big)^{2/p}\end{aligned}$$

by Hölder inequality. Therefore

$$\begin{aligned}&n''(\underline{u}) \cdot h \cdot h\\ &\quad\leq p(p-1)\left[Q(\underline{u}) + (N-1)\Big(\int_{\mathbb{R}^N} |x|^a|\nabla \underline{u}|^p\Big)^{(p-2)/2}\Big(\int_{\mathbb{R}^N} |x|^{a-p}|\underline{u}|^p\Big)^{2/p}\right],\end{aligned}$$

that compared with (4.1) and (4.3) readily leads to the conclusion. □

Proof of Theorem 4.4. Assume that $\underline{u} \in \mathcal{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N; |x|^a\,dx)$ achieves $S_p(a, \lambda, q)$ for some $\lambda < \lambda_{p,a}$. To simplify notations we normalize $\underline{u}$ to have $\int_{\mathbb{R}^N} |x|^{-b_q}|\underline{u}|^q\,dx = 1$. From Lemma 4.6 we infer

$$\int_{\mathbb{R}^N} |x|^a |\nabla \underline{u}|^p \, dx - \lambda \int_{\mathbb{R}^N} |x|^{a-p} |\underline{u}|^p \, dx \leq \gamma \Big(\int_{\mathbb{R}^N} |x|^a |\nabla \underline{u}|^p \, dx \Big)^{(p-2)/p} \Big(\int_{\mathbb{R}^N} |x|^{a-p} |\underline{u}|^p \, dx \Big)^{2/p}, \tag{4.4}$$

where $\gamma := (N-1)(p-1)/(q-p)$. We set

$$s^p := \frac{\int_{\mathbb{R}^N} |x|^a |\nabla \underline{u}|^p \, dx}{\int_{\mathbb{R}^N} |x|^{a-p} |\underline{u}|^p \, dx}, \quad \alpha := \frac{N-p+a}{p}.$$

Notice that $s > \alpha$ by Hardy's inequality. From (4.4) we readily get $\lambda \geq s^p - \gamma s^{p-2}$. Elementary calculus can be used to compute the infimum of $s \to s^p - \gamma s^{p-2}$ on $\{s > \alpha\}$. In this way we get

$$\begin{aligned} \lambda > \alpha^p - \gamma \alpha^{p-2} \quad & \text{if } p = 2 \text{ or } a \geq p - N + \sqrt{\gamma p(p-2)}, \\ \lambda \geq -2\Big(\frac{\gamma}{p}\Big)^{p/2} (p-2)^{(p-2)/2} \quad & \text{otherwise.} \end{aligned}$$

For smaller values of the parameters λ no radially symmetric function achieves $S_p(a, \lambda, q)$. Conversely, if

$$\begin{aligned} \lambda \leq \alpha^p - \gamma \alpha^{p-2} \quad & \text{if } p = 2 \text{ or } a \geq p - N + \sqrt{\gamma p(p-2)}, \\ \lambda < -2\Big(\frac{\gamma}{p}\Big)^{p/2} (p-2)^{(p-2)/2} \quad & \text{otherwise} \end{aligned} \tag{4.5}$$

then $S_p(a, \lambda, q) < S_{p,\mathrm{rad}}(a, \lambda, q)$. □

REMARK 4.7. Some of the results in the present paper can be proved also for a class of noncompact problems on $\mathbb{R}^N$ involving cylindrical weights. For details see [2], [10], [15], [16], [19]–[23] and [28].

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