



Partial Differential Equations — *The limit as $p \rightarrow \infty$ for the p -Laplacian with mixed boundary conditions and the mass transport problem through a given window*, by J. GARCIA-AZORERO, J. J. MANFREDI, I. PERAL and J. D. ROSSI, communicated on 9 January 2009.

ABSTRACT. — In this paper we study the limit as $p \rightarrow \infty$ in a PDE problem involving the p -Laplacian with a right hand side, $-\operatorname{div}(|Du|^{p-2}Du) = f$, with mixed boundary conditions, $u = 0$ on Γ and $|Du|^{p-2}\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega \setminus \Gamma$. We find that this limit is related to an optimal mass transport problem, where the total mass given by f is transported outside the domain through a given window on the boundary Γ .

KEY WORDS: Quasilinear elliptic equations, Dirichlet-Neumann boundary conditions.

MATHEMATICS SUBJECT CLASSIFICATION 2000: 35J65, 35J50, 35J55.

In memoriam of a Master of the XX Century Mathematics, Renato Caccioppoli.

1. INTRODUCTION

The main goal of this article is to study the limit as $p \rightarrow \infty$ in a PDE problem involving the p -Laplacian, $-\operatorname{div}(|Du|^{p-2}Du) = f$, with mixed boundary conditions, $u = 0$ on Γ and $|Du|^{p-2}\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega \setminus \Gamma$, and to connect it with the following mass transport problem: given an amount of material inside a domain, look for the optimal way to transport it outside through a given window on the boundary of the domain.

To formalize this transport problem let $f \geq 0$ be a probability density and let Ω be a convex smooth domain with $\operatorname{supp}(f) \subset \Omega$. Let Γ a smooth submanifold of $\partial\Omega$ (the window) such that if $\bar{\Gamma} \cap \partial\Omega \setminus \bar{\Gamma} \neq \emptyset$ then is a smooth $N - 2$ -dimensional manifold. More precisely, it suffices with C^1 regularity for most of the results in this paper, except for taking limits in viscosity sense, where we need continuity of the normal vector field in $\partial\Omega \setminus \Gamma$.

We want to determine the most efficient way of transport $f(x) dx$ to the window Γ with linear cost; that is, we want to find a function $T : \operatorname{supp}(f) \rightarrow \Gamma$ in such a way that T minimizes

$$L(T) = \int_{\Omega} |x - T(x)| f(x) dx.$$

It turns out that this problem has a very simple solution. Just take as $T(x)$ a point in Γ that realizes the distance; that is, define

$$T(x) = y, \quad \text{for some } y \text{ such that } y \in \Gamma, \text{ dist}(x, \Gamma) = \text{dist}(x, y).$$

Next, we consider a natural way to approximate this problem by taking the limit as $p \rightarrow \infty$ of some PDEs involving the p -Laplacian. We will not need to assume any sign condition on f , but simply that f is bounded.

We will study the limit as $p \rightarrow \infty$ of solutions to the problems

$$(1.1) \quad \begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ |Du|^{p-2} \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

Here $\partial/\partial\nu$ is the outer normal derivative. It is worthy to point out that the solution $u_p \in \mathcal{C}^{1,\alpha}(\Omega) \cap \mathcal{C}^\beta(\bar{\Omega})$ for some $0 < \alpha < 1$ and $0 < \beta < \frac{1}{2}$. This regularity on the interface between both boundary conditions is optimal. See [17].

Solutions to this problem can be easily obtained from a variational argument. In fact, let us consider

$$(1.2) \quad \max \left\{ \int_{\Omega} wf \, dx : w \in W^{1,p}(\Omega), w|_{\Gamma} = 0, \|Dw\|_{L^p(\Omega)} \leq 1 \right\}.$$

From a compactness argument it is easy to check that the maximum is attained and gives a solution to (1.1), up to a Lagrange multiplier. Our first result says that there is a natural variational limit problem as $p \rightarrow \infty$.

THEOREM 1. *The maximizers of (1.2) u_p converge as $p \rightarrow \infty$ along subsequences uniformly in $\bar{\Omega}$ to u_∞ , which is a maximizer of*

$$(1.3) \quad \max \left\{ \int_{\Omega} wf \, dx : w \in W^{1,\infty}(\Omega), w|_{\Gamma} = 0, \|Dw\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

This function u_∞ is a solution to the dual mass transport Kantorovich problem of f_+ to f_- , or to the window Γ , according to the relative mass position between themselves and the boundary. The transport set being given by the union of transport rays that goes from $\text{supp}(f_+)$ or Γ to $\text{supp}(f_-)$ or Γ and are given by segments on which the gradient of u_∞ has modulus exactly one (see [9] for a more precise description of the transport set and rays).

Going back to our original motivation, when f is nonnegative the transport rays are segments along which the distance to the window Γ is realized. Hence u_∞ coincides with the distance to Γ in the transport set (see Remark 2.2). In this case we also have uniqueness of the limit and therefore the limit $\lim_{p \rightarrow \infty} u_p$ exists (see Remark 2.3).

We want to pass to the limit in a PDE verified by u_p . To this end we note that weak solutions to (1.1) are also weak solutions to

$$(1.4) \quad \begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ |Du|^{p-2} \frac{\partial u}{\partial \nu} = \mu_p & \text{on } \partial\Omega, \end{cases}$$

where μ_p is a measure supported in Γ defined by

$$\int_{\Omega} |Du|^{p-2} Du D\varphi \, dx - \int_{\Omega} f \varphi \, dx = \int_{\Gamma} \varphi \, d\mu_p$$

for every $\varphi \in C^1(\bar{\Omega})$. See section 3 below.

Our next aim is to pass to the limit in this weak formulation. In particular we will see that μ_p converges weakly to a measure supported on Γ .

Although the result is independent of a sign condition on f we will assume by simplicity f is nonnegative. Hence u is nonnegative and then we get that μ_p has a sign, $\mu_p \leq 0$.

THEOREM 2. *As $p \rightarrow \infty$, weak solutions to (1.1), u_p , converge uniformly in $\bar{\Omega}$ along subsequences to u_{∞} , a weak solution of*

$$(1.5) \quad \begin{cases} -\operatorname{div}(a(x)Du) = f, & \text{in } \Omega, \\ a(x) \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \setminus \Gamma, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

Here the function $a(x)$ is determined by the fact that the weak limit of $|Du_p|^{p-2} Du_p$, that can be written as $a(x)Du$.

Moreover, the measures μ_p converges weakly along subsequences to a measure μ supported on Γ and it holds that

$$(1.6) \quad \int_{\Omega} a(x)Du(x)D\varphi(x) \, dx - \int_{\Omega} f(x)\varphi(x) \, dx = \int_{\Gamma} \varphi(x) \, d\mu,$$

for all $\varphi \in \mathcal{C}^1(\Omega)$.

Note that u is also a solution to

$$(1.7) \quad \begin{cases} -\operatorname{div}(a(x)Du) = f, & \text{in } \Omega, \\ a(x) \frac{\partial u}{\partial \nu} = \mu, & \text{on } \partial\Omega, \end{cases}$$

that can be obtained (as in Theorem 1) as a maximizer of

$$(1.8) \quad \max_{w \in W^{1,\infty}(\Omega)} \left\{ \int_{\Omega} wf \, dx - \int_{\partial\Omega} w \, d\mu : w|_{\Gamma} = 0, \|Dw\|_{L^{\infty}(\Omega)} \leq 1 \right\}.$$

In general, this last maximization problem is the dual problem associated to the optimal mass transport between $f^+ \, dx + \mu^-$ to $f^- \, dx + \mu^+$. If f is nonnegative then μ is nonpositive and we get a solution to the transport problem between f and μ supported on $\Gamma \subset \partial\Omega$.

Taking $\varphi = 1$ in (1.6) we get the mass balance

$$-\int_{\Omega} f \, dx = \int_{\partial\Omega} d\mu,$$

that is the natural condition when dealing with transport problems.

REMARK 1.1. *If $\int_{\Omega} f \neq 0$ then $\mu \neq 0$. But the case $\int_{\Omega} f = 0$ is very sensitive to the boundary conditions and the geometry associated to the problem. Indeed, in this case we can force the transport from f^+ to f^- by prescribing $\Gamma = \emptyset$, i.e.,*

$$(1.9) \quad \begin{cases} -\Delta_p u_p = f & \text{in } \Omega, \\ |Du_p|^{p-2} \frac{\partial u_p}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

normalizing by $\int_{\Omega} u_p = 0$. This is a different approach to the transport problem that the one in [9], where Dirichlet boundary conditions in a sufficiently large ball are considered.

However, we can prescribe mixed boundary conditions, and then it may happen that $\mu = 0$ or not depending on the geometric configuration of the data f_+ , f_- , Ω and Γ . For example, in [9] it is shown that, even if $\Gamma = \partial\Omega$, it can happen that $\mu = 0$. Actually this occurs for a sufficiently large ball for fixed compactly supported f . Notice that this means that the window is very far away and therefore the optimal transport is realized between f_+ and f_- .

However, if we let f_+ and f_- be far away from each other but concentrated near Γ we can easily see that we have $\mu \neq 0$ regardless that $\int_{\Omega} f \neq 0$ or not.

Now, we turn our attention to the PDE verified by the limit in the viscosity sense. The precise definition of solution in the viscosity sense is given in section 3 below.

THEOREM 3. *Given a continuous function u_{∞} which is the uniform limit of some sequence $\{u_p\}$ of weak solutions to (1.1), we have in the viscosity sense:*

$$|Du_{\infty}| \leq 1; \quad \text{and} \quad -|Du_{\infty}| \geq -1,$$

and moreover,

$$\begin{aligned} \Delta_\infty u_\infty &= 0, && \text{in } \Omega \setminus \text{supp}(f) \\ \min\{|Du_\infty| - 1, -\Delta_\infty u_\infty\} &= 0 && \text{in } \{f > 0\}^o \\ \max\{1 - |Du_\infty|, -\Delta_\infty u_\infty\} &= 0 && \text{in } \{f < 0\}^o \\ -\Delta_\infty u_\infty &\geq 0, && \text{in } \Omega \cap \partial\{f > 0\} \setminus \partial\{f < 0\}, \\ -\Delta_\infty u_\infty &\leq 0, && \text{in } \Omega \cap \partial\{f < 0\} \setminus \partial\{f > 0\}, \end{aligned}$$

together with the boundary conditions

$$\begin{aligned} u_\infty &= 0, && \text{on } \Gamma, \\ \frac{\partial u_\infty}{\partial \nu} &= 0, && \text{on } \partial\Omega \setminus \Gamma. \end{aligned}$$

Let us end the introduction with a brief discussion on some of the existing bibliography. That limits to p -Laplacians are related to mass transport problems was first noticed in [9] and later used in many different contexts, for example, see [1], [5], [10], the book [18] and references therein. See the recent references [10], [11], [12] for other papers dealing with limits as $p \rightarrow \infty$ with different boundary conditions. In [13] the limit as $p \rightarrow \infty$ for the Dirichlet problem was studied with special emphasis on conditions that guarantee uniqueness of the limit. On the other hand, the infinity Laplacian has many applications and has attracted a fair amount of attention in recent years; see for example the survey [2]. Recently, problems involving the infinity Laplacian show a connection between PDEs and probability theory, see [6] and [16].

The rest of the paper is organized as follows: in Section 2 we deal with the variational setting and prove Theorem 1. In Section 3 we deal with the weak formulation (Theorem 2). Finally in Section 4 we look at the PDE satisfied by the limit, proving Theorem 3.

2. VARIATIONAL SETTING. PROOF OF THEOREM 1

This section is devoted to the proof of Theorem 1. First, we note that limits of the solutions to the maximization problem (1.2) coincide with limits of the solutions to the corresponding PDE (1.1) when $p \rightarrow \infty$. In fact, the unique maximizer of (1.2), u_p , is a weak solution to

$$(2.1) \quad \begin{cases} -\Delta_p u_p = \lambda_p f & \text{in } \Omega, \\ |Du_p|^{p-2} \frac{\partial u_p}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ u_p = 0 & \text{on } \Gamma, \end{cases}$$

where λ_p is a Lagrange multiplier. If we take

$$\tilde{u}_p = (\lambda_p)^{-1/(p-1)} u_p$$

we get a solution to (1.1), that is,

$$\begin{cases} -\Delta_p \tilde{u}_p = f & \text{in } \Omega, \\ |D\tilde{u}_p|^{p-2} \frac{\partial \tilde{u}_p}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ \tilde{u}_p = 0 & \text{on } \Gamma. \end{cases}$$

From the weak form of (2.1) and our previous results we get

$$\lim_{p \rightarrow \infty} \lambda_p = \lim_{p \rightarrow \infty} \left(\int_{\Omega} f u_p \right)^{-1} = \left(\int_{\Omega} f u_{\infty} \right)^{-1} \neq 0.$$

Therefore,

$$\lim_{p \rightarrow \infty} \tilde{u}_p = \lim_{p \rightarrow \infty} (\lambda_p)^{-1/(p-1)} u_p = \lim_{p \rightarrow \infty} u_p,$$

and we conclude that the limit points of weak solutions to (1.1) and the maximizers of (1.2) as $p \rightarrow \infty$ coincide.

PROOF OF THEOREM 1. We use ideas from [10], but we include some details for the reader's convenience. Since weak solutions and maximizers give the same limit, we can consider a sequence $\{u_p\}$ of solutions to (1.1). First, we derive some estimates on the family u_p . We have,

$$(2.2) \quad \int_{\Omega} |Du_p|^p = \int_{\Omega} u_p f \leq \left(\int_{\Omega} |u_p|^p \right)^{1/p} \left(\int_{\Omega} |f|^{p'} \right)^{1/p'}$$

where p' is the exponent conjugate to p , that is $1/p' + 1/p = 1$. Recall the following Sobolev inequality, see for example [8],

$$\int_{\Omega} |\phi|^p \leq Cp \left(\int_{\Omega} |D\phi|^p \right),$$

where C is a constant that does not depend on p and ϕ vanishes on Γ . Going back to (2.2), we get,

$$\int_{\Omega} |Du_p|^p \leq \left(\int_{\Omega} |f|^{p'} \right)^{1/p'} C^{1/p} p^{1/p} \left(\int_{\Omega} |Du_p|^p dx \right)^{1/p}.$$

On the other hand, for large p we have

$$|u_p(x) - u_p(y)| \leq C_p |x - y|^{1-N/p} \left(\int_{\Omega} |Du_p|^p dx \right)^{1/p}.$$

Since we are assuming that $u_p = 0$ on Γ , we may choose a point $y \in \Gamma$ such that $u_p(y) = 0$, and hence

$$|u_p(x)| \leq C(p, \Omega) \left(\int_{\Omega} |Du_p|^p dx \right)^{1/p}.$$

The arguments in [8], pages 266–267, show that the constant $C(p, \Omega)$ can be chosen uniformly in p . Hence, we obtain

$$\int_{\Omega} |Du_p|^p \leq \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p'} C^{1/p} p^{1/p} (C_2^p + 1)^{1/p} \left(\int_{\Omega} |Du_p|^p dx \right)^{1/p},$$

with constants independent of p .

Taking into account that $p' = p/(p - 1)$, for large values of p we get

$$\left(\int_{\Omega} |Du_p|^p \right)^{1/p} \leq \alpha_p \left(\int_{\Omega} |f|^{p'} \right)^{1/p}$$

where $\alpha_p \rightarrow 1$ as $p \rightarrow \infty$. Next, fix m , and take $p > m$. We have,

$$\left(\int_{\Omega} |Du_p|^m \right)^{1/m} \leq |\Omega|^{1/m-1/p} \left(\int_{\Omega} |Du_p|^p \right)^{1/p} \leq |\Omega|^{1/m-1/p} \left(\int_{\Omega} |f|^{p'} \right)^{1/p},$$

where $|\Omega|^{1/m-1/p} \rightarrow |\Omega|^{1/m}$ as $p \rightarrow \infty$. Hence, there exists a weak limit in $W^{1,m}(\Omega)$ that we will denote by v_{∞} . This weak limit has to verify

$$\left(\int_{\Omega} |Dv_{\infty}|^m \right)^{1/m} \leq |\Omega|^{1/m}.$$

As the above inequality holds for every m , we get that $v_{\infty} \in W^{1,\infty}(\Omega)$ and moreover, taking the limit $m \rightarrow \infty$,

$$|Dv_{\infty}| \leq 1, \quad \text{a.e. } x \in \Omega.$$

Now let us prove that the subsequence u_{p_i} converges to v_{∞} uniformly in $\bar{\Omega}$. From our previous estimates we know that

$$\left(\int_{\Omega} |Du_p|^p dx \right)^{1/p} \leq C,$$

uniformly in p . Therefore we conclude that u_p is bounded (independently of p) and has a uniform modulus of continuity. Hence u_p converges uniformly to v_{∞} .

Now we are ready to finish the proof of Theorem 1. We have

$$\lim_{p \rightarrow \infty} \int_{\Omega} |Du_p|^p = \lim_{p \rightarrow \infty} \int_{\Omega} u_p f = \int_{\Omega} v_{\infty} f.$$

If we multiply (1.1) by a test function w , we have, for large enough p ,

$$\begin{aligned} \int_{\Omega} wf &\leq \left(\int_{\Omega} |Du_p|^p \right)^{(p-1)/p} \left(\int_{\Omega} |Dw|^p \right)^{1/p} \\ &\leq \left(\int_{\Omega} v_{\infty} f + \delta \right)^{(p-1)/p} \left(\int_{\Omega} |Dw|^p \right)^{1/p}. \end{aligned}$$

As the previous inequality holds for every $\delta > 0$, passing to the limit as $p \rightarrow \infty$ we conclude,

$$\int_{\Omega} wf \leq \left(\int_{\Omega} v_{\infty} f \right) \|Dw\|_{\infty}.$$

Hence, the function v_{∞} verifies,

$$\int_{\Omega} v_{\infty} f = \max \left\{ \int_{\Omega} wf : w \in W^{1,\infty}(\Omega), w|_{\Gamma} = 0, \|Dw\|_{\infty} \leq 1 \right\},$$

This ends the proof. □

On the other hand, taking as a test function in the maximization problem v_{∞} itself we obtain the following corollary.

COROLLARY 2.1. *If $f \not\equiv 0$, then $\|Dv_{\infty}\|_{L^{\infty}(\Omega)} = 1$.*

When f is nonnegative, we can obtain additional information about the structure and uniqueness of the limit.

REMARK 2.2. *Let $w \in W^{1,\infty}(\Omega)$ be any function such that $w|_{\Gamma} = 0$ and $\|Dw\|_{L^{\infty}(\Omega)} \leq 1$. Then,*

$$w(x) \leq \text{dist}(x, \Gamma),$$

and hence, for any $f \geq 0$, we have,

$$\int_{\Omega} w(x)f(x) dx \leq \int_{\Omega} \text{dist}(x, \Gamma)f(x) dx$$

As $\text{dist}(x, \Gamma)$ is an admissible function in the maximization problem (1.3) we conclude that a maximizer u_{∞} verifies,

$$u_{\infty}(x) = \text{dist}(x, \Gamma)$$

in the union of the segments that join $x \in \text{supp}(f)$ with a point $y \in \Gamma$ that realizes $\text{dist}(x, \Gamma)$.

REMARK 2.3. *When $f \geq 0$ we have uniqueness of the limit. To see this, we use the fact that any solution to the mass transport problem verifies*

$$(2.3) \quad |Du_\infty| = 1$$

in the transport set \mathcal{T} (remark that we have $\Gamma \cap \mathcal{T} \neq \emptyset$). Therefore, if we have two limits u_∞ and v_∞ of the family of solutions to (1.1), u_p , we can consider

$$w_\infty = \frac{u_\infty + v_\infty}{2}.$$

By (2.3) we obtain that

$$|Dw_\infty| = \left| \frac{u_\infty + v_\infty}{2} \right| = 1,$$

and hence we conclude that

$$u_\infty = v_\infty + C$$

in \mathcal{T} . As $\Gamma \cap \mathcal{T} \neq \emptyset$ we conclude that $C = 0$ (since $u_\infty = v_\infty = 0$ on Γ). This property can be extended to the whole Ω using the uniqueness for the mixed problem for the infinity Laplacian, recently proved in [6].

Therefore we conclude that the limit is unique and hence there exists the limit $\lim_{p \rightarrow \infty} u_p = u_\infty$.

3. WEAK FORMULATIONS. PROOF OF THEOREM 2

In this section we pass to the limit in the weak form of the equation (1.4) and prove Theorem 2. Notice that it is not obvious, since we have to justify that we get in the limit as $p \rightarrow \infty$ a measure μ supported on the boundary.

PROOF OF THEOREM 2. Recall that we are considering the case $f \geq 0$. If $\bar{\Gamma} \cap \partial\bar{\Omega} \setminus \bar{\Gamma} = \emptyset$ (thus, $\partial\Omega$ is a disconnected set), solutions u_p are $\mathcal{C}^{1,\alpha}(\bar{\Omega})$, and we obtain the result by using the same arguments as section 2 in [9]. However the general case, i.e. $\bar{\Gamma} \cap \partial\bar{\Omega} \setminus \bar{\Gamma} \neq \emptyset$ a smooth $N - 2$ -dimensional manifold, is different. Notice that in the latter case we have a threshold of regularity for the corresponding mixed problem. More concretely, if u_p is the solution to problem (1.1) then $u_p \in \mathcal{C}^{1,\alpha}(\Omega) \cap \mathcal{C}^\beta(\bar{\Omega})$ for some $\beta < 1/2$, see [17].

Fix the solution u_p and define the following linear continuous operator

$$(3.1) \quad \begin{aligned} \mathcal{L}_p : \mathcal{C}^1(\Omega) &\rightarrow \mathbb{R} \\ \varphi &\rightarrow \int_{\Omega} |Du_p|^{p-2} Du_p D\varphi \, dx - \int_{\Omega} f \varphi \, dx \end{aligned}$$

\mathcal{L}_p is a distribution compactly supported in Γ that is (formally) represented by

$$(3.2) \quad \mathcal{L}_p(\varphi) = \int_{\Gamma} \varphi |Du_p|^{p-2} \frac{\partial u_p}{\partial \eta} \, d\sigma$$

where η is the outwards normal to Γ and $d\sigma$ is the surface measure in Γ . At this point, it is necessary to justify the existence of a limit measure when $p \rightarrow \infty$.

The uniform estimate for $p > N$

$$\|u_p\|_{L^\infty(\Omega)} < C,$$

could be obtained in a similar way as in the Dirichlet problem case. To obtain the boundary estimates we argue by approximation, first for fixed p and then for $p \rightarrow \infty$.

Consider $\mathcal{X}_p = |Du|^{p-2}Du$ and

$$\Omega_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \Gamma) \geq \varepsilon\},$$

which for suitable $\varepsilon > 0$ small is a tubular neighborhood of Γ union with some regular open set. Call $\Gamma_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \Gamma) = \varepsilon\}$. Then for all $\varphi \in \mathcal{C}^1(\Omega)$,

$$\int_{\Omega_\varepsilon} \mathcal{X}_p D\varphi \, dx - \int_{\Omega_\varepsilon} f\varphi \, dx = \int_{\Gamma_\varepsilon} \varphi |Du_p|^{p-2} \frac{\partial u_p}{\partial \eta_\varepsilon} d\sigma_\varepsilon \equiv \int_{\Gamma_\varepsilon} \varphi G_{p,\varepsilon} d\sigma_\varepsilon.$$

The function $a(x)$ appears in a similar way as in [9], by the uniform L^∞ estimate. As $G_{p,\varepsilon}$ in particular belongs to $L^1(\Gamma_\varepsilon)$, defines a measure in Ω concentrated in Γ_ε . Notice that taking $\varphi = 1$ and taking into account that $f \geq 0$,

$$-\int_{\Gamma_\varepsilon} G_{p,\varepsilon} d\sigma_\varepsilon = \int_{\Omega} f \, dx,$$

and $G_{p,\varepsilon}$ has a sign. In other words the measures

$$\mu_{\varepsilon,p} = \chi_{\Gamma_\varepsilon} G_{p,\varepsilon} \rightharpoonup \mu_p, \quad \varepsilon \rightarrow 0$$

weakly in the sense of measures. Then, we also have that the total variation is bounded, since

$$|d\mu_p| \leq \liminf_{\varepsilon \rightarrow 0} |d\mu_{\varepsilon,p}| = \int_{\Omega} f \, dx.$$

Hence up to a subsequence we find a measure μ concentrated in Γ such that

$$\mu_p \rightharpoonup \mu, \quad \text{weakly in the sense of measures.}$$

As a consequence in the limit we have the representation

$$\int_{\Omega} a(x) Du D\varphi \, dx - \int_{\Omega} f\varphi \, dx = \int_{\Gamma} \varphi d\mu, \quad \forall \varphi \in \mathcal{C}^1(\Omega).$$

where $a(x)$ is bounded in any compact subset of $\overline{\Omega}$ disjoint with Γ , and is in all $L^r(\Omega)$, $1 \leq r < \infty$. □

We will analyze the structure of the measure μ in order to better understand the associated optimal transport problem.

Given a point $z \in \bar{\Gamma}$, we consider

$$L_z = \{x \in \Omega \mid \text{dist}(x, \Gamma) = \text{dist}(x, z)\},$$

we consider the following classification of $\bar{\Gamma}$ and Ω ,

1.

$$\Gamma_1 = \left\{ z \in \bar{\Gamma} \mid \int_{L_z} dx = 0 \right\},$$

$$\Omega_1 = \{x \in \Omega \mid \exists z \in \Gamma_1, \text{dist}(x, \Gamma) = \text{dist}(x, z)\}.$$

2.

$$\Gamma_2 = \left\{ z \in \bar{\Gamma} \mid \int_{L_z} dx > 0 \right\},$$

$$\Omega_2 = \{x \in \Omega \mid \exists z \in \Gamma_2, \text{dist}(x, \Gamma) = \text{dist}(x, z)\}.$$

By a geometric argument it is not difficult to see that $\bar{\Gamma} \cap \overline{\partial\Omega \setminus \Gamma} = \Gamma_2$ if the boundary and the interface $\bar{\Gamma} \cap \partial\Omega \setminus \Gamma$ are smooth.

Recall that any admissible transport function s satisfies the local conservation of mass property

$$\mu(E) = \int_{s^{-1}(E)} f \, dx, \quad \forall E \subset \Gamma.$$

In particular, in our case the transport function T is given by the rays joining a point of Ω with the point of Γ which realizes the distance, and hence we get

- i) If $p \in \Gamma_1$ then $\mu(p) = 0$, thus, μ is absolutely continuous with respect to the areameasure on Γ_1 and then it could be represented by a element of $L^1(\Gamma_1)$.
- ii) If $p \in \Gamma_2$ then $\mu(p) < 0$ if $\text{supp}(f) \cap \Omega_2 \neq \emptyset$. So it is possible to have a mass concentration on points of the interface $\bar{\Gamma} \cap \overline{\partial\Omega \setminus \Gamma}$.

4. VISCOSITY SETTING. PROOF OF THEOREM 3

Following [3] let us recall the definition of viscosity solution taking into account general boundary conditions.

Assume

$$F : \bar{\Omega} \times \mathbb{R}^N \times \mathbb{S}^{N \times N} \rightarrow \mathbb{R}$$

a continuous function. The associated equation

$$F(x, Du, D^2u) = 0$$

is called (degenerate) elliptic if

$$F(x, \zeta, X) \leq F(x, \zeta, Y) \quad \text{if } X \geq Y.$$

DEFINITION 4.1. Consider the boundary value problem

$$(4.1) \quad \begin{cases} F(x, Du, D^2u) = 0 & \text{in } \Omega, \\ B(x, u, Du) = 0 & \text{on } \partial\Omega. \end{cases}$$

1. A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2(\bar{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \bar{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$, we have the inequality

$$\max\{B(x_0, \phi(x_0), D\phi(x_0)), F(x_0, D\phi(x_0), D^2\phi(x_0))\} \geq 0$$

and if $x_0 \in \Omega$ then we require

$$F(x_0, D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

2. An upper semi-continuous function u is a subsolution if for every $\psi \in C^2(\bar{\Omega})$ such that $u - \psi$ has a strict maximum at the point $x_0 \in \bar{\Omega}$ with $u(x_0) = \psi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality

$$\min\{B(x_0, \psi(x_0), D\psi(x_0)), F(x_0, D\psi(x_0), D^2\psi(x_0))\} \leq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F(x_0, D\psi(x_0), D^2\psi(x_0)) \leq 0.$$

3. Finally, u is a viscosity solution if it is a super and a subsolution.

In the sequel, we will use the same notation as in the definition: ϕ stands for the test functions touching from below the graph of u , and ψ stands for the test functions touching from above the graph of u .

First, we point out that the arguments in [4] could be used to prove the following lemma.

LEMMA 4.2. Given a continuous function u_∞ which is the uniform limit of some sequence $\{u_p\}$ of weak solutions to (1.1), we have in the viscosity sense:

$$|Du_\infty| \leq 1; \quad \text{and} \quad -|Du_\infty| \geq -1.$$

On the other hand, at level p we can pass from weak solutions to solutions in the sense of viscosity:

LEMMA 4.3. Let u_p be a continuous weak solution of (1.1) for $p > 2$. Then u_p is a viscosity solution to (1.1).

PROOF. It follows by the same arguments used in [10], Lemma 2.3. □

PROOF OF THEOREM 3. Let us call ω the support of f .

Next, to look for the equation that u_∞ satisfies in the viscosity sense, assume that $u_\infty - \phi$ has a strict minimum at $x_0 \in \Omega$. Depending on the location of the point x_0 we have different cases.

First, suppose that $x_0 \in \Omega \setminus \omega$. By the uniform convergence of u_{p_i} to u_∞ there exists points x_{p_i} such that $u_{p_i} - \phi$ has a minimum at x_{p_i} with $x_{p_i} \in \Omega \setminus \omega$ for p_i large. Using that u_{p_i} is a viscosity solution to (1.1) we obtain

$$-\Delta_p \phi(x_{p_i}) = -\operatorname{div}(|D\phi|^{p_i-2} D\phi)(x_{p_i}) \geq 0.$$

Therefore

$$-(p_i - 2)|D\phi|^{p_i-4} \Delta_\infty \phi(x_{p_i}) - |D\phi|^{p_i-2} \Delta \phi(x_{p_i}) \geq 0.$$

If $D\phi(x_0) = 0$ we get $-\Delta_\infty \phi(x_0) = 0$. If this is not the case, we have that $D\phi(x_{p_i}) \neq 0$ for large i and then

$$-\Delta_\infty \phi(x_{p_i}) \geq \frac{1}{p_i - 2} |D\phi|^2 \Delta \phi(x_{p_i}) \rightarrow 0, \quad \text{as } p_i \rightarrow \infty.$$

We conclude that

$$(4.2) \quad -\Delta_\infty \phi(x_0) \geq 0.$$

That is, u_∞ is a viscosity supersolution of $-\Delta_\infty u_\infty = 0$ in $\Omega \setminus \omega$.

The fact that it is a viscosity subsolution of $-\Delta_\infty u_\infty = 0$ in $\Omega \setminus \omega$ is completely analogous, using a test function ψ such that $u_\infty - \psi$ has a strict maximum at x_0 .

Now, assume that $x_0 \in \omega$ lies in $\{f > 0\}^o$. Then the sequence x_i also lies in $\{f > 0\}^o$ for large i and hence, we get

$$-(p_i - 2)|D\phi|^{p_i-4} \Delta_\infty \phi(x_{p_i}) - |D\phi|^{p_i-2} \Delta \phi(x_{p_i}) \geq f(x_{p_i}) > 0.$$

Taking limits this means that

$$|D\phi(x_0)| \geq 1 \quad \text{and} \quad -\Delta_\infty \phi(x_0) \geq 0.$$

That is,

$$\min\{|D\phi(x_0)| - 1, -\Delta_\infty \phi(x_0)\} \geq 0.$$

Next, suppose that $u_\infty - \psi$ has a strict maximum at the point x_0 . Then the same arguments as before leads to

$$-(p_i - 2)|D\psi|^{p_i-4} \Delta_\infty \psi(x_{p_i}) - |D\psi|^{p_i-2} \Delta \psi(x_{p_i}) \leq f(x_{p_i}) (> 0).$$

In this case, this means that either

$$|D\psi(x_0)| \leq 1,$$

either

$$|D\psi(x_0)| > 1 \quad \text{and} \quad -\Delta_\infty\psi(x_0) \leq 0.$$

That is,

$$\min\{|D\psi(x_0)| - 1, -\Delta_\infty\psi(x_0)\} \leq 0.$$

Therefore, we get that the equation that u_∞ satisfies in the sense of viscosity in the set $\{f > 0\}^o$ is:

$$\min\{|Du_\infty| - 1, -\Delta_\infty u_\infty\} = 0.$$

In an analogous way we obtain

$$\max\{1 - |Du_\infty|, -\Delta_\infty u_\infty\} = 0 \quad \text{for } x \in \{f < 0\}^o,$$

in the viscosity sense.

The next case to consider, is when $f(x_0) = 0$ and the point x_0 can be reached as a limit of points x_{p_i} that could be contained in the region $\{f > 0\}$ or in the region $\{f = 0\}$. In other words,

$$x_0 \in \Omega \cap \partial\{f > 0\} \cap (\partial\{f < 0\})^C.$$

In this case, if we consider a test function ϕ touching from below the graph of u_∞ at x_0 , then we get a sequence $\{x_{p_i}\}$ converging to x_0 , such that $u_{p_i} - \phi$ has a strict minimum at x_{p_i} . Passing to a subsequence if necessary, we have two possibilities: either $f(x_{p_i}) = 0$, or $f(x_{p_i}) > 0$. If we assume $f(x_{p_i}) = 0$, then

$$-(p_i - 2)|D\phi|^{p_i-4}\Delta_\infty\phi(x_{p_i}) - |D\phi|^{p_i-2}\Delta\phi(x_{p_i}) \geq 0.$$

Then, if $|D\phi(x_{p_i})| \neq 0$ it follows that $-\Delta_\infty\phi(x_0) \geq 0$. On the other hand, if $|D\phi(x_{p_i})| = 0$ for infinitely many indexes, then $-\Delta_\infty\phi(x_0) = 0$.

If we assume $f(x_{p_i}) > 0$, then $|D\phi(x_{p_i})| \neq 0$ and therefore passing to the limit we get $-\Delta_\infty\phi(x_0) \geq 0$.

Concerning the test functions ψ touching from above the graph of u_∞ , when $f(x_{p_i}) = 0$, then we have

$$-(p_i - 2)|D\psi|^{p_i-4}\Delta_\infty\psi(x_{p_i}) - |D\psi|^{p_i-2}\Delta\psi(x_{p_i}) \leq 0.$$

This implies that $-\Delta_\infty\psi(x_0) \leq 0$. But if $f(x_{p_i}) > 0$, then, as in a previous case, we get that $\min\{|D\psi(x_0)| - 1, -\Delta_\infty\psi(x_0)\} \leq 0$, and this condition is always satisfied because $|Du_\infty| \leq 1$.

As a conclusion, if $x_0 \in \Omega \cap \partial\{f > 0\} \cap (\partial\{f < 0\})^C$, we have in the sense of viscosity that $-\Delta_\infty u_\infty \geq 0$ (jointly with the general viscosity estimates on the gradient, valid in all Ω).

In an analogous way, if $x_0 \in \Omega \cap (\partial\{f > 0\})^C \cap \partial\{f < 0\}$, we have in the viscosity sense that $-\Delta_\infty u_\infty \leq 0$ (jointly with the general viscosity estimates on the gradient, valid in the whole domain Ω).

The next region consists on the points $x_0 \in \Omega$ that can be reached as limits of sequences contained either in $\{f > 0\}$, either in $\{f = 0\}$, either in $\{f < 0\}$. That is, $x_0 \in \Omega \cap \partial\{f > 0\} \cap \partial\{f < 0\}$. The same arguments as before give us that in this set the equation satisfied in the sense of viscosity is simply $|Du_\infty| \leq 1$ and $-|Du_\infty| \geq -1$.

Finally, we look at the boundary conditions satisfied by u_∞ in the viscosity sense.

It is clear that $u_\infty = 0$ on Γ .

For $x_0 \in \partial\Omega \setminus \Gamma$, with the same notations as before, the sequence x_i can be contained inside Ω (and in this case the previous arguments give the desired inequality), or it is contained on the boundary $\partial\Omega \setminus \Gamma$. In this last case, taking into account the results in [3], the boundary condition at level p in the viscosity sense gives just

$$\frac{\partial\phi}{\partial\nu}(x_i) \geq 0.$$

Therefore, passing to the limit, we get

$$\frac{\partial\phi}{\partial\nu}(x_0) \geq 0.$$

In an analogous way we can deal with the reverse inequalities, obtaining

$$\frac{\partial u_\infty}{\partial\nu}(x) = 0 \quad \text{for } x \in \partial\Omega \setminus \Gamma,$$

in the viscosity sense. □

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