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Algebraic Geometry. — *Plücker formulae for curves in high dimensions*, by C. T. C. Wall, communicated by F. Catanese on 13 February 2009.

Abstract. — The classical relations of Plücker between the invariants and singularities of a plane curve can be expressed as two linear relations and two involving quadratic terms. The linear relations were generalised to curves in n-space already in the nineteenth century, but true generalisations of the others were obtained only in 3-space. In this article, using the classical method of correspondences, we obtain formulae in n-space corresponding to the original ones in the plane.

Key words: Plücker relations; Correspondences; Weierstrass points.

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Introduction

In 1834, Plücker announced [7] four relations between the degree and class of a plane curve with 'ordinary' singularities and the respective numbers of cusps, double points, flexes and double tangents. Twenty years later, the concept of genus had emerged and the relations were enhanced by Riemann [9] and Clebsch [4] to incorporate the genus. Some years later, the formulae were generalised to arbitrary (reduced) plane curves by Noether [6]: the number of cusps was to be interpreted as a sum over double points of the multiplicity minus the number of branches, and the sum of the numbers of nodes and cusps as the 'double point number': half the sum over all 'infinitely near points' Q of $m_Q(m_Q - 1)$, where m_Q denotes the multiplicity.

For curves in 3-dimensional space, Cayley [3] obtained formulae by applying the above relations to the plane projection of the curve and the dual construct, a plane section of the tangent surface. Here I draw a distinction between the linear relations holding between genus, degree, class and numbers of cusps and flexes and the quadratic relations expressing numbers of double points and double tangents. Cayley's argument gives a successful account of the linear relations, and this was generalised by Veronese [11] to curves in n-space. However, it yields the number of chords through a general point rather than the natural generalisation of double points, the number of tangents meeting the curve again. A formula for the latter number was given by Zeuthen [12], and attributed to Salmon.

An account of the Cayley-Plücker formulae, and of Veronese's work was given in Baker's text [1, §8, Part I]; in [2, §1, Part I] he describes the application of the method of correspondences, and includes a proof of Zeuthen's formula. A well written account in modern language is given by Griffiths & Harris [5].

For C a curve in P^n, in [5, (2.4)] they define degrees $d_k(C)$ and indices $\beta_k(C)$ $(0 \leq k \leq n-1)$, and establish n relations which they call the Plücker formulae, which are the linear relations just mentioned. In the following section [5, (2.5)] they describe the method of correspondences and apply it to these questions for n equal to 2 or 3. Our debt to the account of [5] will be apparent to the reader.

We now introduce the notation to be used below. Let Γ be a curve of genus g and $f : \Gamma \to P^n$ an embedding with image C. The associated curve $f_k : \Gamma \to G(k+1, n+1) \subset P(\Lambda^{k+1}\mathbb{C}^{n+1})$ $(0 \leq k \leq n-1)$ is defined by taking the exterior product of a (locally defined) vector function $F : \Gamma \to \mathbb{C}^{n+1}$ defining f and its first k derivatives with respect to a local parameter on Γ. In particular, the curve $f_{n-1}(\Gamma)$ is the dual curve, which we denote $C^\vee$. We can also regard $f_k(P)$ as a k dimensional subspace of P^n, and as such it is the osculating k-space of C at $f_0(P)$, and will be denoted $O_P^k(C)$. Observe that $O_P^k(C^\vee) = O_P^{n-k-1}(C)^\vee$.

We define the degree $r_k(C)$ to be the degree of the image of f_k, or equivalently the number of osculating k-planes to C meeting a generic $(n-k-1)$-plane. Also, for $P \in \Gamma$, we define $s_k^P(C)$ to be the ramification index of f_k at P, or equivalently if, in some local affine co-ordinates at P, F is given by $x_i = c_i t^{\alpha_i} + \cdots$ with $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n$, we have $s_k^P(C) = \alpha_{k+1} - \alpha_k - 1$. We will omit the (C) from the notations $O_P^k(C)$, $r_k(C)$, $s_k^P(C)$ except to avoid possible confusion.

We call a point for which at least one $s_k^P \neq 0$ a W point of C (here W stands for Weierstraß). For our main results we will need to assume (as did Plücker) that the only W points that occur are those such that, for some i, $s_i^P = 1$ and $s_j^P = 0$ for $j \neq i$: we will call such a point a W_i point, for short, and call these *simple W* points.

Write $s_k(C)$ for the sum of local indices $s_k^P(C)$. It is immediate that $r_k(C^\vee) = r_{n-1-k}(C)$ and $s_k(C^\vee) = s_{n-1-k}(C)$ for each k. Then the linear Plücker formulae are

$$r_{k-1} - 2r_k + r_{k+1} = 2g - 2 - s_k, \quad (0 \leq k \leq n-1),$$

where we set $r_{-1} = r_n = 0$. We outline a proof, using induction and the method of projection, in the next section.

When $n = 2$ these do not include all the original formulae. To generalise the others, we define $D_{k-1,n-k-1}$ to be the set of pairs $(P, Q) \in \Gamma \times \Gamma$ with $P \neq Q$ such that the intersection $O_P^{k-1} \cap O_Q^{n-k-1}$ is non-empty, and write $d_{k-1,n-k-1}$ for its cardinality. Denote also by $D^{k,n-k}$ the set of pairs (P, Q) with $P \neq Q$ where the intersection $O_P^k \cap O_Q^{n-k}$ has dimension ≥ 1 and by $d^{k,n-k}$ its cardinality. In each case, k runs over $1 \leq k \leq n-1$. Each invariant d is symmetric in the suffices, and $d^{k,n-k}(C) = d_{k-1,n-k-1}(C^\vee)$. Our main result Theorem 4.2 states that under certain genericity hypotheses we have, for $0 \leq k \leq \frac{1}{2}(n-2)$,

$$\begin{aligned} d_{k,n-k-2} = {} & r_k r_{n-k-2} - (k+2)(n-k)r_k - (k+1)(n-k+1)r_{n-k-2} \\ & + 2\sum_{n-k-1}^{n-1} r_i - \frac{1}{6}(k+1)(k+2)(3(n-k)^2 - (k+3))(2g-2). \end{aligned}$$

To obtain these relations we use the method of correspondences. For $(P, Q) \in \Gamma \times \Gamma$ define $(P, Q) \in T_{k,n-k-1}$ if $O_P^k \cap O_Q^{n-k-1} \neq \emptyset$. In §2 we determine the numerical properties of the $T_{k,n-k-1}$ and give a careful analysis of the local structure of these correspondences at the W_i points. I am indebted to Don Zagier for arguments giving very clean form for this computation. In §3 we introduce the D points, and study the local structure of the correspondences at these points. We obtain our theorem in §4 by applying the standard formulae for numbers of coincidence points and check it by counting intersection points.

1. The linear formulae

We begin with a proof of

Proposition 1.1. *For C a reduced curve in P^n, we have*

$$r_{k-1} - 2r_k + r_{k+1} = 2g - 2 - s_k, \quad 0 \leq k \leq n-1,$$

where we set $r_{-1} = r_n = 0$.

Proof. The result for $n = 2$ is a consequence of the Plücker formulae for plane curves. We deduce it for higher values of n by induction, following essentially the classical method of [11].

Let C_P denote a projection of C onto an $(n-1)$-plane from a point P. Assume that P does not lie on any osculating space of dimension $n-2$ of C or of dimension $n-1$ at any W point. Then the osculating spaces of C_P are the projections of those of C, so $s_k(C_P) = s_k(C)$ for $0 \leq k \leq n-3$, and C_P still has genus g.

A general point $P \notin C$ will lie on the osculating $(n-1)$-spaces at r_{n-1} points, distinct from each other and from the W points. Their projections give further W_{n-2} points of C_P, so we have $s_{n-2}(C_P) = s_{n-2}(C) + r_{n-1}(C)$.

Now $r_k(C)$ is the number of osculating k-planes to C meeting a generic $(n-k-1)$-plane. If P is generic, a generic $(n-k-1)$-plane through P is a generic $(n-k-1)$-plane. Projecting, we deduce that $r_k(C_P)$ is also the number of osculating k-planes to C_P meeting a generic $(n-k-2)$-plane, hence is equal to $r_k(C)$.

The relations (1.1) for $k < n-1$ for C now follow from those for C_P. The final relation follows by applying this result for the dual curve $C^\vee$. □

The same argument shows that if we project from a general point $P \in C$ we have $s_k(C_P) = s_k(C)$ for $0 \leq k \leq n-3$, and as $r_0(C_P) = r_0(C) - 1$, it follows from the relations (1.1) for C_P that $r_k(C_P) = r_k(C) - (k+1)$ for $0 \leq k \leq n-2$ and again $s_{n-2}(C_P) = s_{n-2}(C) + r_{n-1}(C)$.

The relations (1.1) can be rewritten in numerous ways, some of which we need below. We see by induction that

$$r_k = (k+1)r_0 + \binom{k+1}{2}(2g-2) - \sum_0^{k-1}(k-i)s_i;$$

by symmetry, $r_{n-k-1} = (k+1)r_{n-1} + \binom{k+1}{2}(2g-2) - \sum_0^{k-1}(k-i)s_{n-1-i}$.

Summing the relations (1.1) gives $-(r_0 + r_{n-1}) = n(2g-2) - \sum_0^{n-1} s_k$. Hence

$$r_k + r_{n-k-1} = -(k+1)(n-k)(2g-2) + \sum_0^{n-1} c_{i,k}^n s_i,$$

where the coefficient $c_{i,k}^n$ is most succinctly written as

$$c_{i,k}^n := \min(i+1, k+1, n-k, n-i).$$

We will need several similar formulae below: we collect them now (the verifications are trivial).

Lemma 1.2. (i) $\sum_k^{n-k-1} s_i = -(r_{k-1} + r_{n-k}) + (r_k + r_{n-k-1}) + (n-2k)(2g-2)$

(ii) $\sum_0^{n-1} s_i = (r_0 + r_{n-1}) + n(2g-2)$

(iii) $\sum_0^{n-1} c_{i,k}^n s_i = (r_k + r_{n-k-1}) + (k+1)(n-k)(2g-2)$

(iv) $\sum_0^{n-1} (c_{i,k}^n - 1)^2 s_i = -(r_0 + r_{n-1}) - 2\sum_1^{k-1}(r_i + r_{n-i-1}) + (2k-1)(r_k + r_{n-k-1}) + \{nk^2 - \frac{1}{3}k(k+1)(4k-1)\}(2g-2)$

(v) $\sum_0^{n-1} c_{i,k}^n(c_{i,k}^n - 1)s_i = -2\sum_0^{k-1}(r_i + r_{n-i-1}) + 2k(r_k + r_{n-k-1}) + \{nk(k+1) - \frac{1}{3}k(k+1)(4k+2)\}(2g-2)$

(vi) $\sum_0^{n-1} (c_{i,k}^n)^2 s_i = -2\sum_0^{k-1}(r_i + r_{n-i-1}) + (2k+1)(r_k + r_{n-k-1}) + \{n(k+1)^2 - \frac{1}{3}k(k+1)(4k+5)\}(2g-2)$

(vii) $\sum_0^{n-1} c_{i,k}^n c_{i,k-1}^n s_i = -2\sum_0^{k-2}(r_i + r_{n-i-1}) + (k-1)(r_{k-1} + r_{n-k}) + k(r_k + r_{n-k-1}) + \{nk(k+1) - \frac{1}{3}k(k+1)(4k-1)\}(2g-2)$.

2. The correspondences; local structure at W points

A correspondence on an algebraic curve Γ is an algebraic curve $T \subset \Gamma \times \Gamma$. The degree d_1 and codegree d_2 are the degrees of the projections of T on the first and second factor Γ respectively. T has *valence* v if, denoting by $T(P)$ the divisor of the projection of $T \cap (\{P\} \times \Gamma)$ on the second factor, the divisor class of $T(P) + vP$ is independent of P.

We recall that if T has valence v then T has the divisor class of

$$(d_1 + v)(* \times \Gamma) + (d_2 + v)(\Gamma \times *) - v\Delta(\Gamma),$$

where Δ denotes the diagonal. Since the diagonal has self-intersection number $2 - 2g$, we can calculate all intersection numbers. If we have two correspondences T, T', their mutual intersection number is

$$d_1 d_2' + d_2 d_1' - 2gvv'. \tag{1}$$

We can regard the diagonal as a correspondence with degrees 1 and valence -1. The number of self-corresponding points (traditionally called united points) is the intersection number of T with the diagonal, which is thus

$$d_1 + d_2 + 2gv. \tag{2}$$

The genus formula, in the form $\mu(M) - \chi(M) = [M].([M] + K_E)$, where μ denotes the total Milnor number and χ the (topological) Euler characteristic, gives

$$-\chi(T) = 2d_1d_2 + (2g-2)(d_1+d_2) - 2gv^2 - \mu(T). \tag{3}$$

For $(P,Q) \in \Gamma \times \Gamma$ define $(P,Q) \in T_{k,n-k-1}$ if $O_P^k \cap O_Q^{n-k-1} \neq \emptyset$: more precisely, we take the closure of the set of pairs where this condition holds but $P \neq Q$. Since for subspaces of dimensions k and $n-k-1$ to have a non-empty intersection is a single condition, this defines a correspondence on Γ. The transpose, interchanging the roles of P and Q, gives the correspondence $T_{n-k-1,k}$.

Lemma 2.1. *The correspondence $T_{k,n-k-1}$ has degree $r_{n-k-1} - (k+1)(n-k)$, codegree $r_k - (k+1)(n-k)$, and valence $(k+1)(n-k)$.*

Proof. We have to find out how many points Q correspond to a given general point P. Project from O_P^{k-1}. This is a composite of k projections, each of a curve from a point of itself. If P is a general point, each of these points of projection is also general. Hence, by the remark following Proposition 1.1, the invariants of the image curve D in P^{n-k} are given by $s_i(D) = s_i(C)$ for $0 \le i \le n-2-k$ and $r_i(D) = r_i(C) - k(i+1)$ for $0 \le i \le n-1-k$. Write Y_P for the image of O_P^k in P^{n-k}. We want to know how many Q have $Y_P \in O_Q^{n-k-1}$. These points are counted by the class of D, which is $r_{n-k-1}(D) = r_{n-k-1}(C) - k(n-k)$. But this count includes the point $Q = Y_P$ itself. The intersection number of $O_{Y_P}^{n-k-1}(D)$ with D at Y_P is $(n-k)$. So the correct count is obtained by subtracting this, giving $r_{n-k-1} - (k+1)(n-k)$.

For the valence we argue following [5, p 295]. Write $\pi : \Gamma \to P^{n-k-1}$ for the projection from O_P^k. Then the canonical class is $K_\Gamma = \pi^*(-(n-k)H_{P^{n-k-1}}) + T_{k,n-k-1}(P)$, and $\pi^* H_{P^{n-k-1}} = H_{P^n} - (k+1)P$, so $T_{k,n-k-1}(P) + (k+1)(n-k)P = \pi^* H_{P^{n-k-1}} + (n-k)H_{P^n}$. □

We will apply the above formulae (2)–(3) to the $T_{k,n-k-1}$. The Weierstraß points will play a key role, so we first explore what happens at them. It is here that we need to restrict to simple W points.

Proposition 2.2. *Suppose P a W_i point. Then at (P,P), the curve $T_{k,n-k-1}$ consists of $c_{i,k}^n$ mutually transverse smooth branches. Branches of $T_{k-1,n-k}$ and $T_{k,n-k-1}$ are all mutually transverse at (P,P).*

We begin the proof by taking local co-ordinates at P in which C has parametrisation $x_r = t^r$ for $r \le i$ and $x_r = t^{r+1}$ for $r > i$, modulo higher order terms in t. Consider $x(t)$ as a vector. Then the condition that the point $(t,u) \in T_{k,n-k-1}$ is that the matrix $M_{i,k}^n$ with rows $x(t) - x(u)$, $d^r x(t)/dt^r$ $(1 \le r \le k)$, $d^r x(u)/du^r$ $(1 \le r \le n-1-k)$ is singular.

The determinant of $M_{i,k}^n$ has order

$$\binom{n+2}{2} - 1 - \binom{k+1}{2} - \binom{n-k}{2} - i = (k+1)(n-k) + n - i.$$

We can take out powers of t, u and $t-u$ appearing as factors: the quotient gives the local equation of $T_{k,n-k-1}$, and the terms of least degree give the tangent cone of this curve. These terms are obtained by the same procedure, but setting the 'higher terms' equal to zero. It will thus suffice to analyse these.

I am indebted to Don Zagier for the following elegant treatment of these polynomials, and in particular for the next result.

Write $u(t)$ for the vector $(1, t, t^2, \dots, t^{n-1})$, and let $u^{[r]}(t)$ denote $\frac{1}{r!}d^r u(t)/dt^r$ (and similarly in other cases). For any integers $1 \le a \le n$ define $M_{a,n}(t)$ to be the $a \times n$ matrix with coefficients in $\mathbb{Z}[t]$ having rows $u(t), u^{[1]}(t), \dots, u^{[a-1]}(t)$.

Proposition 2.3 (Zagier). *Let n and $a_1, \dots, a_k$ be positive integers with $a_1 + \dots + a_k = n$, and $x_1, \dots, x_k$ be variables. Let M be the matrix with the rows of the $M_{a_i,n}(x_i)$. Then the determinant of M is $\pm\prod_{1\le i<j\le k}(x_i - x_j)^{a_i a_j}$.*

Proof. Write $V_n(x_1, \dots, x_n)$ for the Vandermonde matrix with rows $u(x_1), \dots, u(x_n)$, hence with determinant $\pm\prod_{1\le i\le j\le n}(x_j - x_i)$.

Write V for the direct sum (with block diagonal terms)

$$V := V_{a_1}(\varepsilon_{1,1}, \dots, \varepsilon_{1,a_1}) \oplus \cdots \oplus V_{a_k}(\varepsilon_{k,1}, \dots, \varepsilon_{k,a_k}),$$

where the $\varepsilon_{i,r}$ ($1 \le i \le k$, $1 \le r \le a_i$) are variables satisfying $\varepsilon_{i,r}^{a_i} = 0$. Then direct calculation gives $VM = V_n(x_1 + \varepsilon_{1,1}, \dots, x_n + \varepsilon_{k,a_k})$.

Equating determinants gives

$$\prod_{i=1}^{k}\prod_{1\le r<s\le a_i}(\varepsilon_{i,r} - \varepsilon_{i,s})\det(M)$$
$$= \pm\prod_{i=1}^{k}\prod_{1\le r<s\le a_i}(\varepsilon_{i,r} - \varepsilon_{i,s})\prod_{1\le i<j\le k}\prod_{r=1}^{a_i}\prod_{s=1}^{a_j}(x_i + \varepsilon_{i,r} - x_j - \varepsilon_{j,s}).$$

Cancelling the common factor $\prod_{i=1}^{k}\prod_{1\le r<s\le a_i}(\varepsilon_{i,r} - \varepsilon_{i,s})$ and then setting all $\varepsilon_{i,r}$ equal to 0 now gives the result. □

Apply this result with $k = 3$, $a_1 = 1$, $a_2 = k+1$, $a_3 = n-k$ and $x_1 = v$, $x_2 = t$, $x_3 = u$. The determinant is then $\pm(v-t)^{k+1}(v-u)^{n-k}(t-u)^{(k+1)(n-k)}$. The matrix has $n+2$ columns, with first row $x(v) = (1, \dots, v^{n+1})$. Expanding by this row gives the sum of $(-v)^{i+1}$ multiplied by the minor in which the first row and column $i+1$ are omitted. In each such minor we can subtract the row $x(u)$ from the first row $x(t)$. The resulting matrix has only one non-zero entry in the first column. Expanding by the first column now gives the $n \times n$ determinant of the matrix with rows $x(t) - x(u)$, $x^{[i]}(t)$, $(1 \le i \le k)$, $x^{[i]}(u)$, $(1 \le i \le n-k-1)$, and the columns labelled by 0 and $i+1$ omitted, and this agrees up to constant multiples with the matrix $M^n_{i,k}$ above. Denote the quotient of this determinant by $(t-u)^{(k+1)(n-k)}$ by $P^n_{i,k}(t,u)$, up to signs, which we fix by the formula

$$(t-v)^{k+1}(u-v)^{n-k} = \sum_{-1}^{n}(-v)^{i+1}P^n_{i,k}(t,u). \qquad (4)$$

To complete the proof of Proposition 2.2, it remains to establish that, other than possible factors t and u, $P^n_{i,k}(t,u)$ has no repeated factor and $P^n_{i,k}(t,u)$ and $P^n_{i,k+1}(t,u)$ have no common factor. The polynomials with $n=3$ are exhibited in the matrix

$$\begin{pmatrix} u^4 & tu^3 & t^2u^2 & t^3u & t^4 \\ 4u^3 & 3tu^2+u^3 & 2t^2u+2tu^2 & t^3+3t^2u & 4t^3 \\ 6u^2 & 3tu+3u^2 & t^2+4tu+u^2 & 3t^2+3tu & 6t^2 \\ 4u & t+3u & 2t+2u & 3t+u & 4t \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We collect a number of properties of these polynomials in the next lemma.

LEMMA 2.4. (i) *$P^n_{i,k}$ is homogeneous of degree $n-i$ in t and u.*

(ii) $P^n_{i,k}(1,1)=\binom{n+1}{i+1}$.

(iii) $P^n_{-1,k}(t,u)=t^{k+1}u^{n-k}$, $P^n_{n,k}(t,u)=1$.

(iv) $P^n_{i,-1}(t,u)=\binom{n+1}{i+1}u^{n-i}$, $P^n_{i,n}(t,u)=\binom{n+1}{i+1}t^{n-i}$.

(v) $P^n_{i,k}(t,u)=\sum_r\binom{k+1}{r}\binom{n-k}{n-i-r}t^ru^{n-i-r}$.

(vi) *$P^n_{i,k}(t,u)$ is divisible by $t^{\max(0,k-i)}u^{\max(0,n-i-k+1)}$.*

(vii) $\binom{n}{k+1}t^iP^n_{i,k}(t,u)=\binom{n}{i+1}t^kP^n_{k,i}(t,u)$.

(viii) $P^n_{i,k}(t,u)=P^n_{i,n-1-k}(u,t)$, $P^n_{n-i-1,k}(t,u)=t^{k+i+1-n}u^{i-k}P^n_{i,k}(u,t)$.

(ix) $(n-k)P^n_{i,k+1}(t,u)-(i-k)P^n_{i,k}(t,u)=(i+2)tP^n_{i+1,k}(t,u)$.

(x) $(k+2)P^n_{i,k}(t,u)-(i+k-n+2)P^n_{i,k+1}(t,u)=(i+2)uP^n_{i+1,k+1}(t,u)$.

PROOF. (i) is immediate from the definition (4); (ii) follows by substituting $t=u=1$. For (iii) we pick out the coefficients of v^0 and v^{n+1} on both sides; (iv) follows by taking $k=-1$ or $k=n$.

To obtain (v) we expand the left hand side of (4) by the binomial theorem to obtain

$$\Big(\sum_{r=0}^{k+1}\binom{k+1}{r}t^r(-v)^{k+1-r}\Big)\Big(\sum_{s=0}^{n-k}\binom{n-k}{s}u^s(-v)^{n-k-s}\Big),$$

and equate coefficients of t^ru^{n-i-r}. The range of summation is defined by $0\le r\le k+1$ and $0\le n-i-r\le n-k$, thus $\max(0,k-i)\le r\le\min(n-i,k+1)$. Hence $P^n_{i,k}(t,u)$ is divisible by $t^{\max(0,k-i)}u^{\max(0,n-i-k+1)}$, proving (vi).

The coefficient of t^ru^{n-i-r} in $\binom{n}{k+1}P^n_{i,k}(t,u)$ is $n!/(r!(k+1-r)!(n-i-r)!\cdot(i+r-k)!)$; that of t^su^{n-k-s} in $\binom{n}{i+1}P^n_{k,i}(t,u)$ is $n!/(s!(i+1-s)!(n-k-s)!\cdot(k+s-i)!)$, and these are equal if $s=i+r-k$, proving (vii).

The first identity (viii) is immediate from the definition (4); the second follows by combining it with (vi).

We prove (ix) by direct calculation: it suffices to verify that, for each r with $\max(0,k-i)\le r\le\min(n-i,k+2)$, the coefficient of t^ru^{n-i-r} in $(n-k)P^n_{i,k+1}(t,u)-(i-k)P^n_{i,k}(t,u)-(i+2)tP^n_{i+1,k}(t,u)$ vanishes. (For the extreme values of r, one of the binomial coefficients below must be interpreted as 0: this does not affect our calculation.) We see from (v) that this coefficient is

$$(n-k)\binom{k+2}{r}\binom{n-k-1}{n-i-r} - (i-k)\binom{k+1}{r}\binom{n-k}{n-i-r} - (i+2)\binom{k+1}{r-1}\binom{n-k}{n-i-r}.$$

Multiplying each term by $r!(k+2-r)!(n-i-r)!(i+r-k)!/(k+1)!(n-k)!$, this becomes $(k+2)(i+r-k) - (i-k)(k+2-r) - (i+2)r$, which indeed reduces to 0.

We can prove (x) similarly, or deduce it from (ix) by substituting $k = n-s-2$ and applying the symmetry (viii). □

We now prove, by downward induction on i, that $P^n_{i,k}(t,u)$ and $P^n_{i,k+1}(t,u)$ have no common factor other than powers of t and u: by (iii) this holds if $i = -1$ or if $i = n$. But it follows from (ix) and (x) that any common factor of $P^n_{i,k}(t,u)$ and $P^n_{i,k+1}(t,u)$ other than t and u divides also $P^n_{i+1,k}(t,u)$ and $P^n_{i+1,k+1}(t,u)$.

Now suppose $u - \lambda t$ appears as a repeated factor of $P^n_{i,k}(t,u)$, and hence also of $\det(M^n_{i,k})$. Then the derivative of $\det(M^n_{i,k})$ with respect to t also vanishes when $u = \lambda t$. Hence so does the determinant obtained from $M^n_{i,k}$ by replacing the row $d^k x(t)/dt^k$ by $d^{k+1}x(t)/dt^{k+1}$. Hence the $n \times (n+1)$ matrix obtained by adjoining this row has rank $\leq n-1$ when $u = \lambda t$. Thus $u - \lambda t$ also divides $\det(M^n_{i,k+1})$, hence is a common factor of $P^n_{i,k}(t,u)$ and $P^n_{i,k+1}(t,u)$, contradicting what we have just proved. This completes the proof of Proposition 2.2.

Some of the above properties are simpler in terms of the following modification $p^n_{i,k}(t,u)$ of $P^n_{i,k}(t,u)$. First divide by $t^{\max(k-i,0)}u^{\max(n-i-k+1,0)}$, giving a polynomial of degree $c^n_{i,k}$ not divisible by t or u (replacing (i) and (vi)); divide by a constant factor (if necessary) to achieve $p^n_{i,k}(1,1) = \binom{n+1}{c^n_{i,k}}$, replacing (ii). Then (iii) and (iv) become $p^n_{i,k}(t,u) = 1$ if $c^n_{i,k} = 0$, and the symmetry properties (vii), (viii) become

$$p^n_{i,k}(t,u) = p^n_{k,i}(t,u) = p^n_{n-1-i,k}(u,t) = p^n_{i,n-1-k}(u,t).$$

However, (v) and the recurrence relations (ix), (x) become more complicated.

It is not always true that the branches of distinct $T_{k,n-k-1}$ are mutually transverse at (P,P): for example, if $n = 2m+1$ and P has type W_m, the above symmetry property (viii) gives $P^{2m+1}_{m,k}(t,u) = t^{k-m}u^{m-k}P^{2m+1}_{m,2m-k}(t,u)$. Also, each $P^{2m+1}_{m,2r}$ is divisible by $t+u$. These may perhaps be the only exceptions.

Write $\Delta(W)$ for the of points (P,P) with P a W point. Then

Lemma 2.5. *If all W points are simple, the intersections of $T_{k,n-k-1}$ with $\Delta(\Gamma)$ occur only at $\Delta(W)$. The intersection multiplicity at a W_i point is $c^n_{i,k}$.*

Proof. For $T_{k,n-k-1}$, (2) gives $r_k + r_{n-k-1} - 2(k+1)(n-k) + 2g(k+1)(n-k)$ for the number of united points. By Lemma 1.2(iii), this equals $\sum_0^{n-1} c^n_{i,k}s_i$. But by Proposition 2.2, this is equal to the sum of the intersection multiplicities of $T_{k,n-k-1}$ with $\Delta(\Gamma)$ at the points of $\Delta(W)$. □

3. The invariants; local structure at D points

It follows by counting dimensions that if $1 \leq k \leq n-1$, the set $D_{k-1,n-k-1}$ of pairs $(P,Q) \in \Gamma \times \Gamma$ where $P \neq Q$ and the intersection $O_P^{k-1} \cap O_Q^{n-k-1}$ is non-empty consists of a finite number $d_{k-1,n-k-1}$ of points. Also, again if $1 \leq k \leq n-1$, the set $D^{k,n-k}$ of pairs (P,Q) with $P \neq Q$ and the intersection $O_P^k \cap O_Q^{n-k}$ of dimension at least 1 consists of a finite number $d^{k,n-k}$ of points. We call (P,Q) a D point if it belongs to one of these sets. Since $(P,Q) \in D_{k-1,n-k-1} \Leftrightarrow (Q,P) \in D_{n-k-1,k-1}$ and $D^{k,n-k}(C^\vee) = D_{n-k-1,k-1}(C)$, each invariant d is symmetric in the suffices, and $d^{k,n-k}(C^\vee) = d_{k-1,n-k-1}(C)$.

Lemma 3.1. *The points where the projection of $T_{k,n-k-1} \subset \Gamma \times \Gamma$ on the* first *factor is not a local bijection are as follows: $\Delta(\Gamma) \cap T_{k,n-k-1}$, points $(P,Q) \in T_{k,n-k-1}$ with $s_{n-k-1}^Q > 0$, and points of $D_{k,n-k-2}$ and $D^{k,n-k}$.*

Proof. A general point $P \in C$ corresponds to $r_{n-k-1} - (k+1)(n-k)$ points Q, in general distinct: we want cases when two of these coincide. The points Q are those whose images in $f_{n-k-1}(\Gamma)$ lie on a certain hyperplane H_P in $P(\Lambda^{n-k}\mathbb{C}^{n+1})$.

If $f_{n-k-1}(Q)$ is a singular point of $f_{n-k-1}(\Gamma)$, equivalently, $s_{n-k-1}^Q > 0$, then any intersection of H_P with $f_{n-k-1}(\Gamma)$ at $f_{n-k-1}(Q)$ is multiple.

Otherwise, $f_{n-k-1}(Q)$ is a smooth point of $f_{n-k-1}(\Gamma)$, and we require the hyperplane to contain the tangent line at this point. By the discussion on [5, p 272], this tangent line is the Schubert cycle of k-planes in P^n containing O_Q^{n-k-2} and contained in O_Q^{n-k}. If $P \neq Q$, either $O_P^k \cap O_Q^{n-k-2}$ is non-empty and $(P,Q) \in D_{k,n-k-2}$ or $O_P^k \cap O_Q^{n-k}$ contains a line and $(P,Q) \in D^{k,n-k}$. □

Take local co-ordinates t_p, t_q for Γ at P, Q respectively; write $\psi(t_p,t_q) = 0$ for a local equation of $T_{k,n-k-1}$ at (P,Q), so that $\psi(0,0) = 0$. Thus Lemma 3.1 gives a necessary and sufficient condition for the coefficient of t_q in ψ to be zero.

Interchanging the factors, it follows that the projection of $T_{k,n-k-1} \subset \Gamma \times \Gamma$ on the *second* factor is not a local bijection at (P,Q), or equivalently, the coefficient of t_p in ψ is zero, if and only if $(P,Q) \in \Delta(\Gamma)$, $s_k^P > 0$, $(P,Q) \in D_{k-1,n-k-1}$ or $(P,Q) \in D^{k+1,n-k-1}$. We observe that $T_{k,n-k-1}$ is singular at (P,Q) if and only if both coefficients vanish, i.e. neither projection is a local isomorphism.

We will need further details; to prepare for the argument to follow, we reprove the above.

Theorem 3.2. *Let $(P,Q) \in D_{k,n-k-2}$ with $s_k^P = 0$, $(P,Q) \notin D_{k-1,n-k-1}$ and $(P,Q) \notin D^{k+1,n-k-1}$. Then*

(i) *the coefficient of t_p in ψ is non-zero;*

(ii) *the coefficient of t_q^2 in ψ is non-zero if and only if $(P,Q) \notin D_{k+1,\,n-k-3}$, $(P,Q) \notin D^{k,n-k}$ and $s_{n-k-1}^Q = 0$;*

(iii) *more generally, if* $(P,Q) \notin D_{k+1,n-k-3}$ *and* $(P,Q) \notin D^{k,n-k}$, $\psi(0,t_q)$ *has order* $s^Q_{n-k-1}+2$.

PROOF. Since $(P,Q) \notin D_{k-1,n-k-1}$, we have $O_P^{k-1} \cap O_Q^{n-k-1} = \emptyset$, so we can take (projective) co-ordinates $(x_0,\dots,x_n)$ such that if $0 \le i \le k-1$, O_P^i is defined by $x_r = 0$ for $r > i$, and if $0 \le j \le n-k-1$, O_Q^j is defined by $x_r = 0$ for $r < n-j$.

Thus the expansions of a point of Γ near P are given in local co-ordinates by $x_i = a_i t_p^{\alpha_i} +$ higher terms $(a_i \ne 0)$ for $i \le k-1$, while for $i \ge k$, we write $x_i = t_p^{\alpha_k}(a_i + a_i' t_p + \cdots)$. Thus if A is the point with co-ordinates $x_i = 0$ for $i \le k-1$ and $x_i = a_i$ for $i \ge k$, and A' the same with a_i replaced by a_i', O_P^k is spanned by O_P^{k-1} and A, and O_P^{k+1} is spanned by O_P^{k-1}, A and A'. Here the sequence α_i is increasing; indeed, unless $s_i^P > 0$ for some $i < k$, we have $\alpha_i = i$ for $i \le k$. We may also take $a_i = 1$ for $i \le k-1$. Observe that since $s_k^P = 0$, $\alpha_{k+1} = \alpha_k + 1$.

Similarly at a point near Q we can write $x_{n-i} = t_q^{\beta_i} +$ higher terms for $i \le n-k-1$, while for $i \ge n-k$, x_{n-i} is divisible by $t_q^{\beta_{n-k}}$, and we will denote the coefficient of $t_q^{\beta_{n-k}}$ by b_{n-i}. Thus if B is the point with co-ordinates $x_i = b_i$ for $i \le k$ and $x_i = 0$ for $i > k$, O_Q^{n-k} is spanned by O_Q^{n-k-1} and B. Here β_i is increasing, and $\beta_i = i$ for $i \le n-k-1$ unless $s_j^Q > 0$ for some $j < n-k-1$.

As in the proof of Proposition 2.2, the local equation of $T_{k,n-k-1}$ is given by the vanishing of the determinant of the matrix whose r^{th} row is $d^r x(t_p)/dt_p^r$ if $1 \le r \le k$, $d^{n-r}x(t_q)/dt_q^{n-r}$ if $k+1 \le r \le n-1$ and $x(t_p) - x(t_q)$ if $r = n$. Here we must use affine co-ordinates, so modify the above by setting $x_0' = x_0 + x_n$ (which takes the value 1 at both P and Q), and taking co-ordinates $x_i' = x_i/x_0'$ for $1 \le i \le n$: their local expansions have the same form as before, and we now denote the determinant by $\psi(t_p,t_q)$.

(i) Since $(P,Q) \in D_{k,n-k-2}$, A must lie in O_Q^{n-k-2}, so $a_k = a_{k+1} = 0$. Further, O_P^{k+1} is spanned by O_P^{k-1}, A and A'; since $(P,Q) \notin D^{k+1,n-k-1}$, we must have $A' \notin O_Q^{n-k-1}$, so $a_k' \ne 0$.

To obtain the coefficient of t_p in the determinant it suffices to replace t_q by 0 in the calculation. Let us first consider the general case when $\alpha_i = i$ for $i \le k$ and $\beta_i = i$ for $i \le n-k-1$. Then in the last $(n-k)$ rows of the matrix, all terms below the main diagonal vanish, and those on the diagonal are non-zero constants.

It remains to consider the submatrix formed by the first k rows and columns. Now the k^{th} row is divisible by t_p. The non-zero constant terms in the other rows all lie on the principal diagonal. Hence the only term linear in t_p is a non-zero multiple of the (k,k) entry, which is $a_k' t_p$.

To allow for differing α_i and β_i it is convenient to modify the matrix as follows. In place of $d^r x(t_p)/dt_p^r$, we take $t_p^{-\alpha_r}\prod_{i=0}^{r-1}(t_p\partial/\partial t_p - \alpha_i)x(t_p)$. Note that the linear span of the operators $\prod_{i=0}^{r-1}(t_p\partial/\partial t_p - \alpha_i)$ $(1 \le r \le k)$ is the same as that of the $t_p^r(\partial/\partial t_p)^r$, so we have divided the result by a power of t_p. However the new row r has zero entries in the columns x_i for $i \le r$. Perform a corresponding modification for t_q. The above argument now applies to the resulting matrix.

(ii) Again we first consider the case when $\alpha_i = i$ for $i \le k$ and $\beta_i = i$ for $i \le n-k-1$. To obtain the desired coefficient we may substitute $t_p = 0$. We claim that to obtain a coefficient of t_q^2 we must take the elements in columns k,

$k+1$, $k+2$ in the rows with the same numbers: these entries yield, on dividing by $k!(n-k)!(n-k-1)!$,

$$\begin{vmatrix} 0 & 0 & a_{k+2} \\ b_k t_q & b_{k+1} & 0 \\ \frac{1}{2} b_k t_q^2 & b_{k+1} t_q & b_{k+2}/(n-k-1) \end{vmatrix}.$$

For in column k, the only terms not divisible by t_q^3 are those in these 3 rows. The entry in row $(k+2)$ can only be taken together with constant terms, which are either on the diagonal or in row k. The term in row $(k+1)$ is divisible by t_q. The only term in column $(k+1)$ and not in row $(k+1)$ which is not divisible by t_q^2 is in row $(k+2)$. The claim follows.

As in (i), we now have the product of the non-zero constant determinant formed from the first $(k-1)$ rows and columns, the determinant formed from the last $(n-k-2)$, which has non-zero constant term, and the above, which reduces to $\frac{1}{2} b_k b_{k+1} a_{k+2} t_q^2$. It remains to consider $b_k a_{k+2}$.

Now $a_{k+2} = 0$ if and only if $A \in O_Q^{n-k-3}$, if and only if $(P, Q) \in D_{k+1,n-k-3}$.

Also $b_k = 0$ if and only if $B \in O_P^{k-1}$, and we easily see that this is equivalent to $O_P^k \cap O_Q^{n-k}$ containing a line (necessarily AB), i.e. to $(P, Q) \in D^{k,n-k}$.

To allow arbitrary α_i, β_r we proceed as in (i). The one point to note is that to obtain the t_q^2, at the point where we prove $b_k \neq 0$ above, we now require $\beta_{n-k} = \beta_{n-k-1} + 1$, i.e. $s_{n-k-1}^Q = 0$.

(iii) If $s_{n-k-1}^Q \neq 0$, then since $\beta_{n-k} = \beta_{n-k-1} + s_{n-k-1}^Q + 1$, the expansion of $x_k(t_q)$ starts at a power of t_q increased by s_{n-k-1}^Q, so the effect on the above matrix is to multiply the k^{th} column by $t_q^{s_{n-k-1}^Q}$ (and to alter the numerical coefficients). Essentially the same argument as above now applies in this case. □

It follows by duality that a result corresponding to Theorem 3.2 holds also for $D^{k,n-k}$. As the hypothesis appears somewhat clumsy, we now present an alternative viewpoint.

The sequence of osculating spaces O_P^k at a point P of Γ defines a complete flag of subspaces of $\mathbb{C}^{n+1}$. Write $G := GL_{n+1}(\mathbb{C})$ and B for the Borel subgroup consisting of upper triangular matrices: then P defines a coset $g_P B \subset G/B$, so a pair of points $P, Q \in \Gamma$ defines a double coset $B g_P^{-1} g_Q B$. It is well known that each double coset contains a unique permutation matrix α (representing an element of the Weyl group of G), so the double cosets $B\alpha B$, the Schubert cells, partition G/B. The dimension of the cell is equal to the number of inversions (i.e. pairs with $i < j$ and $\alpha(i) > \alpha(j)$) of α. As we are interested in cells of low codimension, introduce the reversal permutation ρ with $\rho(i) = n - i$ for each i, and for each permutation τ denote by $\mathcal{S}_\tau$ the Schubert cell corresponding to $\alpha = \tau\rho$: this has codimension the number of inversions of τ.

Suppose the pair $(P, Q) \in \Gamma \times \Gamma$ determines the permutation α. Then there exists a basis $\{e_i\}$ of $\mathbb{C}^{n+1}$ such that. for each k, O_P^k is spanned by $e_0, \ldots, e_k$ and O_Q^k by $e_{\alpha(0)}, \ldots, e_{\alpha(k)}$: thus the pair of flags belongs to $\mathcal{S}_\tau$. The condition that

$(P, Q) \in T_{k,n-k-1}$ is now that $\{0, 1, \ldots, k\} \cap \{\tau(n), \ldots, \tau(k+1)\} \neq \emptyset$, hence that for some $i \leq k$ we have $k+1 \leq \tau^{-1}(i)$. The case of lowest codimension satisfying this is $\tau = (k, k+1)$, with just one reversal. It is convenient also to write $\sigma = \tau^{-1}$.

Similarly we have $(P, Q) \in D_{k,n-k-2}$ if and only if, for some $i \leq k$, we have $\sigma(i) \geq k+2$; the generic case here $\sigma = (k, k+2, k+1)$, with codimension 2.

Also, $(P, Q) \in D^{k,n-k}$ if and only if there exist $i < j \leq k$ with $\sigma(i), \sigma(j) \geq k$. We see in succession that this is equivalent to

$$\#(\sigma[0,k] \cap [k,n]) \geq 2, \quad \#(\sigma[0,k] \cap [0,k-1]) \leq k-1,$$
$$\#(\sigma^{-1}[0,k-1] \cap [0,k]) \leq k-1, \quad \#(\sigma^{-1}[0,k-1] \cap [k+1,n]) \geq 1;$$

and hence to: for some $i \leq k-1$, we have $\tau(i) \geq k+1$. Here the generic case is $\tau = (k-1, k+1, k)$.

Although in fact we can only construct it locally, we can think of a map $\pi : \Gamma \times \Gamma \to G/B$, and we expect that away from the diagonal this is transverse to the stratification by Schubert varieties, hence in particular that we only encounter those of codimension at most 2. We note also that the cohomology ring of G/B is generated by the classes of the Schubert varieties of codimension 1, and $\mathscr{S}_{(k,k+1)}$ and $\mathscr{S}_{(k+1,k+2)}$ intersect transversely along $\mathscr{S}_{(k,k+1,k+2)}$ and $\mathscr{S}_{(k,k+2,k+1)}$.

The assumptions made in Theorem 4.2 are somewhat weaker than this. Let $(P, Q) \in \Gamma \times \Gamma$ correspond to the permutations τ and $\sigma = \tau^{-1}$. For $j \leq k$, denote by $[j, k]$ the set of integers i with $0 \leq i \leq n$ and $j \leq i \leq k$. Then

Lemma 3.3. *A point $(P, Q) \in T_{k,n-k-1}$ lies in none of $D_{k,n-k-2}$, $D_{k-1,n-k-1}$, $D^{k,n-k}$, $D^{k+1,n-k-1}$ if and only if $\tau(k) = k+1$, $\tau(k+1) = k$, and τ permutes $[0, k-1]$ and $[k+2, n]$.*

If $(P, Q) \in D_{k,n-k-2}$, then it lies in none of $D_{k-1,n-k-1}$, $D^{k+1,n-k-1}$, $D^{k,n-k}$, $D^{k+2,n-k-2}$ and $D_{k+1,n-k-3}$ if and only if $\tau(k) = k+2$, $\tau(k+1) = k$, $\tau(k+2) = k+1$, and τ permutes $[0, k-1]$ and $[k+3, n]$.

Proof. Since $(P, Q) \in T_{k,n-k-1}$, we have $i_0 \leq k$ with $\sigma(i_0) \geq k+1$.

Since $(P, Q) \notin D_{k,n-k-2}$, $i \leq k \Rightarrow \sigma(i) \leq k+1$, so $\sigma(i_0) = k+1$.

Since $(P, Q) \notin D_{k-1,n-k-1}$, $i \leq k-1 \Rightarrow \sigma(i) \leq k$, so $i_0 = k$.

Since $(P, Q) \notin D^{k,n-k}$, $\#(\sigma([0,k]) \cap [k,n]) \leq 1$; we already have $\sigma(k) = k+1$, so $i \leq k-1 \Rightarrow \sigma(i) \leq k-1$, i.e. σ permutes $[0, k-1]$.

Since $(P, Q) \notin D^{k+1,n-k-1}$, $\#(\sigma([0,k+1]) \cap [k+1,n]) \leq 1$; we already have $\sigma(k) = k+1$, so $\sigma(k+1) \leq k$, and it now follows that $\sigma(k+1) = k$, and hence σ permutes the remaining elements $[k+2, n]$.

Next, since $(P, Q) \in D_{k,n-k-2}$, for some $i_0 \leq k$ we have $\sigma(i_0) \geq k+2$.

Since $(P, Q) \notin D_{k-1,n-k-1}$, $i \leq k-1 \Rightarrow \sigma(i) \leq k$; hence $i_0 = k$.

Since $(P, Q) \notin D_{k+1,n-k-3}$, $i \leq k+1 \Rightarrow \sigma(i) \leq k+2$; hence $\sigma(i_0) = k+2$.

Now as $(P, Q) \notin D^{k,n-k}$, there can be no $j \leq k$ other than k with $\sigma(j) \geq k$, so σ induces a permutation of $[0, k-1]$.

As $(P, Q) \notin D^{k+1,n-k-1}$, k is the only number $j \leq k+1$ with $\sigma(j) \geq k+1$, so $\sigma(k+1) = k$.

Finally as $(P, Q) \notin D^{k+2,n-k-2}$, k is the only number $j \leq k+2$ with $\sigma(j) \geq k+2$, so $\sigma(k+2) = k+1$. $\square$

To apply Theorem 3.2, we must restrict $D_{k,n-k-2}$ to be disjoint from $D_{k-1,n-k-1}$, $D^{k+1,n-k-1}$, $D_{k+1,n-k-3}$ and $D^{k,n-k}$. Interchanging the suffices, we see that we also need it disjoint from $D^{k+2,n-k-2}$, so the hypothesis of Lemma 3.3 holds. We call a $D_{k,n-k-2}$ point *neat* if this is the case. It now follows from the lemma that the condition that all the D strata are neat is equivalent to restricting each permutation to be a product of disjoint cycles of the form $(k-1, k)$, $(k-1, k+1, k)$ and $(k-1, k, k+1)$.

We can deal with the singular points of $T_{k,n-k-1}$ other than D points and $\Delta(W)$ points by an argument similar to the above.

Proposition 3.4. *If $(P, Q) \in T_{k,n-k-1}$ with $s_k^P = 1$ and $s_{n-k-1}^Q = 1$ is not a D point, then at (P, Q) the curve $T_{k,n-k-1}$ has an ordinary double point, with neither branch tangent to either axis. More precisely, the coefficients of t_p^2 and t_q^2 in ψ are non-zero, while the coefficient of $t_p t_q$ vanishes.*

Proof. By Lemma 3.3, (P, Q) corresponds to a permutation which preserves the subsets $[0, k-1]$ and $[k+2, n]$ and interchanges k and $k+1$. We can thus take co-ordinates such that the leading terms in the local expansion at P are $x_i = t_p^{\alpha_i}$ for $i \leq k+1$ and $a_i t_p^{\alpha_{k+2}}$ for $i > k+1$, and at Q are $x_{n-i} = t_q^{\beta_i}$ for $i < n-k-1$, $x_{k+1} = b_{k+1} t_q^{\beta_{n-k}}$, $x_k = b_k t_q^{\beta_{n-k-1}}$ and $x_i = b_i t_q^{\beta_{n-k+1}}$ for $i < k$; where b_k, b_{k+1} are non-zero and the other a_i and b_i may contain powers of t_p and t_q respectively.

It will be convenient first to suppose $\alpha_i = i$ for $i \leq k$ and $\beta_i = i$ for $i \leq n-k-1$. Since $s_k^P = 1$ and $s_{n-k-1}^Q = 1$, we then have $\alpha_{k+1} = k+2$ and $\beta_{n-k} = n-k+1$.

As in the proof of Theorem 3.2, the equation ψ is given by a determinant, the rows of which are derivatives of the rows $x(t_p)$ and $x(t_q)$. First set $t_q = 0$ to find the coefficient of t_p^2. Then in the last $(n-k-2)$ rows the non-zero entries are those in the main diagonal; in row $(k+1)$ we just have the entry in column k. In the $(k+1)^{st}$ column, all entries are divisible by t_p^2; indeed all of these except the entry in row k are divisible by t_p^3, so for the desired coefficient we must use this entry. There remain the first $(k-1)$ rows and columns: here the entries with non-zero constant term are just those on the principal diagonal. Hence the desired coefficient is non-zero.

Similarly, setting $t_p = 0$, the non-zero entries in the first k rows are just those on the principal diagonal. In column $(k+1)$, all entries are divisible by t_q^2, so the coefficient of t_q^2 in ψ comes only from elements of the principal diagonal, so this too is non-zero. Indeed, this result also follows from the first by interchanging the roles of P and Q.

Since each element of column $(k+1)$ is divisible either by t_p^2 or by t_q^2, the coefficient of $t_p t_q$ in ψ vanishes.

As in the proof of Theorem 3.2, we can infer that the supposition that $\alpha_i = i$ for $i \leq k$ and $\beta_i = i$ for $i \leq n-k-1$ is not essential for the result. $\square$

4. The main theorem

From now on, we assume that all W points are simple and all D points are neat. Then by Lemma 2.5, all the intersections of $T_{k,n-k-1}$ with $\Delta(\Gamma)$ occur at W points, i.e. at $\Delta(W)$, and by Lemma 3.1 at any other singular point of $T_{k,n-k-1}$ we have (a) one of $s^Q_{n-k-1} > 0$, $D_{k,n-k-2}$ and $D^{k,n-k}$, and also (b) one of $s^P_k > 0$, $D_{k-1,n-k-1}$ and $D^{k+1,n-k-1}$; hence either we have a D point or (c) the situation $s^P_k = 1$ and $s^Q_{n-k-1} = 1$ of Proposition 3.4. Thus

Lemma 4.1. *If all D points of $T_{k,n-k-1}$ are simple, and $T_{k,n-k-1}$ contains no point (P,Q) with $s^P_k > 0$ and $s^Q_{n-k-1} > 0$, the singular points of $T_{k,n-k-1}$ are the (P,P) with P a W_i point (and $c^n_{i,k} > 1$.)*

For at any other point at least one of the projections is a local isomorphism.

It follows from Proposition 2.2 that the Milnor number of $T_{k,n-k-1}$ at a W_i point is $(c^n_{i,k} - 1)^2$. Hence the total Milnor number $\mu(T_{k,n-k-1}) = \sum_i (c^n_{i,k} - 1)^2 s_i$, which was evaluated in Lemma 1.2(iv).

Further, since all D points are neat, by Theorem 3.2, at a point of $D_{k,n-k-2}$, provided $s^P_k \neq 0$, the first projection of T is an isomorphism (the coefficient of t_p is non-zero) and the second projection has a point of ramification of multiplicity $(s^Q_{n-k-1} + 1)$ (i.e. $\psi(0, t_q)$ has order $(s^Q_{n-k-1} + 2)$).

We are now ready for our main result.

Theorem 4.2. *Suppose all W points of C are simple, all D points of C are neat, and for each k, $s^P_k = 0$ for each $(P,Q) \in D_{k,n-k-2} \cup D^{k-1,n-k+1}$: then for $0 \le k \le \frac{1}{2}(n-2)$, we have*

$$d_{k,n-k-2} = r_k r_{n-k-2} - (k+2)(n-k)r_k - (k+1)(n-k+1)r_{n-k-2} + 2\sum_{n-k-1}^{n-1} r_i - \frac{1}{6}(k+1)(k+2)(3(n-k)^2 - (k+3))(2g-2).$$

The values of the remaining invariants follow from the symmetries $d_{k,\ell}(C) = d_{\ell,k}(C)$ and $d^{k,\ell}(C) = d_{n-1-k,n-1-\ell}(C^\vee)$. In particular, for $1 \le k \le \frac{1}{2}n$, we have

$$d^{k,n-k} = r_k r_{n-k} + 2\sum_0^{k-1} r_i - k(n-k+2)r_k - (k+1)(n-k+1)r_{n-k} - \frac{1}{6}k(k+1)(3(n-k+1)^2 - (k+2))(2g-2).$$

Note that these formulae give incorrect values if k does not satisfy the stated condition.

Proof. We will calculate $\chi(T_{k,n-k-1})$ in two different ways. First we suppose that, for each k, $T_{k,n-k-1}$ contains no point (P,Q) with $s^P_k > 0$ and $s^Q_{n-k-1} > 0$, so is singular only at $\Delta(W)$.

To simplify the appearance of the next calculation write, for now, K_k for $(k+1)(n-k)$. On one hand, applying (3) gives

$$\begin{aligned}-\chi(T_{k,n-k-1}) &= 2(r_k - K_k)(r_{n-k-1} - K_k) + (2g-2)(r_k + r_{n-k-1} - 2K_k) \\ &\quad - 2gK_k^2 - \mu(T_{k,n-k-1}).\end{aligned}$$

On the other hand, the projection of $T_{k,n-k-1}$ on the first factor has degree $r_{n-k-1} - K_k$. Since $\chi(\Gamma) = 2 - 2g$, $\chi(T_{k,n-k-1})$ is equal to $(r_{n-k-1} - K_k)(2-2g)$, diminished by the effect of ramification. According to Lemma 3.1, we have three cases to consider.

For $(P, Q) \in D_{k,n-k-2}$ or $D^{k,n-k}$ we have an ordinary branch point; it follows from Theorem 3.2 that, provided (in the former case) $s^Q_{n-k-1} = 0$, such branching gives a term $d_{k,n-k-2} + d^{k,n-k}$.

For $P = Q$ a W_i point, $T_{k,n-k-1}$ has a singular point at (P, P) with $c^n_{i,k}$ mutually transverse branches. Hence this contributes $c^n_{i,k} - 1$ to the Euler characteristic calculation; the total such contribution is thus $\sum_i (c^n_{i,k} - 1)s_i$ (strictly, if $c^n_{i,k} = 1$ the point is not singular, but the contribution to the sum is 0).

For Q a W_{n-k-1} point and $P \neq Q$, we again have an ordinary branch point. Now Q corresponds in principle to $r_k - K_k$ points P; however we know that Q itself counts here with multiplicity $c^n_{n-k-1,k} = c^n_{k,k}$. Hence the total contribution from such pairs is $(r_k - K_k - c^n_{k,k})s_{n-k-1}$.

If there exists a point $(P, Q) \in D_{k,n-k-2}$ with s^Q_{n-k-1} non-zero (hence equal to 1), then by Theorem 3.2(iii), $T_{k,n-k-1}$ is defined in terms of local co-ordinates at (P, Q) by an equation $t_p = \phi(t_q)$ where ϕ has order 3 at 0. In this case, while the contribution of the point P to the calculation is increased by 1, that of Q is decreased by 1, since one of the points $(P, Q) \in T_{k,n-k-1}$ now coincides with P. Thus the total contribution is unchanged.

Putting these results together, we have

$$\begin{aligned}\chi(T_{k,n-k-1}) &= (r_{n-k-1} - K_k)(2-2g) - (d_{k,n-k-2} + d^{k,n-k}) \\ &\quad - (r_k - K_k - c^n_{k,k})s_{n-k-1} - \sum_i (c^n_{i,k} - 1)s_i.\end{aligned}$$

Comparing our two calculations of $\chi(T_{k,n-k-1})$ gives

$$\begin{aligned}&2(r_k - K_k)(r_{n-k-1} - K_k) + (2g-2)(r_k + r_{n-k-1} - 2K_k) - 2gK_k^2 \\ &\quad - \sum_i (c^n_{i,k} - 1)^2 s_i + (r_{n-k-1} - K_k)(2-2g) - (d_{k,n-k-2} + d^{k,n-k}) \\ &\quad - (r_k - K_k - c^n_{k,k})s_{n-k-1} - \sum_i (c^n_{i,k} - 1)s_i = 0.\end{aligned}$$

Substituting for s_{n-k-1} and collecting terms, we have

$$\begin{aligned}d_{k,n-k-2} + d^{k,n-k} &= r_k(r_{n-k-2} + r_{n-k}) - K_k(2r_k + r_{n-k-2} + r_{n-k}) \\ &\quad + c^n_{k,k}(2r_{n-k-1} - r_{n-k-2} - r_{n-k} + (2g-2)) \\ &\quad - (2g-2)K_k^2 - \sum_i c^n_{i,k}(c^n_{i,k} - 1)s_i.\end{aligned}$$

We can now replace K_k by $(k+1)(n-k)$ and substitute $\sum_i c^n_{i,k}(c^n_{i,k}-1)s_i = \frac{1}{3}k(k+1)(3n-4k-2)(2g-2) - \sum_0^{k-1} 2(r_i + r_{n-i-1}) + 2k(r_k + r_{n-k-1})$ from Lemma 1.2(v).

We may now consider the case when $T_{k,n-k-1}$ contains a point (P,Q) with $s^P_k = 1$ and $s^Q_{n-k-1} = 1$. Then by Proposition 3.4, the curve $T_{k,n-k-1}$ has an ordinary double point at (P,Q), with neither branch tangent to either axis. The effect of this on the calculations is to increase our estimate of $\chi(T_{k,n-k-1})$ by 1, but also to increase $\mu(T_{k,n-k-1})$ by 1. These cancel out, so the result is unchanged.

This yields equations E_k, say, for $0 \le k \le n-1$, where E_k gives an explicit value for $d_{k,n-k-2} + d^{k,n-k}$. Since $d_{k-1,n-k-1}$ and $d^{k,n-k}$ are only defined for $1 \le k \le n-1$, E_0 has only one term on the left, and gives $d_{0,n-2}$ explicitly; dually E_{n-1} gives $d^{n-1,1} = d^{1,n-1}$. Now E_1 gives $d_{1,n-3} + d^{1,n-1}$ and hence $d_{1,n-3}$. Continuing by induction, we can determine all the $d_{k-1,n-k-1}$ and $d^{k,n-k}$.

It will thus suffice to verify that the stated formula gives the correct values of $d_{k,n-k-2} + d^{k,n-k}$. Note that if $n = 2m$ is even, we have $2m$ equations for $2m$ variables; if $n = 2m+1$ is odd, there are $2m+1$ equations for $2m$ variables: there is a consistency requirement that $\sum_0^{2m}(-1)^k E_k$ vanish identically, which has been a useful check in my calculations. Given the explicit formula stated above, all that remains is a rather trivial verification. We note a few points which clarify how to do this.

It is simpler to split each equation into 3 terms: (q) quadratic in the r_i, (l) linear in the r_i, and (c) independent of the r_i (but divisible by $(2g-2)$).

The easiest is

$$d_{k,n-k-2}(q) + d^{k,n-k}(q) = r_k(r_{n-k-2} + r_{n-k});$$

now by induction we find $d_{k,n-k-2}(q) = r_k r_{n-k-2}$ and $d^{k,n-k}(q) = r_k r_{n-k}$.

For the equations $E_k(l)$, we need to distinguish cases $k \le n-k-2$, $k = n-k-1$ and $k \ge n-k$. All are similar. For $k \le n-k-2$, $E_k(l)$ gives

$$\begin{aligned} d_{k,n-k-2}(l) + d^{k,n-k}(l) = &-(k+1)(n-k)(r_{n-k-2} + 2r_k + r_{n-k}) \\ &-(k+1)(r_{n-k-2} - 2r_{n-k-1} + r_{n-k}) \\ &+\sum_0^{k-1} 2(r_i + r_{n-i-1}) - 2k(r_k + r_{n-k-1}), \end{aligned}$$

and the desired result follows easily.

Finally, dividing by $2g-2$ gives, for $2k \le n-1$,

$$d_{k,n-k-2}(c) + d^{k,n-k}(c) = k + 1 - (k+1)^2(n-k)^2 - \frac{1}{3}k(k+1)(3n-4k-2).$$

As the right hand side of $E_k(c)$ is unaltered by interchanging k and $n-k-1$, the only verification required in this case is that this expression is the sum of $-\frac{1}{6}(k+1)(k+2)(3(n-k)^2 - (k+3))$ and the expression obtained from this by replacing k by $(k-1)$. □

As a check on the calculation, we apply (1) to the correspondences $T_{k,n-k-1}$ and $T_{k-1,n-k}$ for $1 \le k \le n-1$. The common points are $\Delta(W)$ and points of $D_{k-1,n-k-1}$ and $D^{k,n-k}$. By Proposition 2.2, the intersection number at a W_i point is $c^n_{i,k}c^n_{i,k-1}$. It follows from Theorem 3.2 that at a point $(P,Q) \in D_{k-1,n-k-1}$, $T_{k-1,n-k}$ touches $\{P\} \times \Gamma$ and $T_{k,n-k-1}$ touches $\Gamma \times \{Q\}$, so the intersection number is 1; similarly for $D^{k,n-k}$. Hence

$$\begin{aligned}
d_{k-1,n-k-1} + d^{k,n-k} + \sum_i c^n_{i,k}c^n_{i,k-1}s_i
&= (r_{n-k-1} - (k+1)(n-k))(r_{k-1} - k(n-k+1)) \\
&\quad + (r_k - (k+1)(n-k))(r_{n-k} - k(n-k+1)) \\
&\quad - 2gk(k+1)(n-k)(n-k+1) \\
&= r_{n-k-1}r_{k-1} + r_k r_{n-k} - (k+1)(n-k)(r_{k-1} + r_{n-k}) \\
&\quad - k(n-k+1)(r_{n-k-1} + r_k) - (2g-2)k(k+1)(n-k)(n-k+1).
\end{aligned}$$

Substituting for $\sum_i c^n_{i,k}c^n_{i,k-1}s_i$ from Lemma 1.2(vii), we obtain

$$\begin{aligned}
d_{k-1,n-k-1} + d^{k,n-k} = r_{n-k-1}r_{k-1} + r_k r_{n-k} + 2\sum_0^{k-2}(r_i + r_{n-i-1}) \\
- ((k+1)(n-k) + k - 1)(r_{k-1} + r_{n-k}) \\
- (k(n-k+1) + k)(r_k + r_{n-k-1}) \\
- \{k(k+1)(n-k)(n-k+1) + nk(k+1) \\
- \frac{1}{3}k(k+1)(4k-1)\}(2g-2),
\end{aligned}$$

while substituting from Theorem 4.2 gives

$$\begin{aligned}
d_{k-1,n-k-1} &= r_{k-1}r_{n-k-1} - (k+1)(n-k+1)r_{k-1} - k(n-k+2)r_{n-k-1} \\
&\quad + 2\sum_{n-k}^{n-1} r_i - \frac{1}{6}k(k+1)(3(n-k+1)^2 - (k+2))(2g-2), \\
d^{k,n-k} &= r_k r_{n-k} + 2\sum_0^{k-1} r_i - k(n-k+2)r_k - (k+1)(n-k+1)r_{n-k} \\
&\quad - \frac{1}{6}k(k+1)(3(n-k+1)^2 - (k+2))(2g-2),
\end{aligned}$$

giving the same result.

5. Further comments

In Theorem 3.2, we needed to consider the conditions $s^P_k = 0$ and $s^Q_{n-k-1} = 0$ at a point $(P,Q) \in D_{k,n-k-2}$; in Theorem 4.2 we had to exclude the first case, but permitted the second. To illustrate this, note that in the case of plane curves, this means that we exclude a singular point with two branches and a cusp at one of them, but permit a flecnode. Observe also that since our hypotheses bear only on pairs of points of Γ, we do not exclude the case of a triple (or higher multiple) point with transverse smooth branches. We do exclude the case $\sigma = (0, 2)$, giving a double point with coincident tangents (a tacnode).

For curves in 3-space, the 'neat' hypothesis allows the permutation $\sigma = (0, 1)(2, 3)$ corresponding to the situation $P \in O_Q^2$, $Q \in O_P^2$. Presumably there is here, as well as in the plane, a way of counting multiplicities that will make our formula correct in general. Finding this seems to be interesting but difficult problem.

For curves which are ordinary in the sense of maps $\Gamma \to P^n$, we have $s_i = 0$ for $0 \le i \le n-2$, hence $r_k = (k+1)r_0 + \binom{k+1}{2}(2g-2)$ for $0 \le k \le n-1$. Substituting in Theorem 4.2 gives an expression for $d_{k,n-k-1}$, quadratic in r_0 and g, with coefficients depending on k and n, which can be reduced to

$$\frac{1}{2}(k+1)(n-k-1)\Big\{2r_0^2 + (n-2)r_0(2g-2) + \frac{1}{2}k(n-k-2)(2g-2)^2 - 2(n+1)r_0 - (n^2 - nk + k^2 - n + 2k)(2g-2)\Big\}.$$

For low values of n, Theorem 4.2 gives

$$\begin{array}{ll}
n=2 & d_{0,0} = r_0^2 - 7r_0 + 2r_1 - 3(2g-2) \\
n=3 & d_{0,1} = r_0 r_1 - 6r_0 - 4r_1 + 2r_2 - 8(2g-2) \\
n=4 & d_{0,2} = r_0 r_2 - 8r_0 - 5r_2 + 2r_3 - 15(2g-2) \\
n=4 & d_{1,1} = r_1^2 - 17r_1 + 2r_2 + 2r_3 - 23(2g-2) \\
n=5 & d_{0,3} = r_0 r_3 - 10r_0 - 6r_3 + 2r_4 - 24(2g-2) \\
n=5 & d_{1,2} = r_1 r_2 - 12r_1 - 10r_2 + 2r_3 + 2r_4 - 44(2g-2) \\
n=6 & d_{0,4} = r_0 r_4 - 12r_0 - 7r_4 + 2r_5 - 35(2g-2) \\
n=6 & d_{1,3} = r_1 r_3 - 15r_1 - 12r_3 + 2r_4 + 2r_5 - 71(2g-2) \\
n=6 & d_{2,2} = r_2^2 - 31r_2 + 2r_3 + 2r_4 + 2r_5 - 86(2g-2)
\end{array}$$

and formulae for the $d^{k,n-k}$ are easily read off, e.g.

$$n=5 \quad d^{2,3} = r_2 r_3 - 12r_3 - 10r_2 + 2r_1 + 2r_0 - 44(2g-2).$$

In the case $n = 2$ this does indeed give the traditional relations, on noting that r_0 is the degree, r_1 the class, s_0 the number of cusps, s_1 the number of flexes, $d_{0,0}$ is double the number of nodes (since $D_{0,0}$ was a set of ordered pairs of points of Γ, each node contributes 2: a similar comment applies to $d_{k,k}$ in general), and $d^{1,1}$ is twice the number of bitangents. The result for $n = 3$ is, of course, equivalent to the formula given by Zeuthen [12].

The method can in principle be extended to obtain further formulae. In [5, (2.5)], the correspondence on a space curve defined by having the chord PQ meet the curve again is considered. In general one may consider the condition on a set of points $P_i \in \Gamma$ $(1 \leq i \leq N)$ that the osculating spaces $O_{P_i}^{k_i}$ lie in a hyperplane, or more generally a subspace of dimension $n - D$. In principle, this imposes $c := D(\sum(k_i + 1) - (n + 1) + D)$ conditions, so if $c = N - 1$, it defines a correspondence between P_1 and P_2. The cases when $c = N$ each define a finite number of N^{tuples}, and studying these correspondences will give information about these numbers.

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