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Partial Differential Equations — *A Calderon-Zygmund theory for infinite energy minima of some integral functionals*, by Lucio Boccardo, communicated on 8 May 2009.

"... onore e lume
vagliami 'l lungo studio e 'l grande amore
che m'ha fatto cercar lo tuo volume."
(*Inferno I,* 82–84)[1]

Abstract. — A Calderon-Zygmund theory in Lebesgue and Marcinkiewicz spaces for infinite energy minima of some integral functionals is proved.

Key words: Calderon-Zygmund theory; Infinite energy minima.

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1. Introduction

1.1. Finite energy solutions. We recall the following regularity theorem by G. Stampacchia concerning solutions of linear Dirichlet problems in Ω, bounded subset of $\mathbb{R}^N$, $N > 2$, with right hand side a measurable function $f(x)$.

Consider a bounded elliptic matrix $M(x)$ with ellipticity constant $\alpha > 0$ and the related boundary value problem

$$(1)\qquad \begin{cases} -\mathrm{div}(M(x)Du) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

[1] *Always remembering (after* 40 *years) what I have learned from my teacher of "Istituzioni di Analisi Superiore" and "Analisi Superiore"* ([7] *pg.* 1, [8] *pg.* 1)

[2] *... e piú d'onore ancora assai mi fenno,*
ch'e' sí mi fecer de la loro schiera,
sí ch'io fui sesto tra cotanto senno.
(*Inferno IV,* 100–102)

In [20] (see also[3] [21] and [16]), the following result is proved about the solution $u \in W_0^{1,2}(\Omega)$ (recall that the coefficients of $M(x)$ are not smooth[4]) under the assumption that f belongs to the Marcinkiewicz space $M^m(\Omega)$[5]:

$$\text{(2)} \qquad \begin{cases} \text{if } m > N/2, \text{ then } u \in L^\infty(\Omega); \\ \text{if } 2N/(N+2) < m < N/2, \text{ then } u \in M^{m^{**}}(\Omega), \end{cases}$$

where $m^{**} = (m^*)^* = \frac{mN}{N-2m}$ (see [14] for new contributions in this field). The fundamental tool for the proofs of (2) by Stampacchia is the use of the test function $[u - T_k(u)]$, where $T_k(u)$ is the truncation at the levels $+k$, $-k$.

Note that the proofs of (2) do not use the linearity of the differential operator. Only the ellipticity is used, so that the results of (2) still hold for boundary value problems with nonlinear operators like

$$\text{(3)} \qquad \begin{cases} -\operatorname{div}(a(x,u,Du)) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

under the ellipticity assumption $a(x,s,\xi)\xi \geq \alpha|\xi|^2$, $\alpha > 0$.

If the matrix M is symmetric, the solution u of (1) is the minimum of the functional

$$J(v) = \frac{1}{2}\int_\Omega M(x)DvDv - \int_\Omega fv, \quad v \in W_0^{1,2}(\Omega).$$

Thus the regularity theorem by Stampacchia can be stated in the following way: if $f \in M^m(\Omega)$ and $m > N/2$, the minimum u of J belongs to $L^\infty(\Omega)$; if $2N/(N+2) < m < N/2$, the minimum belongs to $M^{m^{**}}(\Omega)$. Moreover, the proof of (2) can be easily adapted to the study of minima u of more general integral functionals like

$$\int_\Omega j(x,v,Dv) - \int_\Omega fv, \quad v \in W_0^{1,2}(\Omega).$$

The first result concerning the summability (in Lebesgue spaces) of u, solution of (1), is again due to G. Stampacchia: u belongs to $L^{m^{**}}(\Omega)$ if $f \in L^m(\Omega)$, $2N/(N+2) \leq m < N/2$. The proof uses (2), linear interpolation theory and the Marcinkiewicz–Zygmund Theorem. However, a summability result for weak (finite energy) solutions is proved in [11], [12] by a direct method which uses as test function a suitable power of u.

In [13] the regularity results for minima of functionals are extend to the Lebesgue framework.

[3] but for me see "Appunti del corso di Analisi Superiore—Università di Roma—a.a. 1969–70"

[4] (Calderon-Zygmund without derivatives)

[5] $M^m(\Omega)$, $m > 0$, is the space of measurable functions v on Ω such that

$$\exists C \geq 0 : \text{meas}\{x \in \Omega : |v(x)| \geq t\} \leq Ct^{-m}, \quad \forall t > 0$$

1.2. Infinite energy solutions. If the datum f belongs to larger spaces ($L^m(\Omega)$, $1 \le m < 2N/(N+2)$ or $M^m(\Omega)$, $1 < m < 2N/(N+2)$) the regularity of the distributional (infinite energy) solutions u of (3) and of Du (nonlinear Calderon–Zygmund Theory) is proved in [13] in the Lebesgue framework and in [6] in the Marcinkiewicz framework (see also [17]).

If the datum f belongs to larger spaces as above, it is not possible to use the definition of minimum, because the associated functional is not well defined on the "energy space" $W_0^{1,2}(\Omega)$.

A possible way to handle minimization problems is then the use of T-minima, introduced in [2]. Minimization problems for integral functionals with nonregular data are also studied in [3], [5], [4], [19] (in these papers the function j of (4) can also depend on u) and [18], where existence of minima is proved also for functionals with measure data, using the definition of "weak minimum" introduced by Iwaniec and Sbordone [15].

Of course, it is possible to work with the same proofs if the standard framework is $W_0^{1,p}(\Omega)$ instead of $W_0^{1,2}(\Omega)$; that is: if the assumption of coercivity is $\alpha|\xi|^p \le j(x,\xi) \le \beta|\xi|^p$, $1 < p \le N$, instead of (5) (see below).

1.3. Assumptions. Let $j(x,\xi)$ be a function defined in $\Omega \times \mathbb{R}^N$. On $j(x,\xi)$ we assume the standard hypotheses of the integrands in the Calculus of Variations, which lead to existence and uniqueness of minima in $W_0^{1,2}(\Omega)$ of $\int_\Omega j(x,Dv) - \int_\Omega f(x)v(x)$, if $f \in L^2(\Omega)$, that is:

$$\text{(4)} \qquad \begin{cases} \text{the function } j(x,\xi) \text{ is measurable with respect to } x \\ \text{and strictly convex with respect to } \xi \end{cases}$$

there exist $\alpha, \beta > 0$ such that

$$\text{(5)} \qquad \alpha|\xi|^2 \le j(x,\xi) \le \beta|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. in } \Omega.$$

Recalling the definition of truncation $T_k : \mathbb{R} \mapsto \mathbb{R}$

$$T_k(t) = \begin{cases} t, & |t| \le k, \\ k\frac{t}{|t|}, & |t| > k, \end{cases}$$

we give the definition and the existence theorem for T-minima.

Definition 1.1 ([2]). *Let $f \in L^1(\Omega)$. A measurable function u is a T-minimum for the functional*

$$\text{(6)} \qquad J(v) = \int_\Omega j(x,Dv) - \int_\Omega f(x)v(x)$$

if

$$\text{(7)} \qquad \begin{cases} T_i(u) \in W_0^{1,2}(\Omega), \quad \forall i > 0: \\ \displaystyle\int_{\{|u-\varphi|\le i\}} j(x,Du) - \int_\Omega f(x)T_i[u-\varphi] \le \int_{\{|u-\varphi|\le i\}} j(x,D\varphi), \\ \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \quad \forall i > 0. \end{cases}$$

Proposition 1 ([2]). *Under the assumptions* (4) *and* (5) *there exists a T-minimum u of the functional J defined in* (6). *Moreover the T-minimum u is unique and, in the case of differentiability of $j(x,\xi)$ with respect to ξ, the T-minimum is the entropy solution* (*see* [1]) *of the Euler-Lagrange equation for J.*

Remark 1. In [2] it is also proved that $u \in W_0^{1,q}(\Omega)$, $\forall q < N/(N-1)$.

2. A Calderon-Zygmund theory in Lebesgue spaces for infinite energy minima of some integral functionals

For Dirichlet problems with measure (or L^1) data, existence of distributional solutions is proved in [9]; while in [10] it is proved that the assumption $f \in L^m(\Omega)$, $1 < m < 2N/(N+2)$, yields more summability for the solutions and their gradients.

We use the following definitions, for $k \in \mathbb{R}^+$,

$$A_k = \{x \in \Omega : k \le |u(x)|\}, \quad B_k = \{x \in \Omega : k \le |u(x)| < k+1\}.$$

Theorem 2.1 (Calderon-Zygmund theory for functionals 1). *Under the assumptions* (4) *and* (5), *if $f \in L^m(\Omega)$, $1 < m < 2N/(N+2)$, then there exists a positive constant C_f such that the T-minimum u of the functional J of* (6) *satisfies the estimates $\|u\|_{L^{m^{**}}(\Omega)} \le C_f$ and $\|Du\|_{L^{m^*}(\Omega)} \le C_f$.*

Proof. Let $k > 0$ be fixed. In the definition of T-minimum we use as test function $\varphi = T_k(u)$ and $i = 1$. We then obtain

$$\int_{\{|u-T_k(u)|\le 1\}} j(x,Du) \le \int_{\{|u-T_k(u)|\le 1\}} j(x,DT_k(u)) + \int_\Omega f(x)T_1[u-T_k(u)],$$

which implies, since $j(x,0) = 0$,

$$\alpha \int_{B_k} |Du|^2 \le \int_{B_k} j(x,Du) \le \int_\Omega fT_1[u-T_k(u)] \le \int_{A_k} |f| \tag{8}$$

Define $\theta = \frac{m^{**}}{2^*}$, so that $\theta 2^* = (2\theta - 1)m' = m^{**}$. Then a consequence of (8) is that

$$\alpha \int_{B_k} \frac{|Du|^2}{(1+|u|)^{2(1-\theta)}} \le \alpha \int_{B_k} \frac{|Du|^2}{(1+k)^{2(1-\theta)}} \le \int_{A_k} \frac{|f|}{(1+k)^{2(1-\theta)}}, \tag{9}$$

which implies, summing on k ranging from 0 to $M-1$,

$$\begin{aligned} \alpha \sum_{k=0}^{k=M-1} \int_{B_k} \frac{|Du|^2}{(1+|u|)^{2(1-\theta)}} &\le \sum_{k=0}^{k=M-1} \int_{A_k} \frac{|f|}{(1+k)^{2(1-\theta)}} \\ &\le \sum_{k=0}^{k=M} \int_{A_k} \frac{|f|}{(1+k)^{2(1-\theta)}}. \end{aligned} \tag{10}$$

Observe that $A_k = \bigcup_{h=k}^{h=\infty} B_h$, and that $\{|u| \le M\} = \bigcup_{k=0}^{k=M-1} B_k$, so that a consequence of (10) is

$$\alpha \int_\Omega \frac{|DT_M(u)|^2}{(1+|T_M(u)|)^{2(1-\theta)}} \le \sum_{k=0}^{M} \sum_{h=k}^{h=\infty} \int_{B_h} \frac{|f|}{(1+k)^{2(1-\theta)}}.$$

Exchanging the summation order in the right hand side, we obtain

$$(11)\quad \begin{aligned} &\sum_{k=0}^{k=M} \sum_{h=k}^{h=\infty} \int_{B_h} \frac{|f|}{(1+k)^{2(1-\theta)}} = \sum_{k=0}^{k=M} \frac{1}{(1+k)^{2(1-\theta)}} \sum_{h=k}^{h=\infty} \int_{B_h} |f| \\ &= \sum_{h=0}^{h=\infty} \int_{B_h} |f| \sum_{k=0}^{k=T_M(h)} \frac{1}{(1+k)^{2(1-\theta)}} \le \sum_{h=0}^{h=\infty} \frac{1}{2\theta-1} \int_{B_h} |f|(1+T_M(h))^{2\theta-1} \\ &\le \frac{1}{2\theta-1} \int_\Omega |f|[1+|T_M(u)|]^{2\theta-1}, \end{aligned}$$

since one has

$$\begin{aligned} \sum_{k=0}^{k=T_M(h)} \frac{1}{(1+k)^{2(1-\theta)}} &\le \sum_{k=0}^{k=T_M(h)} \int_k^{k+1} \frac{dx}{x^{2(1-\theta)}} = \int_0^{1+T_M(h)} \frac{dx}{x^{2(1-\theta)}} \\ &= \frac{[1+T_M(h)]^{2\theta-1}}{2\theta-1}. \end{aligned}$$

Thus we have

$$(12)\qquad \alpha \int_\Omega \frac{|DT_M(u)|^2}{(1+|T_M(u)|)^{2(1-\theta)}} \le \frac{1}{2\theta-1} \int_\Omega |f|[1+|T_M(u)|]^{2\theta-1},$$

and (thanks to the Sobolev inequality),

$$\begin{aligned} &\|(1+|T_M(u)|)^\theta\|_{L^{2^*}(\Omega)} \le \|[(1+|T_M(u)|)^\theta - 1]\|_{L^{2^*}(\Omega)} + C_\Omega \\ &\le C_1 \left[\int_\Omega |f|[1+|T_M(u)|]^{2\theta-1}\right]^{1/2} + C_\Omega. \end{aligned}$$

Using the Hölder inequality, we then have

$$\int_\Omega (1+|T_M(u)|)^{\theta 2^*} \le C_2 \|f\|_{L^m(\Omega)}^{2^*/2} \left[\int_\Omega [1+|T_M(u)|]^{(2\theta-1)m'}\right]^{2^*/2m'} + C_2.$$

Note that $\frac{1}{2} < \theta < 1$ since $1 < m < \frac{2N}{N+2}$. Thus we proved the inequality

$$\left[\int_\Omega [1+|T_M(u)|]^{m^{**}}\right]^{1/m^{**}} \le C_f \|f\|_{L^m(\Omega)} + C_f,$$

which implies, as $M \to \infty$ (thanks to Fatou Lemma),

$$\left[\int_\Omega [1+|u|]^{m^{**}}\right]^{1/m^{**}} \le C_f \|f\|_{L^m(\Omega)} + C_f, \tag{13}$$

that is the first part of the result.

Now we use (12), the above estimate (13) and Hölder inequality.

$$\int_\Omega \frac{|DT_M(u)|^2}{(1+|T_M(u)|)^{2(1-\theta)}} \le C_4\|f\|_{L^1(\Omega)} + C_4\|f\|_{L^m(\Omega)}\left[\int_\Omega |u|^{m^{**}}\right]^{1/m'} = C_0.$$

Then the use of (once more) estimate (13) and Fatou Lemma (as $M \to \infty$) in the inequality

$$\left|\begin{aligned} &\int_\Omega |DT_M(u)|^{m^*} = \int_\Omega \frac{|DT_M(u)|^{m^*}}{(1+|T_M(u)|)^{m^*(1-\theta)}}(1+|T_M(u)|)^{m^*(1-\theta)} \\ &\le {C_0}^{m^*/2}\left[\int_\Omega (1+|u|)^{m^{**}}\right]^{(2-m^*)/2} \end{aligned}\right. \tag{14}$$

gives the second part of the result. □

Theorem 2.2 (A borderline case). *Under the assumptions* (4) *and* (5), *if*

$$\int_\Omega |f|\log(1+|f|) < \infty,$$

then the T-minimum u of the functional J of (6) *belongs to* $W_0^{1,1^*}(\Omega)$.

Proof. Since one has

$$\sum_{k=0}^{k=T_M(h)} \frac{1}{(1+k)} \le 1 + \sum_{k=1}^{k=T_M(h)} \int_k^{k+1} \frac{dx}{x} = 1 + \log[1+T_M(h)],$$

if we put $\theta = 1/2$ in (11), inequality (12) becomes

$$\alpha \int_\Omega \frac{|DT_M(u)|^2}{(1+|T_M(u)|)} \le \int_\Omega |f| + \int_\Omega |f|\log(1+|T_M(u)|). \tag{15}$$

Then (thanks to Sobolev and Young inequalities) we have

$$\left|\begin{aligned} &\left[\int_\Omega (1+|T_M(u)|)^{N/(N-2)}\right]^{(N-2)/2N} \\ &= \|(1+|T_M(u)|)^{1/2}\|_{L^{2N/(N-2)}(\Omega)} \le \|(1+|T_M(u)|)^{1/2} - 1\|_{L^{2N/(N-2)}(\Omega)} + C_1 \\ &\le C_1 + C_2\left[\int_\Omega |f|\right]^{1/2} + C_2\left[\int_\Omega |f|\log(1+|f|)\right]^{1/2} + C_2\left[\int_\Omega e^{\log(1+|T_M(u)|)}\right]^{1/2}. \end{aligned}\right.$$

Thus we proved the inequality

$$\int_\Omega |T_M(u)|^{N/(N-2)} \le C_f,$$

which implies (as $M \to \infty$)

$$\int_\Omega |u|^{N/(N-2)} \le C_f.$$

As a consequence of this estimate, of inequality (15) and of Fatou Lemma (as $M \to \infty$) we have

$$\int_\Omega \frac{|Du|^2}{(1+|u|)} \le C_0.$$

Here we repeat (14) with $m = 1$ and we prove the result concerning the summability of the gradient. □

3. A Calderon-Zygmund theory in Marcinkiewicz spaces for infinite energy minima of some integral functionals

Theorem 3.1 (Calderon-Zygmund theory for functionals 2). *Under the assumptions* (4) *and* (5), *if* $f \in M^m(\Omega)$, $1 < m < 2N/(N+2)$, *then there exists a positive constant* C_f *such that the* T*-minimum* u *of the functional* J *of* (6) *satisfies the estimates*

$$\text{meas}\{k \le |u|\} \le \frac{C_f}{k^{m^{**}}}$$

and

$$\text{meas}\{t \le |Du|\} \le \frac{C_f}{t^{m^*}}.$$

Proof. We start as in Theorem 2.1, we use (9) with $\theta = \frac{m^{**}}{2^*}$. If we sum these inequalities, with k ranging now between $j \ge 1$ and M, we obtain

$$\alpha \int_{j \le |u| < M} \frac{|Du|^2}{|u|^{2(1-\theta)}} = \alpha \sum_{k=j}^{k=M-1} \int_{B_k} \frac{|Du|^2}{|u|^{2(1-\theta)}} \le \sum_{k=j}^{k=M} \int_{A_k} \frac{|f|}{k^{2(1-\theta)}}. \tag{16}$$

Exchanging the summation order, as in (11), we have

$$\alpha \int_{j \le |u|} \frac{|DT_M(u)|^2}{|T_M(u)|^{2(1-\theta)}} \le C_\theta \int_{j \le |u|} |f|\,|T_M(u)|^{2\theta-1},$$

which implies

$$\left[\int_{j\le|u|}|T_M(u)|^{\theta 2^*}\right]^{2/2^*}\le C_f\left[\int_{j\le|u|}|T_M(u)|^{(2\theta-1)m'}\right]^{1/m'}, \tag{17}$$

so that, since $\frac{2}{2^*}>\frac{1}{m'}$

$$\left[\int_{j\le|u|}|T_M(u)|^{m^{**}}\right]^{1/m^{**}}\le C_1$$

and, as $M\to\infty$,

$$j[\text{meas}\{j\le|u|\}]^{1/m^{**}}\le\left[\int_{j\le|u|}|u|^{m^{**}}\right]^{1/m^{**}}\le C_1,$$

that is the Marcinkiewicz estimate on u:

$$\text{meas}\{j\le|u|\}\le\frac{C_0}{j^{m^{**}}}. \tag{18}$$

With respect to the gradient, from (8) and (18) we have, since our assumption on m implies $\frac{m^{**}}{m'}<1$,

$$\left|\begin{aligned}&\alpha\int_\Omega|DT_M(u)|^2=\alpha\sum_{k=0}^{k=M-1}\int_{B_k}|Du|^2\le\sum_{k=0}^{k=M}\int_{A_k}|f|\\&\le\int_\Omega|f|+\sum_{k=1}^{k=M}\frac{\tilde{C}_f}{k^{m^{**}/m'}}\le C_1M^{(2N-2m-mN)/(N-2m)}\end{aligned}\right.$$

Here we follow a technique of [1]. The previous estimate also implies

$$t^2\,|\{|u|<k\}\cap\{|Du|\ge t\}|\le\int_{\{|u|<k\}\cap\{|Du|\ge t\}}|Du|^2\le\frac{C_1}{\alpha}k^{1-m^{**}(1-1/m)}$$

On the other hand the inequality

$$|\{|Du|\ge t\}|\le|\{|Du|\ge t,|u|<k\}|+|\{|u|\ge k\}|$$

and (18) give

$$|\{|Du|\ge t\}|\le\frac{C_1}{\alpha}\,\frac{k^{1-m^{**}(1-1/m)}}{t^2}+C_0\frac{1}{k^{m^{**}}}.$$

Note that

$$m^{**}\Big(1-\frac{1}{m}\Big)=\frac{(m-1)N}{N-2m},\quad 1-m^{**}\Big(1-\frac{1}{m}\Big)=\frac{2N-m(N+2)}{N-2m}\in(0,1].$$

The minimization with respect to k gives (choose $k=t^{(N-2m)/(N-m)}$)

$$|\{|Du| \geq t\}| \leq \frac{C_f}{t^{m^*}}, \tag{19}$$

as desired. □

LEMMA 3.2 (A borderline case). *Under the assumptions* (4) *and* (5), *if* $f \in M^{2N/(N+2)}(\Omega)$, *then there exists a positive constant* C_f *such that the T-minimum u of the functional J of* (6) *satisfies the estimates*

$$\text{meas}\{k \leq |u|\} \leq \frac{C_f}{k^{2N/(N-2)}} \tag{20}$$

PROOF. Let $1 < m < 2N/(N+2)$ and $\theta = \frac{m^{**}}{2^*}$. We start as in Theorem 3.1, we get (17), we use Hölder and Young inequalities and we have

$$\left|\begin{aligned} &\left[\int_{j\leq|u|} |T_M(u)|^{m^{**}}\right]^{2/2^*} \leq C_m \left[\int_{j\leq|u|} |f|^m\right]^{1/m} \left[\int_{j\leq|u|} |T_M(u)|^{m^{**}}\right]^{1/m'} \\ &\leq \frac{1}{2}\left[\int_{j\leq|u|} |T_M(u)|^{m^{**}}\right]^{2/2^*} + C_1 \left[\int_{j\leq|u|} |f|^m\right]^{(N-2)/(N-2m)}. \end{aligned}\right.$$

Now, thanks to Fatou Lemma (as $M \to \infty$), we have

$$\left[\int_{j\leq|u|} |u|^{m^{**}}\right]^{2/2^*} \leq 2C_1 \left[\int_{j\leq|u|} |f|^m\right]^{(N-2)/(N-2m)}.$$

Now we use Hölder inequality in Marcinkiewicz framework and we have

$$\left[\int_{j\leq|u|} |u|^{m^{**}}\right]^{2/2^*} \leq C_2[\text{meas}\{j \leq |u|\}]^{[2N-m(N+2)](N-2)/2N(N-2m)}$$

which implies

$$j^{2m^{**}/2^*}[\text{meas}\{j \leq |u|\}]^{(N-2)/N} \leq C_2[\text{meas}\{j \leq |u|\}]^{[2N-m(N+2)]m^{**}/mN2^*}$$

that is

$$\text{meas}\{j \leq |u|\} \leq \frac{C_f}{j^{2N/(N-2)}},$$

that is the Marcinkiewicz estimate on u. □

REMARK 2. We are not able to prove the Marcinkiewicz estimate in $M^2(\Omega)$ on the gradient, under the same assumptions of the previous lemma. Note that, if we put together the above estimate and (8), we have

$$\alpha \int_{B_k} |Du|^2 \leq \frac{C_f}{k},$$

which implies (with the previous techniques)

$$\alpha \int_{\Omega} |DT_M(u)|^2 \leq C_0 \log(M).$$

Then, if we follow the second part of the proof of Theorem 3.1, we can show

$$\text{meas}\{t \leq |Du|\} \leq C_f \frac{\log(t)}{t^2},$$

but we are not able to prove that

$$\text{meas}\{t \leq |Du|\} \leq C_f \frac{1}{t^2}. \tag{21}$$

We point out that the similar bordeline case for the Dirichlet problems involving equations has been recently treated in [17], where the estimate (21) is proved.

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