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Algebraic Geometry — *Some (big) irreducible components of the moduli space of minimal surfaces of general type with $p_g = q = 1$ and $K^2 = 4$*, by ROBERTO PIGNATELLI, communicated on 11 June 2009.

ABSTRACT. — This paper is devoted to the irregular surfaces of general type having the smallest invariants, $p_g = q = 1$. We consider the still unexplored case $K^2 = 4$, classifying those whose Albanese morphism has general fibre of genus 2 and such that the direct image of the bicanonical sheaf under the Albanese morphism is a direct sum of line bundles. We find 8 unirational families, and we prove that all are irreducible components of the moduli space of minimal surfaces of general type. This is unexpected because the assumption on the direct image bicanonical sheaf is a priori only a closed condition. One more unexpected property is that all these components have dimension strictly bigger than the expected one.

KEY WORDS: Surfaces of general type; Fibrations; Moduli.

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INTRODUCTION

Minimal surfaces of general type with $p_g = q$ (*i.e* with $\chi(\mathcal{O}) = 1$, the minimal possible value) have attracted the interest of many authors, but we are very far from a complete classification of them. Bombieri's theorem on pluricanonical maps ensures that there is only a finite number of families of such surfaces, but recent results show that the number of these families is huge (see for instance [PK], [BCG], [BCGP] for the case $p_g = q = 0$, [Pol1], [Pol2] for the case $p_g = q = 1$, [Zuc] and [Pen] for the case $p_g = q = 2$).

The irregular case is possibly more affordable, and in fact there is a complete classification of the case $p_g = q \geq 3$ ([HP], [Pir], see also [BCP] for more on surfaces with $\chi(\mathcal{O}) = 1$).

We are interested in the case $p_g = q = 1$. A classification of the minimal surfaces of general type with $p_g = q = 1$ and $K^2 \leq 3$ has been obtained ([Cat1], [CC1], [CC2], [CP]) by looking at the Albanese morphism, which, for a surface with $q = 1$, is a fibration onto an elliptic curve.

In this paper we begin the analysis of the next case $K^2 = 4$, by studying the surfaces whose general Albanese fibre has the minimal possible genus, i.e., genus 2.

We proved the following

THEOREM 0.1. *Let $\mathcal{M}$ be the algebraic subset of the moduli space of minimal surfaces of general type given by the set of isomorphism classes of minimal surfaces S with $p_g = q = 1$, $K_S^2 = 4$, whose Albanese fibration α is such that*

- *the general fibre of α is a genus 2 curve;*
- *$\alpha_* \omega_S^2$ is a direct sum of line bundles.*

Then

- *$\mathcal{M}$ has 8 connected components, all unirational, one of dimension 5 and the others of dimension 4;*
- *these are also irreducible components of the moduli space of minimal surfaces of general type;*
- *the general surface in each of these components has ample canonical class.*

We find noteworthy that all these families have bigger dimension than expected. Standard deformation theory says that any irreducible component of the moduli space of minimal surfaces of general type containing a surface S has dimension at least $-\chi(\mathcal{T}_S) = 10\chi(\mathcal{O}_S) - 2K_S^2$, but by the general principle "Hodge theory kills the obstruction" (stated in [Ran] and later made precise in [Cle]) this bound is not sharp for irregular surfaces. By applying this principle as in [CS], (proof of theorem 5.10), if $q = 1$ a better lower bound is $10\chi(\mathcal{O}_S) - 2K_S^2 + p_g = 11p_g - 2K^2$. This new bound is sharp, and in fact ([Cat1], [CC1], [CC2], [CP]) all irreducible components of the moduli space of surfaces with $p_g = q = 1$ and $K^2 \leq 3$ attain it. For $K^2 = 4$ this bound is 3, and all our families have strictly bigger dimension.

For technical reasons we assume $\alpha_* \omega_S^2$ to be a sum of line bundles. This is a closed assumption, and it is rather surprising that all the families we find are irreducible components of the moduli space of minimal surfaces of general type. Since [CC1] (thm. 1.4 and prop. 2.2) shows that the number of direct summands of $\alpha_* \omega_S$ is a topological invariant, we ask the following

Question: is the number of direct summands of $\alpha_* \omega_S^2$ a deformation or a topological invariant?

The author knows of constructions of minimal surfaces with $p_g = q = 1$ and $K^2 = 4$ by Catanese ([Cat2]), Polizzi ([Pol2]) and Rito ([Rit1], [Rit2]). Only one of these constructions gives a family of dimension at least 4, one of Polizzi's families. But these are obtained by resolving the singularities of a surface with 4 nodes; since each of our 8 families contains a surface with ample canonical class, the general surface in each of them is new. In section 5 we show that the 4-dimensional family constructed by Polizzi is a proper subfamily of our "bigger" family, the one of dimension 5.

The proof uses three main tools.

The first one is the study of the relative canonical algebra of genus 2 fibrations (introduced in [Rei] after the results obtained in [Hor], [Xia], [CC1]) and in particular the structure theorem for genus 2 fibrations of [CP]. The assumption on the direct image of the bicanonical sheaf is a natural assumption in view of the results of [CP].

The second step consists in the analysis of several cases a priori possible: some of these are excluded through the investigation of the geometry of a certain conic bundle, which is obtained as the quotient of our surface by the involution which induces the hyperelliptic involution on each fibre. Contradictions are derived by comparing a "very negative" section s with the branch locus (for example, by showing that s is contained in its divisorial part, which is reduced, with multiplicity 2).

Finally, to show that all our families are irreducible components of the moduli space of minimal surfaces of general type we need to bound from above the dimension of the first cohomology group of the tangent sheaf. To do that, we relate it with the dimension of a subsystem of the bicanonical system which we can explicitly compute.

The paper is organized as follows.

In section 1 we recall the structure theorem for genus 2 fibrations.

In section 2, we apply it to construct 8 families of minimal surfaces of general type with $p_g = q = 1$, $K^2 = 4$ whose Albanese fibration α has fiber of genus 2 and $\alpha_*\omega_S^2$ is a sum of line bundles. We also remark that each family contains surfaces with ample canonical class.

In sections 3 and 4 we show that we have constructed all surfaces with the above properties. In other words the image of our families in the moduli space of surface of general type equals the scheme $\mathcal{M}$ in theorem 0.1.

In section 5 we first remark that $\mathcal{M}$ has 8 unirational connected components (one for each family) and compute the dimension of each component. We prove then that they all are irreducible components of the moduli space of minimal surfaces of general type by investigating their bicanonical system as mentioned above.

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1. The structure theorem for genus 2 fibrations

1.1. The relative bicanonical map. In this section we recall results of [CP] (section 4) without giving any proof. The goal is to explain the structure theorem for genus 2 fibration (4.13 there).

Let $f : S \to B$ be a relatively minimal fibration of a smooth compact complex surface to a smooth curve whose general fibre has genus 2. We denote by F_p the fibre $f^{-1}(p)$.

Consider the relative dualizing sheaf $\omega_{S|B} := \omega_S \otimes f^*\omega_B^{-1}$. The direct images $V_n := f_*\omega_{S|B}^n$ are vector bundles on B whose fibre over any point p is canonically isomorphic to $H^0(\omega_{F_p}^n)$. Therefore the induced rational maps $\varphi_n : S \dashrightarrow \mathbb{P}(V_n) :=$ **Proj**(Sym V_n) (cf. [Har], chapter 2, section 7) map each fibre F_p to the corresponding fibre of $\mathbb{P}(V_n)$ by its own n-canonical map.

We remember to the reader that the canonical map of a smooth genus 2 curve F is a double cover of $\mathbb{P}^1$ and that its bicanonical map is the composition of this map with the 2-Veronese embedding of $\mathbb{P}^1$ onto a conic in $\mathbb{P}^2$, defined by the isomorphism $\mathrm{Sym}^2(H^0(\omega_F)) \cong H^0(\omega_F^2)$. The relative analog is an injective morphism of sheaves $\sigma_2 : \mathrm{Sym}^2 V_1 \hookrightarrow V_2$ (surjectivity fails on the stalks of points p such that F_p is not 2-connected) giving a *relative* 2-*Veronese* $v : \mathbb{P}(V_1) \dashrightarrow \mathbb{P}(V_2)$ birational onto a conic bundle $\mathscr{C}$, the image of φ_2. In fact $\varphi_2 = v \circ \varphi_1$.

The main point is that φ_2 is always a morphism. More precisely, φ_2 is a quasi-finite morphism of degree 2 contracting exactly the (-2) curves contained in fibres. In other words, if we substitute S with its relative canonical model (the surface obtained contracting that curves), φ_2 becomes a finite morphism of degree 2. Moreover $\mathscr{C}$ can only have singularities of type A_n or D_n, that are Rational Double Points.

The structure theorem proves that to reconstruct the pair (S, f) one only needs to know σ_2 (that gives at once $\mathscr{C}$ and the isolated branch points of φ_2) and the divisorial part Δ of the branch locus of φ_2. It gives moreover a concrete *recipe* to construct all possible pairs (σ_2, Δ).

We now introduce the 5 *ingredients* (B, V_1, τ, ξ, w), and then explain how to *cook* σ_2 and Δ from them.

1.2. The 5 ingredients. Their order is important, since each ingredient is given as an element in a space that depends on the previously given ingredients. They are

(B): Any curve.
(V_1): Any rank 2 vector bundle over B.
(τ): Any effective divisor on B.
(ξ): Any extension class

$$\xi \in \mathrm{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau, \mathrm{Sym}^2(V_1))/\mathrm{Aut}_{\mathcal{O}_B}(\mathcal{O}_\tau)$$

such that the central term, say V_2, of the corresponding short exact sequence

$$0 \to \mathrm{Sym}^2(V_1) \xrightarrow{\sigma_2} V_2 \to \mathcal{O}_\tau \to 0 \tag{1}$$

is a vector bundle.
(w): A nontrivial element of

$$\mathrm{Hom}((\det V_1 \otimes \mathcal{O}_B(\tau))^2, \mathscr{A}_6)/\mathbb{C}^*.$$

where $\mathscr{A}_6$ is a vector bundle determined by the other 4 ingredients as we explain in the following.

Consider the map v in the natural short exact sequence

$$0 \to (\det V_1)^2 \xrightarrow{v} \mathrm{Sym}^2(\mathrm{Sym}^2(V_1)) \to \mathrm{Sym}^4(V_1) \to 0;$$

given locally, if x_0, x_1 are generators of the stalk of V_1 in a point, by

$$v((x_0 \wedge x_1)^{\otimes 2}) = (x_0)^2(x_1)^2 - (x_0 x_1)^2. \tag{2}$$

$\mathscr{A}_6$ is the cokernel of the (automatically injective) composition of maps

$$(3) \qquad (\det V_1)^2 \otimes V_2 \xrightarrow{v \otimes id_{V_2}} \mathrm{Sym}^2(\mathrm{Sym}^2(V_1)) \otimes V_2$$
$$\xrightarrow{\mathrm{Sym}^2(\sigma_2) \otimes id_{V_2}} \mathrm{Sym}^2(V_2) \otimes V_2 \xrightarrow{\mu_{2,1}} \mathrm{Sym}^3(V_2).$$

In other words, writing i_3 for the composition of the maps in (3), we have an exact sequence

$$(4) \qquad 0 \to (\det V_1)^2 \otimes V_2 \xrightarrow{i_3} \mathrm{Sym}^3(V_2) \to \mathscr{A}_6 \to 0.$$

The 5 ingredients are required to satisfy some open conditions, just to ensure that what you cook is *eatable*. We need first to give the recipe.

1.3. THE RECIPE. The conic bundle $\mathscr{C}$ comes from the first 4 ingredients, and more precisely is the image of the *relative* 2-*Veronese* $\mathbb{P}(V_1) \dashrightarrow \mathbb{P}(V_2)$ given by the map σ_2 in the exact sequence (1).

We give an *equation* defining $\mathscr{C}$ as a divisor in $\mathbb{P}(V_2)$. A conic bundle in a projective bundle $\mathbb{P}(V)$ is given by an injection of a line bundle to $\mathrm{Sym}^2 V$; in this case the map $\mathrm{Sym}^2(\sigma_2) \circ v : (\det V_1)^2 \to \mathrm{Sym}^2 V_2$.

Now we explain how to get Δ from w. The curve Δ is *locally* (on B) the complete intersection of $\mathscr{C}$ with a relative cubic in $\mathbb{P}(V_2)$. In other words, a divisor in the linear system associated to the restriction to $\mathscr{C}$ of the line bundle $\mathcal{O}_{\mathbb{P}(V_2)}(3) \otimes \pi^* \mathscr{L}^{-1}$ for π being the projection on B, $\mathscr{L}$ a line bundle on B.

Why a map from a line bundle to the vector bundle $\mathscr{A}_6$ gives such a divisor? The *equation* of a divisor $\mathscr{G} \in |\mathcal{O}_{\mathbb{P}(V_2)}(3) \otimes \pi^* \mathscr{L}^{-1}|$ is an injective map $\mathscr{L} \hookrightarrow \mathrm{Sym}^3 V_2$. Intersecting it with $\mathscr{C}$ we do not obtain all divisor in that linear system since in general they are not all complete intersections of the form $\mathscr{C} \cap \mathscr{G}$. To get the complete linear system we need to consider injections $\mathscr{L} \hookrightarrow \mathscr{A}_6$ where $\mathscr{A}_6$ is the quotient of $\mathrm{Sym}^3 V_2$ by the subbundle *of the relative cubics vanishing on* $\mathscr{C}$, that is exactly the image of the map i_3 in the exact sequence (4).

1.4. THE OPEN CONDITIONS. We need to impose that

- $\mathscr{C}$ has only Rational Double Points as singularities;
- the curve Δ has only simple singularities, where "simple" means that the germ of double cover of $\mathscr{C}$ branched on it is either smooth or has a Rational Double Point.

DEFINITION 1.1. The map σ_2 gives isomorphisms of the respective fibres of $\mathrm{Sym}^2 V_1$ and V_2 over points not in $\mathrm{supp}(\tau)$. On the points of $\mathrm{supp}(\tau)$ it defines a rank 2 matrix, whose image defines a pencil of lines in the corresponding $\mathbb{P}^2$, thus having a base point. We denote by $\mathscr{P}$ the union of these (base) points. So $\mathscr{P}$ is in natural bijection with $\mathrm{supp}(\tau)$.

REMARK 1.2. By theorem 4.7 of [CP], $\mathscr{P} \subset \mathrm{Sing}(\mathscr{C})$ is the set of isolated branch points of ψ_2, so in particular $\Delta \cap \mathscr{P} = \emptyset$.

Remark 1.3. By remark 4.14 in [CP], if τ is a reduced divisor and every fibre of $\mathscr{C} \to B$ is reduced (it is enough to check the preimages of points of τ, the other fibres being smooth) then the first open condition is fulfilled. More precisely automatically $\mathrm{Sing}(\mathscr{C}) = \mathscr{P}$ and these points are A_1 singularities of $\mathscr{C}$.

It follows that if moreover Δ is smooth and $\Delta \cap \mathscr{P} = \emptyset$ both open conditions are fulfilled and the relative canonical model of the surface is smooth.

1.5. The dish. What we get is a genus 2 fibration $f : S \to B$ (the base is the first ingredient) with $V_1 \cong f_* \omega_{S|B}$ and $V_2 \cong f_* \omega^2_{S|B}$. The structure theorem says that any relatively minimal genus 2 fibration is obtained in this way.

Denoting by b the genus of the base curve B

$$\chi(\mathcal{O}_S) = \deg V_1 + (b-1) \quad K_S^2 = 2 \deg V_1 + 8(b-1) + \deg \tau$$

2. The families

In this section we construct 8 families of surfaces of general type with $p_g = q = 1$, $K^2 = 4$ and Albanese of genus 2 using the recipe described in the section 1. We need then to give the ingredients, quintuples (B, V_1, τ, ξ, w) with B elliptic curve and (by 1.5) $\deg V_1 = 1$, $\deg \tau = 2$.

As first ingredient we take any elliptic curve B. For later convenience we fix a group structure on B and denote by $\eta_0 = 0$ its neutral element, and by η_1, η_2 and η_3 the nontrivial 2-torsion points.

The choice of the next 3 ingredients for the 8 families is summarized in the tables 1 and 2, which we are going to explain.

As second ingredient, V_1, we need a vector bundle of rank 2. V_1 can be sum of line bundles (table 1) or indecomposable (table 2).

In the decomposable case we take $V_1 \cong \mathcal{O}_B(p) \oplus \mathcal{O}_B(0 - p)$ where p is a t-torsion point for some t $\in \{2, 3, 4, 6\}$, $V_2 := \mathcal{O}_B(D_1) \oplus \mathcal{O}_B(D_2) \oplus \mathcal{O}_B(D_3)$ for D_1, D_2 and D_3 suitable divisors on B. Since

$$V_1 \cong \mathcal{O}_B(p) \oplus \mathcal{O}_B(0-p) \Rightarrow \mathrm{Sym}^2 V_1 \cong \mathcal{O}_B(2 \cdot p) \oplus \mathcal{O}_B(0) \oplus \mathcal{O}_B(2 \cdot 0 - 2 \cdot p)$$

the splitting of the source and the target of σ_2 as sum of line bundles allows to represent σ_2 by a 3×3 matrix whose entries are global sections of line bundles over B. The table 1 give 4 families of choices of t, D_1, D_2, D_3 and σ_2. The pair (a_i, b_i) must be taken general in the sense of 1.4, and we will later show that this open condition is nonempty. The linear system on which τ varies depends on the other data, and can be computed by (1): we wrote the result on the last column.

Otherwise we take V_1 to be the only indecomposable rank 2 vector bundle on B with $\det V_1 = \mathcal{O}_B(0)$. By [Ati], (as shown for the analogous case $K_S^2 = 3$ in [CP]) it follows that also in this case $\mathrm{Sym}^2 V_1$ is sum of line bundles, and more precisely

$$\mathrm{Sym}^2 V_1 \cong \mathcal{O}_B(\eta_1) \oplus \mathcal{O}_B(\eta_2) \oplus \mathcal{O}_B(\eta_3).$$

Table 1. $\sigma_2 : \mathrm{Sym}^2 V_1 \to \bigoplus_{i=1}^3 \mathcal{O}_B(D_i)$ for $V_1 \cong \mathcal{O}_B(p) \oplus \mathcal{O}_B(0-p)$, p t-torsion

family	t	D_1	D_2	D_3	σ_2	$\lvert\tau\rvert$
$\mathscr{M}_{2,3}$	2	$2\cdot 0$	$2\cdot 0$	0	$\begin{pmatrix} 0 & 0 & a_1 \\ 1 & 0 & b_1 \\ 0 & 1 & 0 \end{pmatrix}$	$\lvert 2\cdot 0\rvert$
$\mathscr{M}_{4,2}$	4	$2\cdot 0$	$2\cdot p$	0	$\begin{pmatrix} 0 & 0 & a_2 \\ 1 & 0 & b_2 \\ 0 & 1 & 0 \end{pmatrix}$	$\lvert 2\cdot p\rvert$
$\mathscr{M}_{3,1}$	3	$0+p$	$2\cdot p$	0	$\begin{pmatrix} 0 & 0 & a_3 \\ 1 & 0 & b_3 \\ 0 & 1 & 0 \end{pmatrix}$	$\lvert 2\cdot 0\rvert$
$\mathscr{M}_{6,1}$	6	$4\cdot p - 2\cdot 0$	$2\cdot p$	0	$\begin{pmatrix} 0 & 0 & a_4 \\ 1 & 0 & b_4 \\ 0 & 1 & 0 \end{pmatrix}$	$\lvert 2\cdot 0\rvert$

Therefore also in this case, writing $V_2 := \mathcal{O}_B(D_1) \oplus \mathcal{O}_B(D_2) \oplus \mathcal{O}_B(D_3)$ we can represent σ_2 by a matrix. The table 2 give 4 families of choices of D_1, D_2, D_3 and σ_2, and the resulting τ (it moves in a pencil in all cases but the first); in the last row σ denotes a nontrivial 3-torsion point of B. a_i, b_i, c_i, d_i are general in the sense of 1.4.

Table 2. $\sigma_2 : \mathrm{Sym}^2 V_1 \to \bigoplus_{i=1}^3 \mathcal{O}_B(D_i)$ for V_1 indecomposable, $\det V_1 \cong \mathcal{O}_B(0)$

family	D_1	D_2	D_3	σ_2	τ
$\mathscr{M}_{i,3}$	$2\cdot 0$	$2\cdot 0$	η_3	$\begin{pmatrix} a_5 & 0 & 0 \\ 0 & d_5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$=\eta_1+\eta_2$
$\mathscr{M}_{i,2}$	$2\cdot 0$	$\eta_1+\eta_2$	η_3	$\begin{pmatrix} a_6 & b_6 & 0 \\ c_6 & d_6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\in \lvert 2\cdot 0\rvert$
$\mathscr{M}'_{i,2}$	$2\cdot 0$	$0+\eta_1$	η_3	$\begin{pmatrix} a_7 & b_7 & 0 \\ c_7 & d_7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\in \lvert 0+\eta_2\rvert$
$\mathscr{M}_{i,1}$	$0+\sigma$	$2\cdot\sigma$	η_3	$\begin{pmatrix} a_8 & b_8 & 0 \\ c_8 & d_8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\in \lvert 0+\eta_3\rvert$

Table 3. $\mathscr{C}$ and $\Delta = \mathscr{C} \cap \mathscr{G}$

family	$\mathscr{C}$	$\mathscr{G}$
$\mathscr{M}_{2,3}$	$y_2(a_1y_1 + b_1y_2) = y_3^2$	$\sum_0^3 k_iy_1^{3-i}y_2^i = 0$
$\mathscr{M}_{4,2}$	$y_2(a_2y_1 + b_2y_2) = y_3^2$	$y_1(k_0y_1^2 + k_2y_2^2) = 0$
$\mathscr{M}_{3,1}$	$y_2(a_3y_1 + b_3y_2) = y_3^2$	$k_0y_1^3 + k_3y_2^3 = 0$
$\mathscr{M}_{6,1}$	$y_2(a_4y_1 + b_4y_2) = y_3^2$	$k_0y_1^3 + k_3y_2^3 = 0$
$\mathscr{M}_{i,3}$	$a_5^2y_1^2 + d_5^2y_2^2 + y_3^2 = 0$	$\sum_0^3 k_iy_1^{3-i}y_2^i = 0$
$\mathscr{M}_{i,2}$	$(a_6y_1 + c_6y_2)^2 + (b_6y_1 + d_6y_2)^2 + y_3^2 = 0$	$y_1(k_0y_1^2 + k_2y_2^2) = 0$
$\mathscr{M}'_{i,2}$	$(a_7y_1 + c_7y_2)^2 + (b_7y_1 + d_7y_2)^2 + y_3^2 = 0$	$y_1(k_0y_1^2 + k_2y_2^2) = 0$
$\mathscr{M}_{i,1}$	$(a_8y_1 + c_8y_2)^2 + (b_8y_1 + d_8y_2)^2 + y_3^2 = 0$	$k_0y_1^3 + k_3y_2^3 = 0$

Now that we have the first 4 ingredients, we can construct the conic bundle. The splitting of V_2 as sum of line bundles gives relative coordinates on $\mathbb{P}(V_2)$, by taking the injections $y_i : \mathcal{O}_B(D_i) \hookrightarrow V_2$. We can use these coordinates to give equations of $\mathscr{C} \subset \mathbb{P}(V_2)$.

Lemma 2.1. *The conic bundle $\mathscr{C}$ obtained by the ingredients given in a row of the table* 1 *or* 2 *following the recipe in* 1.3 *has the equation given in the first column* (*and corresponding row*) *of the table* 3.

Proof. As explained in 1.3, an equation of $\mathscr{C}$ is given by the map $\mathrm{Sym}^2(\sigma_2) \circ v$, where v is given in (2).

In the cases of table 1 V_1 is sum of two line bundles, so we can use the splitting to give two generators x_0, x_1 on each stalk. When we write $\mathrm{Sym}^2 V_1 \cong \mathcal{O}_B(2 \cdot p) \oplus \mathcal{O}_B(0) \oplus \mathcal{O}_B(2 \cdot 0 - 2 \cdot p)$ the first summand correspond to x_0^2, the second to x_0x_1, the third to x_1^2. So by the expression of σ_2

$$\begin{cases} x_0^2 \mapsto y_2 \\ x_0x_1 \mapsto y_3 \\ x_1^2 \mapsto a_iy_1 + b_iy_2 \end{cases}$$

and the equation $(x_0)^2(x_1)^2 = (x_0x_1)^2$ maps to $y_2(a_iy_1 + b_iy_2) = y_3^2$.

In the cases of table 2, V_1 is indecomposable so we do not have "global" x_0, x_1. Anyway, as noticed in remark 6.13 of [CP], the map $v : \mathcal{O}_B(2 \cdot 0) \to \mathrm{Sym}^2(\bigoplus \mathcal{O}_B(\eta_i))$ is given by a 6×1 matrix whose entries are

– 0 the three entries corresponding to the "mixed terms" ($\mathcal{O}_B(\eta_i + \eta_j)$ for $i \neq j$), since $i \neq j \Rightarrow \mathrm{Hom}(\mathcal{O}_B(2 \cdot 0), \mathcal{O}_B(\eta_i + \eta_j)) = 0$
– isomorphisms the three entries corresponding to the pure powers ($\mathcal{O}_B(\eta_i + \eta_i)$) since the Veronese image of $\mathbb{P}^1$ in $\mathbb{P}^2$ has rank 3.

It follows that the equation of the relative Veronese embedding $\mathbb{P}(V_1) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 V_1)$ is $z_1^2 + z_2^2 + z_3^2 = 0$ for suitable choice of coordinates $z_i : \mathcal{O}_B(\eta_i) \hookrightarrow \mathrm{Sym}^2 V_1$ on $\mathbb{P}(\mathrm{Sym}^2 V_1)$. Composing with σ_2 we get the equations in the table. □

We still have to give the last ingredient. Since in each case $\mathcal{O}_B(2 \cdot \tau) \cong \mathcal{O}_B(4 \cdot 0)$, we have $(\det V_1 \otimes \mathcal{O}_B(\tau))^2 \cong \mathcal{O}_B(6 \cdot 0)$, therefore w is the class (modulo $\mathbb{C}^*$) of a map $\mathcal{O}_B(6 \cdot 0) \to \mathcal{A}_6$, where $\mathcal{A}_6$ is a quotient of $\mathrm{Sym}^3 V_2$ as in (4).

We choose this map as composition of a general map $\overline{w} : \mathcal{O}_B(6 \cdot 0) \to \mathrm{Sym}^3 V_2$ with the projection to the quotient. This geometrically means that we take $\Delta = \mathcal{C} \cap \mathcal{G}$ for a relative cubic $\mathcal{G} \subset \mathbb{P}(V_2)$ whose equation is given by $\overline{w}$. Since $\mathrm{Sym}^3 V_2$ is sum of line bundles whose maximal degree is 6, the nonzero entries of $\overline{w}$ are constants and correspond to the summands of the target isomorphic to $\mathcal{O}_B(6 \cdot 0)$. In the table 3 we give the exact equation of $\mathcal{G}$ in each case. The parameters $k_i \in \mathbb{C}$ must be taken general in the sense of 1.2, requiring that Δ has only simple singularities.

Proposition 2.2. *Cooking the ingredients given above (B general elliptic curve, V_1, τ, ξ given by a row of the table* 1 *or* 2*, w by the corresponding row in the table* 3*) following the recipe* 1.3*, one finds* 8 *unirational families of minimal surfaces of general type with $p_g = q = 1$, $K^2 = 4$, Albanese morphism α with fibres of genus* 2 *and $\alpha_*\omega_S^2$ sum of line bundles. The general element in each family has ample canonical class.*

Proof. By the recipe (1.3) and remark 1.3, if we show that all these families of ingredients contain one element such that

on τ: coker $\sigma_2 \cong \mathcal{O}_\tau$ for τ reduced divisor;
on $\mathcal{C}$: all fibres of $\mathcal{C} \to B$ are reduced conics;
on Δ: Δ is smooth and $\Delta \cap \mathcal{P} = \emptyset$.

then all these examples give families of genus 2 fibrations $f : S \to B$ with (by 1.5) $K_S^2 = 4$ and $\chi(\mathcal{O}_S) = 1$ with smooth relative canonical model. Since B has genus 1, $q(S) \geq 1$, so $p_g = q = 1$. By the universal property of the Albanese morphism $\alpha = f$, and therefore $\alpha_*\omega_S \cong V_1$, $\alpha_*\omega_S^2 \cong \mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \oplus \mathcal{O}(\eta_3)$.

So we only need to find an element in each family satisfying the three conditions. Since all conditions are open and each family irreducible, it is enough to show that each condition (separately) is fulfilled by some choice of the parameters. This is easy, we sketch a way to do it.

On τ: we need to choose the entries of the matrix of σ_2 so that the determinant is not a perfect square.
On $\mathcal{C}$: a conic of the form $y_3^2 = q(y_1, y_2)$ is a double line if and only if $q = 0$. By the equation of $\mathcal{C}$ in the table 3 we see that in the first 4 cases it is enough to choose a_i, b_i without common zeroes, whereas in the last 4 cases it is enough $\det \sigma_2 \neq 0$.
On Δ: in 5 cases the linear system $|\mathcal{G}|$ has fixed locus $\{y_1 = y_2 = 0\}$ which do not intersect $\mathcal{C}$. So $|\Delta|$ is free and therefore we can conclude by Bertini. In the

remaining cases $\mathcal{M}_{4,2}$, $\mathcal{M}_{i,2}$ and $\mathcal{M}'_{i,2}$ the fixed part of $|\Delta|$ is $\{y_1 = 0\} \cap \mathcal{C}$ and the general element of the movable part of $|\Delta|$ do not intersect the fixed part. So we only need to check $\{y_1 = 0\} \cap \mathcal{C}$ smooth and not containing $\mathcal{P}$. For $\mathcal{M}_{4,2}$ if we take $b_2 \neq 0$ we get smoothness, and the other condition comes automatically since $\mathcal{P} \subset \{y_2 = y_3 = 0\}$. In the other two cases $c_i^2 + d_i^2$ square free gives the smoothness, and from (*e.g.*) $a_i b_i c_i d_i \neq 0$ follows $\{y_1 = 0\} \cap \mathcal{C} \not\supset \mathcal{P}$. □

We end the section by explaining the choice of the indices of the name of each family.

The first index remembers us which V_1 we have chosen: i stands for "V_1 indecomposable", a number t means "V_1 has a t-torsion bundle as direct summand".

The second index gives the number of connected components of the curve Δ for a surface in the family. Let us show this decomposition.

The equation of $\mathcal{G}$ is homogeneous of degree 3 in two variables (with constant coefficients), so we can formally decompose it as product of three linear factors. When $D_1 = D_2$ ($\mathcal{M}_{2,3}$ and $\mathcal{M}_{i,3}$) each factor gives a map of a line bundle ($\mathcal{O}_B(2 \cdot 0)$) to V_2, so a relative hyperplane of $\mathbb{P}(V_2)$: these three relative hyperplanes cut on $\mathcal{C}$ three components of Δ that pairwise they do not intersect.

When $\mathcal{O}_B(D_1) \not\cong \mathcal{O}_B(D_2)$ a factor $cy_1 + c'y_2$ determines a relative hyperplane only if $cc' = 0$. In the cases $\mathcal{M}_{4,2}$, $\mathcal{M}_{i,2}$, $\mathcal{M}'_{i,2}$ one can then decompose Δ as union of its fixed part $\{y_1 = 0\}$ and its movable part.

3. Direct image of the canonical sheaf decomposable

In this section we prove the following

Proposition 3.1. *All minimal surfaces of general type S with $K_S^2 = 4$, $p_g = q = 1$ such that the general fibre of the Albanese morphism α has genus 2 and $\alpha_*\omega_S$, $\alpha_*\omega_S^2$ are direct sum of line bundles belong to $\mathcal{M}_{2,3}$, $\mathcal{M}_{4,2}$, $\mathcal{M}_{3,1}$ or $\mathcal{M}_{6,1}$.*

By the structure theorem of genus 2 fibrations, we need to classify 5-tuples (B, V_1, τ, ξ, w) with B elliptic curve, $\deg V_1 = 1$, $\deg \tau = 2$ such that V_1 and V_2 are sum of line bundles.

Since $h^0(V_1) = h^0(\omega_S) = p_g$ we can assume up to translations $V_1 \cong \mathcal{O}_B(p) \oplus \mathcal{O}_B(0 - p)$ for some $p \neq 0$. We write $V_2 = \mathcal{O}_B(D_1) \oplus \mathcal{O}_B(D_2) \oplus \mathcal{O}_B(D_3)$, with D_i divisors of degree d_i, $d_3 \leq d_2 \leq d_1$. We consider relative coordinates in V_1 and V_2 as follows: x_i correspond to the summand of degree i in V_1, y_j correspond to the summand $\mathcal{O}_B(D_j)$ in V_2.

Lemma 3.2. *$d_1 = d_2 = 2$, $d_3 = 1$.*

Proof. By the exact sequence (1), since $\mathrm{Sym}^2 V_1$ is direct sum of three line bundles of respective degrees 0, 1 and 2, $d_1 + d_2 + d_3 = 5$, $d_i \geq 3 - i$.

Since $d_3 \leq d_2 \leq d_1$ to show $d_3 = 1$ we assume by contradiction $d_3 = 0$. Then the summands of positive degree in $\mathrm{Sym}^2 V_1$ map trivially on $\mathcal{O}_B(D_3)$. In other words $\sigma_2(x_1^2), \sigma_2(x_0x_1) \in \mathrm{Span}(y_1, y_2)$. In particular, the equation of $\mathscr{C}$ being $\sigma_2(x_0^2)\sigma_2(x_1^2) = \sigma_2(x_0x_1)^2$, the section $s := \{y_1 = y_2 = 0\}$ is contained in $\mathscr{C}$.

We consider s as Weil divisor in $\mathscr{C}$. Note that $\mathscr{C}$ has only canonical singularities, so s is $\mathbb{Q}$-Cartier, and the self-intersection number s^2 is well defined, as the numbers $s \cdot D \in \mathbb{Z}$ for any Cartier divisor D on $\mathscr{C}$, including $K_{\mathscr{C}}$.

We denote by H the numerical class of a divisor in $\mathcal{O}_{\mathbb{P}(V_2)}(1)$, by F the class of a fiber of the map $\mathbb{P}(V_2) \to B$. Then s, as a cycle in $\mathbb{P}(V_2)$, has numerical class $(H - d_1F)(H - d_2F) = H^2 - 5HF$. Δ is Cartier on $\mathscr{C}$, and the corresponding line bundle is the restriction to $\mathscr{C}$ of a line bundle in $\mathbb{P}(V_2)$ whose numerical class is $3H - 6F$ (since $\deg(\det V_1 \otimes \mathcal{O}_B(\tau))^2 = 6$). It follows $\Delta \cdot s = (3H - 6F) \cdot (H^2 - 5HF) = -6 < 0$. Being s irreducible, $s < \Delta$.

Consider now a minimal resolution of the singularities $\rho : \tilde{\mathscr{C}} \to \mathscr{C}$ and let $\tilde{s}$ be the strict transform of s. Then $\tilde{s}$ is a smooth elliptic curve and $\tilde{s} = \rho^*s - e$ for some exceptional $\mathbb{Q}$-divisor e, so $s^2 + K_{\mathscr{C}}s \geq s^2 + e^2 + K_{\mathscr{C}}s = \tilde{s}^2 + K_{\tilde{C}}\tilde{s} = 0$. Since the class of $\mathscr{C}$ is $2H - 2F$ it follows $-s^2 \leq K_{\mathscr{C}}s = (-H + 3F)(H^2 - 5HF) = 3$ and $(\Delta - s)s = -s^2 - 6 \leq -3 < 0$. It follows so $2s < \Delta$, contradicting 1.4.

Then $d_3 = 1$ and to conclude we can assume by contradiction $d_2 = 1$, then $\sigma_2(x_1)^2 \in \mathrm{Span}(y_1)$. It follows that the equation of $\mathscr{C}$ is a square modulo y_1. In other words the relative hyperplane $\{y_1 = 0\}$ cut $2 \cdot s'$ where s is a section of the map $\mathbb{P}(V_2) \to B$. The class of s' is $H^2 - 4HF$: repeating the above argument we find $\Delta \cdot s' = -3$, $(\Delta - s') \cdot s' \leq -1 \Rightarrow 2s' < \Delta$, the same contradiction as above. □

LEMMA 3.3. $\sigma_2(x_0x_1) \notin \mathrm{Span}(y_1, y_2)$.

PROOF. Since $\sigma_2(x_1^2) \in \mathrm{Span}(y_1, y_2)$, if also $\sigma_2(x_0x_1) \in \mathrm{Span}(y_1, y_2)$, then the section $s := \{y_1 = y_2 = 0\}$ is contained in $\mathscr{C}$. The numerical class of s is $H^2 - 4HF$ so (as in the previous proof) $\Delta \cdot s = -3$, $(\Delta - s) \cdot s \leq -1 \Rightarrow 2s < \Delta$, a contradiction. □

REMARK 3.4. The lemma 3.3 says that the composition of σ_2 with the projection onto the summand $\mathcal{O}_B(D_3)$ is different from zero. Since any nonzero morphism between line bundles of the same degree is an isomorphism, it follows $\mathcal{O}_B(D_3) \cong \mathcal{O}_B(0)$.

LEMMA 3.5. *The exact sequence* (4) *splits.*

PROOF. By the lemma 3.3 and remark 3.4 the coefficient of the term y_3^2 in the relative conic $\sigma_2(x_0^2)\sigma_2(x_1)^2 - \sigma_2(x_0x_1)^2$ defining $\mathscr{C}$ is a nonzero constant. Then each relative conic can be uniquely decomposed as a sum of a multiple of this equation with an equation where the multiples of y_3^2 $(y_1y_3^2, y_2y_3^2, y_3^3)$ do not appear.

Since the multiples of the equation of $\mathscr{C}$ define exactly the image of i_3, this means that the restriction of the projection $\mathrm{Sym}^3 V_2 \to \mathscr{A}_6$ to $\mathrm{Sym}^3(\mathcal{O}_B(D_1) \oplus \mathcal{O}_B(D_2)) \oplus (\mathrm{Sym}^2(\mathcal{O}_B(D_1) \oplus \mathcal{O}_B(D_2)) \otimes \mathcal{O}_B(D_3))$ is an isomorphism. Its inverse splits the exact sequence (4). □

In particular every morphism to $\mathscr{A}_6$ lift to a morphism to $\operatorname{Sym}^3 V_2$, and therefore the last "ingredient" w comes from a map $\overline{w} : (\det V_1 \otimes \mathcal{O}_B(\tau))^2 \to \operatorname{Sym}^3 V_2$. It follows

Corollary 3.6. *$\mathscr{T} := \mathcal{O}_B(D_1 - D_2)$ is a* t*-torsion bundle for some* $\mathrm{t} \in \{1, 2, 3\}$, *and up to exchange D_1 and D_2, $\mathcal{O}_B(0 + \tau)^2 \cong \mathcal{O}_B(D_1)^3$.*

Proof. The source of $\overline{w}$ is the line bundle $\mathcal{O}_B(0 + \tau)^2$ of degree 6. Since $\operatorname{Sym}^3 V_2$ is sum of line bundles of degree at most 6, its image is contained in the sum of those having exactly degree 6, $\operatorname{Sym}^3(\mathcal{O}_B(D_1) \oplus \mathcal{O}_B(D_2))$, and more precisely in those summands isomorphic to $\mathcal{O}_B(0 + \tau)^2$.

So $\Delta = \mathscr{C} \cap \mathscr{G}$ with $\mathscr{G} = \sum k_i y_1^{3-i} y_2^i$ where k_i are constant that can be different form 0 only when $\mathcal{O}_B((3 - i)D_1 + iD_2) \cong \mathcal{O}_B(0 + \tau)^2$. The claim follows since Δ is reduced, and then at least two k_i's are different from 0. □

Proof of proposition 3.1. By remark 3.4 and corollary 3.6 $V_2 \cong \mathscr{T}(D_2) \oplus \mathcal{O}_B(D_2) \oplus \mathcal{O}_B(0)$ for some t-torsion line bundle $\mathscr{T}$, $\mathrm{t} \in \{1, 2, 3\}$. Moreover, by exact sequence (1) and corollary 3.6

$$\mathscr{T}(2 \cdot D_2 + 0) \cong \mathcal{O}_B(3 \cdot 0 + \tau) \quad \mathcal{O}_B(2 \cdot 0 + 2 \cdot \tau) \cong \mathscr{T}^3(3 \cdot D_2).$$

equivalently

$$\mathcal{O}_B(D_2) \cong \mathscr{T}(2 \cdot 0) \quad \mathcal{O}_B(\tau) \cong \mathscr{T}^3(2 \cdot 0) \tag{5}$$

Moreover, by the injectivity of σ_2, $2p$ must be linearly equivalent to D_1 or D_2, *i.e.*

$$\mathcal{O}_B(2 \cdot p) \cong \mathscr{T}(2 \cdot 0) \quad \text{or} \quad \mathcal{O}_B(2p) \cong \mathscr{T}^2(2 \cdot 0) \tag{6}$$

If $\mathrm{t} = 1$: $\mathscr{T} \cong \mathcal{O}_B$ and the two alternatives in (6) are identical: $\mathcal{O}_B(2 \cdot p) \cong \mathcal{O}_B(2 \cdot 0)$. Since $p \neq 0$, p is a 2-torsion point. We can choose coordinates in V_2 such that $y_2 = \sigma_2(x_1^2)$ and (by lemma 3.3) $y_3 = \sigma_2(x_0 x_1)$. We can also assume $\sigma_2(x_0^2) \in \operatorname{Span}(y_1, y_2)$ by changing the coordinates (x_0, x_1): we have found the family $\mathscr{M}_{2,3}$.

If $\mathrm{t} = 2$: If $\mathcal{O}_B(2 \cdot p) \cong \mathscr{T}(2 \cdot 0)$, p is a 4-torsion point. Changing coordinates in V_1 and V_2 as above we find the family $\mathscr{M}_{4,2}$.
Else $\mathcal{O}_B(2 \cdot p) \cong \mathscr{T}^2(2 \cdot 0)$. In this case $(\mathcal{O}_B(D_2) \not\cong \mathcal{O}_B(2 \cdot p))$ $\sigma_2(x_1^2) \in \operatorname{Span}(y_1)$, therefore (see definition 1.1) $\mathscr{P} \subset \{y_1 = 0\}$. On the other hand $\mathscr{G} = \{y_1(k_0 y_1^2 + k_2 y_2^2) = 0\}$, so the fixed part of $|\Delta|$ contains $\mathscr{P}$, contradicting remark 1.2: this case do not occur.

If $\mathrm{t} = 3$: If $\mathcal{O}_B(2 \cdot p) \cong \mathscr{T}(2 \cdot 0)$, p is either a 3-torsion point or a 6-torsion point. Changing coordinates as above we find respectively the families $\mathscr{M}_{3,1}$ and $\mathscr{M}_{6,1}$. The other case $\mathcal{O}_B(2 \cdot p) \cong \mathscr{T}^2(2 \cdot 0)$ gives the same families (with D_1 and D_2 exchanged). □

4. Direct image of the canonical sheaf indecomposable

In this section we prove the following

Proposition 4.1. *All minimal surfaces of general type S with $K_S^2 = 4$, $p_g = q = 1$ such that the general fibre of the Albanese morphism α has genus 2, $\alpha_*\omega_S$ is an indecomposable vector bundle and $\alpha_*\omega_S^2$ is a direct sum of line bundles belong to $\mathcal{M}_{i,3}$, $\mathcal{M}_{i,2}$, $\mathcal{M}'_{i,2}$ or $\mathcal{M}_{i,1}$.*

We need to classify 5-tuples (B, V_1, τ, ξ, w) with B elliptic curve, V_1 indecomposable of degree 1, $\deg\tau = 2$ such that V_2 is sum of three line bundles.

B can be any elliptic curve and by Atiyah's classification of the vector bundles on an elliptic curves [Ati], we can assume (up to translations) $V_1 = E_0(2,1)$, that is the only indecomposable vector bundle over B whose determinant is $\mathcal{O}_B(0)$.

From Atiyah's results follows $\mathrm{Sym}^2(V_1) \cong \mathcal{O}_B(\eta_1) \oplus \mathcal{O}_B(\eta_2) \oplus \mathcal{O}_B(\eta_3)$. As in the previous case we write $V_2 = \mathcal{O}_B(D_1) \oplus \mathcal{O}_B(D_2) \oplus \mathcal{O}_B(D_3)$, with D_i divisors of degree d_i, $d_3 \leq d_2 \leq d_1$.

Remark 4.2. As shown in the proof of lemma 2.1, in this case the relative 2-Veronese $\mathbb{P}(V_1) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 V_1)$ has equation $z_1^2 + z_2^2 + z_3^2 = 0$ for a suitable choice of coordinates $z_i : \mathcal{O}_B(\eta_i) \hookrightarrow \mathrm{Sym}^2 V_1$.

It follows that, in these coordinates, $\mathscr{C}$ is defined by the polynomial $\sum_{i=1}^3 \sigma_2(z_i)^2$.

Lemma 4.3. *We can assume $D_3 = \eta_3$, and we can choose coordinates in V_2 so that $\sigma_2(z_3) = y_3$. Moreover the exact sequence* (4) *splits.*

Proof. Since $\sum d_i = 5$ and by the injectivity of σ_2, $\forall i\ d_i \geq 1$, $d_3 = 1$. The injectivity of σ_2 forces now one of the induced maps $\mathcal{O}_B(\eta_i) \to \mathcal{O}_B(D_3)$ to be an isomorphism and then (renaming the torsion points) we have $D_3 = \eta_3$. Changing coordinates in V_2 we can assume $\sigma_2(\mathcal{O}_B(\eta_3)) = \mathcal{O}_B(D_3)$.

By remark 4.2 the coefficient of the term y_3^2 in the equation of $\mathscr{C}$ is a nonzero constant and we can conclude as in the proof of lemma 3.5. $\square$

Lemma 4.4. *$d_1 = d_2 = 2$.*

Proof. We assume by contradiction $d_2 = 1$, $d_1 = 3$. By lemma 4.3 the curve Δ is a complete intersection $\mathscr{G} \cap \mathscr{C}$ for a relative cubic $\mathscr{G}$ defined by an immersion $\overline{w}$ of a line bundle of degree 6 to $\mathrm{Sym}^3 V_2$.

The image of $\overline{w}$ is then contained in $\mathcal{O}_B(D_1)^2 \otimes V_2$ since all other summands have degree strictly smaller than 6. In other words the equation of $\mathscr{G}$ is divisible by y_1^2. In particular Δ contains $\{y_1 = 0\} \cap \mathscr{C}$ with multiplicity 2, contradicting 1.4. $\square$

It follows, as in the previous case

Corollary 4.5. *$\mathscr{T} := \mathcal{O}_B(D_1 - D_2)$ is a* t*-torsion bundle for some* $\mathrm{t} \in \{1,2,3\}$ *and, up to exchange D_1 and D_2, $\mathcal{O}_B(0 + \tau) \cong \mathcal{O}_B(D_1)^3$.*

PROOF. Identical to the proof of the analogous corollary 3.6. □

PROOF OF PROPOSITION 4.1. By lemma 4.3 and corollary 4.5, $V_2 \cong \mathcal{T}(D_2) \oplus \mathcal{O}_B(D_2) \oplus \mathcal{O}_B(\eta_3)$, and, by the exact sequence (1) and corollary 4.5

$$\mathcal{T}(2 \cdot D_2 + \eta_3) \cong \mathcal{O}_B(3 \cdot 0 + \tau) \quad \mathcal{O}_B(2 \cdot 0 + 2 \cdot \tau) \cong \mathcal{T}^3(3 \cdot D_2),$$

equivalently

$$\mathcal{O}_B(D_2) \cong \mathcal{T}(2 \cdot 0) \quad \mathcal{O}_B(\tau) \cong \mathcal{T}^3(0 + \eta_3). \tag{7}$$

Recall that by lemma 4.5 we can choose $y_3 = \sigma_2(z_3)$ and since $d_1 = d_2 = 2$, $\sigma_2(z_1), \sigma_2(z_2) \in \mathrm{Span}(y_1, y_2)$. In other words the matrix of σ_2 is as the matrices in the last three rows of table 2.

If t = 1: $\mathcal{T} \cong \mathcal{O}_B$, $\mathcal{O}_B(D_1) \cong \mathcal{O}_B(D_2) \cong \mathcal{O}_B(2 \cdot 0)$ and $\mathcal{O}_B(\tau) \cong \mathcal{O}_B(0 + \eta_3)$. In fact, since $D_1 = D_2$ we can change coordinates in V_2 to add to one of the first two rows any multiple of the other and diagonalize the matrix: this is the family $\mathcal{M}_{i,3}$. Note that $\tau = \eta_1 + \eta_2$ cannot move.

If t = 2: then either $\mathcal{T} \cong \mathcal{O}_B(\eta_3)$ or we can rename η_1 and η_2 to get $\mathcal{T} \cong \mathcal{O}_B(\eta_1)$. This gives respectively the families $\mathcal{M}_{i,2}$ and $\mathcal{M}'_{i,2}$.

If t = 3: Then $\mathcal{T} \cong \mathcal{O}_B(0 - \sigma)$ for some 3-torsion point σ. This is the family $\mathcal{M}_{i,1}$. □

5. MODULI

In this section we consider the scheme $\mathcal{M}$ in theorem 0.1, subscheme of the moduli space of the minimal surfaces of general type given by the surfaces with $p_g = q = 1$, $K^2 = 4$ whose Albanese fibration α has general fibre a genus 2 curve and such that $\alpha_* \omega_S^2$ is sum of line bundles.

We have constructed 8 unirational families of such surfaces in proposition 2.2, labeled $\mathcal{M}_{2,3}$, $\mathcal{M}_{4,2}$, $\mathcal{M}_{3,1}$, $\mathcal{M}_{6,1}$, $\mathcal{M}_{i,3}$, $\mathcal{M}_{i,2}$, $\mathcal{M}'_{i,2}$ and $\mathcal{M}_{i,1}$. Their parameter spaces have a natural map to $\mathcal{M}$.

REMARK 5.1. $\mathcal{M}$ has 8 connected components, that with a natural abuse of notation we will denote by $\mathcal{M}_{2,3}$, $\mathcal{M}_{4,2}$, $\mathcal{M}_{3,1}$, $\mathcal{M}_{6,1}$, $\mathcal{M}_{i,3}$, $\mathcal{M}_{i,2}$, $\mathcal{M}'_{i,2}$ and $\mathcal{M}_{i,1}$. Each component is the image of the parameter space of the namesake family, in particular is unirational.

PROOF. The map from the parameter space of our families to $\mathcal{M}$ is surjective by propositions 3.1 and 4.1.

There are many way to show that the closure of the images of two of these parameter spaces do not intersect. For example, since the number of direct summands of V_1 is a topological invariant by [CC1],

$$(\overline{\mathcal{M}_{2,3} \cup \mathcal{M}_{4,2} \cup \mathcal{M}_{3,1} \cup \mathcal{M}_{6,1}}) \cap (\overline{\mathcal{M}_{i,3} \cup \mathcal{M}_{i,2} \cup \mathcal{M}'_{i,2} \cup \mathcal{M}_{i,1}}) = \emptyset.$$

The closure of two of the first 4 families cannot intersect because the degree 0 summand of V_1 is in all cases a torsion line bundle but with different torsion order. To show $\overline{\mathcal{M}_{i,2}} \cap \overline{\mathcal{M}'_{i,2}} = \emptyset$ we apply the same argument to $(\det V_1)^2 \otimes \mathrm{hol}_B(-\tau)$. Finally the same argument applied to $\mathrm{hol}_B(D_1 - D_2)$ shows that also the closures of the remaining pairs of families do not intersect. □

PROPOSITION 5.2. $\dim \mathcal{M}_{2,3} = 5$. *All other components of* $\mathcal{M}$ *have dimension* 4.

PROOF. The natural way to compute the dimension of each component is computing the dimension of the corresponding parameter space, and then subtract to the result the dimension of the general fibre of the map into $\mathcal{M}$. These fibres correspond to orbits for the action of certain automorphism groups.

Aut V_1 and Aut V_2 do not act on our data, since in the tables 1 and 2 we require the matrix of σ_2 to have special form. But in fact in all cases this "special" form is the form of a general morphism $\mathrm{Sym}^2 V_1 \to V_2$ in suitable coordinates (for V_1 and V_2). It is then equivalent (but easier to compute) to consider σ_2 general in $\mathrm{Hom}(\mathrm{Sym}^2 V_1, V_2)$ and act on it with the full group $\mathrm{Aut}\, V_1 \times \mathrm{Aut}\, V_2$.

Are there other automorphisms to consider? We can forget the action of Aut B since we have fixed a point of B by choosing $\det V_1 \cong \mathcal{O}_B(0)$, so only a finite subgroup of Aut B act on our data, and quotienting by it do not affect the dimension. The other automorphism to consider is (since we are interested in Δ and not in its equation) "multiply the equation of $\mathcal{G}$ by a constant leaving the other data fixed". If you prefer, that's the action of the automorphisms of the line bundle $(\det V_1 \otimes \mathcal{O}_B(\tau))^2$. Anyway, multiplying V_1 by λ and V_2 by λ^2 do not change σ_2 but multiply the equation of $\mathcal{G}$ by λ^{-6}: this shows that we can restrict to consider the action of $\mathrm{Aut}\, V_1 \times \mathrm{Aut}\, V_2$.

We leave to the reader the check that the subgroup of $\mathrm{Aut}\, V_1 \times \mathrm{Aut}\, V_2$ fixing our data is finite. It follows (the moduli space of elliptic curves has dimension 1) that the dimension of each family is

$$1 + h + \delta - \alpha_1 - \alpha_2$$

where h, δ, α_i are respectively the dimensions of $\mathrm{Hom}(\mathrm{Sym}^2 V_1, V_2)$, $\mathrm{Hom}((\det V_1 \otimes \mathcal{O}_B(\tau))^2, \mathrm{Sym}^3 V_2)$ and Aut V_i.

Now the computation is easy:

$$\begin{aligned}
\dim \mathcal{M}_{2,3} &= 1 + 10 + 4 - 3 - 7 = 5\\
\dim \mathcal{M}_{4,2} &= 1 + 9 + 2 - 3 - 5 = 4\\
\dim \mathcal{M}_{3,1} &= 1 + 9 + 2 - 3 - 5 = 4\\
\dim \mathcal{M}_{6,1} &= 1 + 9 + 2 - 3 - 5 = 4\\
\dim \mathcal{M}_{i,3} &= 1 + 7 + 4 - 1 - 7 = 4\\
\dim \mathcal{M}_{i,2} &= 1 + 7 + 2 - 1 - 5 = 4\\
\dim \mathcal{M}'_{i,2} &= 1 + 7 + 2 - 1 - 5 = 4\\
\dim \mathcal{M}_{i,1} &= 1 + 7 + 2 - 1 - 5 = 4
\end{aligned}$$

□

PROPOSITION 5.3. *All connected components of $\mathcal{M}$ are irreducible components of the moduli space of minimal surfaces of general type.*

PROOF. We need to show that for the general surface in each component, $h^1(\mathcal{T}_S)$ is not greater than the dimension of the family, say d. By proposition 5.2, $d \in \{4, 5\}$ and more precisely $d = 5$ only for the family $\mathcal{M}_{2,3}$.

Equivalently (by Serre duality and since $h^0(\mathcal{T}_S) = 0$ for a surface of general type) we can show $h^0(\Omega^1_S \otimes \omega_S) = 2K_S^2 - 10\chi(\mathcal{O}_S) + h^1(\mathcal{T}_S) \le d - 2$.

For a fibration $f : S \to B$, we denote by $\mathrm{Crit}(f) \subset S$ the scheme of its critical points, $\mathcal{D} \subset \mathrm{Crit}(f)$ its divisorial part. By definition $\mathcal{D}$ is supported on the non-reduced components of the singular fibres.

Then (cf. [Cat3] lect. 9) computing kernel and cokernel of the natural map $\xi' : \Omega^1_S \to \omega_{S|B}$ locally defined by $\xi'(\eta) = (\eta \wedge dt) \otimes (dt)^{-1}$ (for t a local parameter on B) one finds an exact sequence

$$0 \to \mathcal{O}_S(f^*\omega_B + \mathcal{D}) \to \Omega^1_S \to \omega_{S|B} \to \mathcal{O}_{\mathrm{Crit}(f)}(\omega_{S|B}) \to 0 \tag{8}$$

By the proof of proposition 2.2, the Albanese fibration α of a general element S in each of our families factors as composition of

- a conic bundle $\mathscr{C} \to B$ with two singular fibres, both reduced, with $\mathrm{Sing}(\mathscr{C})$ consisting in two nodes, at the vertices of the two singular fibres;
- a finite double cover $S \to \mathscr{C}$ branched on the two nodes of $\mathscr{C}$ and on a smooth curve Δ not passing through the nodes.

It follows that each component of each fibre of α is reduced, so $\mathcal{D} = \emptyset$. Since $\omega_B = \mathcal{O}_B$ twisting the exact sequence (8) by ω_S we get the exact sequence

$$0 \to \omega_S \to \Omega^1_S \otimes \omega_S \to \omega_S^2 \to \mathcal{O}_{\mathrm{Crit}(\alpha)}(\omega_S^2) \to 0$$

Since $p_g = 1$ the required inequality $h^0(\Omega^1_S \otimes \omega_S) \le d - 2$ follows if we show $\dim\ker(H^0(\omega_S^2) \to H^0(\mathcal{O}_{\mathrm{Crit}(\alpha)}(\omega_S^2))) = d - 3$. In other words we must show that

1) the set of bicanonical curves containing the 0-dimensional scheme $\mathrm{Crit}(\alpha)$ of the general surface in $\mathcal{M}_{2,3}$ is a pencil;
2) the general surface in each of the other families has only one bicanonical curve containing $\mathrm{Crit}(\alpha)$.

We study the bicanonical system of S. The involution on a surface induced by a genus 2 fibration (acting as the hyperelliptic involution on any fibre) acts on $H^0(2K_S)$ as the identity. In our cases, at least for a general surface as above (the relative canonical model is smooth and minimal), the quotient by this involution is $\mathscr{C}$. So the bicanonical system of S is the pull-back of a linear system on $\mathscr{C}$, more precisely ($\omega_S = \omega_{S|B}$) the restriction of $|\mathcal{O}_{\mathbb{P}(V_2)}(1)|$.

We study the critical points of α. Since $\mathscr{C}$ has only reduced fibres the critical points of α must be fixed points for the involution on S. The isolated fixed points are the preimages of the two nodes of $\mathscr{C}$, and they are critical for α (in suitable local coordinates $\alpha(x, y) = xy$). The other critical points of α lies on the divisorial fixed locus of the involution, where the involution has the form $(x, y) \mapsto (x, -y)$: they are critical for α if and only if $\frac{\partial \alpha}{\partial x} = 0$. In other words we need their image on $\mathscr{C}$ to be a ramification point for the map $\Delta \to B$.

So we need to compute the dimension of the subsystem of $|\mathcal{O}_{\mathbb{P}(V_2)}(1)|$ containing the nodes of $\mathscr{C}$ and the critical points of the map $\Delta \to B$. Note that by the local computation above this is true schematically: we need H to contain the zero dimensional scheme $\mathrm{Sing}(\mathscr{C}) \cup \mathrm{Crit}(\Delta \to B)$.

In all cases (see table 3) $\mathscr{C} = \{q(y_1, y_2) + y_3^2 = 0\}$: in particular the nodes of $\mathscr{C}$ lie in $\{y_3 = 0\}$. Moreover $\Delta = \mathscr{C} \cap \mathscr{G}$ for $\mathscr{G} = \{G(y_1, y_2) = 0\}$. $\mathrm{Crit}(\Delta \to B)$ is defined by

$$\mathrm{rank}\begin{pmatrix} \frac{\partial q}{\partial y_1} & \frac{\partial q}{\partial y_2} & 2y_3 \\ \frac{\partial G}{\partial y_1} & \frac{\partial G}{\partial y_2} & 0 \end{pmatrix} \leq 1$$

therefore (being q and G homogeneous in the y_i's) $\mathrm{Crit}(\Delta \to B) = \Delta \cap \{y_3 = 0\}$.

We have shown that $(\mathrm{Sing}(\mathscr{C}) \cup \mathrm{Crit}(\Delta \to B)) \subset \{y_3 = 0\}$. First consequence is that any relative hyperplane of the form $\{fy_3 = 0\}$ contains the nodes of $\mathscr{C}$ and $\mathrm{Crit}(\Delta \to B)$.

Choosing $f \in H^0(\mathcal{O}_B(D_3))$, $\mathcal{O}_B(D_3)$ being the direct summand of V_2 given by the coordinate y_3, we find a curve whose pull-back is a bicanonical curve through $\mathrm{Crit}(\alpha)$. Note that $\deg D_3 = 1$ so in all cases we have found exactly one bicanonical curve through $\mathrm{Crit}(\alpha)$.

If there are further bicanonical curves through $\mathrm{Crit}(\alpha)$, then in the corresponding system of relative hyperplanes in $\mathbb{P}(V_2)$ there is an element H not containing $\{y_3 = 0\}$ and $H \cap \mathscr{C} \cap \{y_3 = 0\}$ contains the 0-dimensional scheme $\Delta \cap \{y_3 = 0\}$. If $H \cap \mathscr{C} \cap \{y_3 = 0\}$ is also 0-dimensional, then by intersection computation both $H \cap \mathscr{C} \cap \{y_3 = 0\}$ and $\Delta \cap \{y_3 = 0\}$ have length 6, so they must be equal, a contradiction since $\mathrm{Sing}\,\mathscr{C} \subset H \cap \mathscr{C} \cap \{y_3 = 0\}$ but $\mathrm{Sing}(\mathscr{C}) \not\subset \Delta$. Therefore, if there are further bicanonical curves through $\mathrm{Crit}(\alpha)$, then $H \cap \mathscr{C} \cap \{y_3 = 0\}$ contains a curve.

To conclude the proof we must now argue differently according to the family.

$(\mathscr{M}_{i,1}, \mathscr{M}'_{i,2}, \mathscr{M}_{i,3})$
We set $b_5 := c_5 := 0$ to treat these cases together. If a_j, b_j, c_j, d_j have no common zeroes, $\mathscr{C} \cap \{y_3 = 0\}$ has a finite map of degree 2 onto B and then, if it is reducible, its components are cut on $\{y_3 = 0\}$ by two relative hyperplanes $\{a'y_1 + b'y_2 = 0\}$ and $\{c'y_1 + d'y_2 = 0\}$ and $(a_jy_1 + c_jy_2)^2 + (b_jy_1 + d_jy_2)^2 = (a'y_1 + b'y_2)(c'y_1 + d'y_2)$.

This is impossible for general choice of a_j, b_j, c_j, d_j. In fact, take for simplicity $b_j = c_j = 0$, $a_jd_j \neq 0$. Then the only possible formal decomposition (up to $\mathscr{C}^*$ is $(a_jy_1)^2 + (d_jy_2)^2 = (a_jy_1 + id_jy_2)(a_jy_1 - id_jy_2)$ (here $i = \sqrt{-1}$). But, since "a_jy_1" is a map from $\mathcal{O}_B(D_1 - \eta_1)$ to V_2 and "d_jy_2" is a map from $\mathcal{O}_B(D_2 - \eta_2)$ to V_2, these factors make sense as relative hyperplanes only when $\mathcal{O}_B(D_1 - D_2) \cong \mathcal{O}_B(\eta_1 - \eta_2)$, that is not the case.

It follows that $\mathscr{C} \cap \{y_3 = 0\}$ is irreducible, then $H \cap \mathscr{C} \cap \{y_3 = 0\}$ is 0-dimensional and therefore there are no further bicanonical curves through $\mathrm{Crit}(\alpha)$ and $h^1(\mathscr{T}_S) \leq 4$.

$(\mathscr{M}_{i,2})$
The difference with the previous cases is that $\mathscr{O}_B(D_1 - D_2) \cong \mathscr{O}_B(\eta_1 - \eta_2)$, so, setting as above $b_6 = c_6 = 0$, $a_6 d_6 \neq 0$ we can obtain that $H \cap \mathscr{C} \cap \{y_3 = 0\}$ contains a curve by taking $H := \{a_6 y_1 \pm i d_6 y_2 = 0\}$. But then $H \cap \Delta$ is 0-dimensional of length 3 so $H \cap \mathscr{C} \cap \{y_3 = 0\}$ cannot contain $\Delta \cap \{y_3 = 0\}$, that has length 6. It follows that there are no further bicanonical curves through $\mathrm{Crit}(\alpha)$ and $h^1(\mathscr{T}_S) \leq 4$.

$(\mathscr{M}_{6,1}, \mathscr{M}_{3,1}, \mathscr{M}_{4,2})$
$\mathscr{C} \cap \{y_3 = 0\}$ reduces as union of $\{y_2 = 0\}$ and $\{a_j y_1 + b_j y_2 = 0\}$, that are irreducible for a_j, b_j without common zeroes. The first component do not intersect Δ, so to find a bicanonical curve we need to take H containing $\{a_j y_1 + b_j y_2 = 0\}$. This is possible only when $\mathscr{O}_B(2 \cdot 0 - 2 \cdot p)$ is the trivial bundle.

Since this is not the case for the three families under consideration, arguing as above there are no further bicanonical curves through $\mathrm{Crit}(\alpha)$ and $h^1(\mathscr{T}_S) \leq 4$.

$(\mathscr{M}_{2,3})$
Arguing exactly as above we find that the only possibility to get a further bicanonical curve through $\mathrm{Crit}(\alpha)$ is by choosing $H := \{a_1 y_1 + a_2 y_2 = 0\}$. It follows that the set of bicanonical curves through $\mathrm{Crit}(\alpha)$ is a pencil and therefore $h^1(\mathscr{T}_S) \leq 5$. □

Proof of theorem 0.1. The first statement comes from remark 5.1 and proposition 5.2. The second statement is proposition 5.3. The last statement was shown in proposition 2.2. □

Remark 5.4. As mentioned in the introduction the biggest family of minimal surfaces with $K^2 = 4$, $p_g = q = 1$ constructed by Polizzi is a subfamily of $\mathscr{M}_{2,3}$. We can be more precise, by looking at the properties of these surfaces (that we will claim without proof, all follow from the description in [Pol2]).

It is a family of nodal surfaces obtained as quotient of a product of curves by an action of $\mathbb{Z}_{/2\mathbb{Z}} \times \mathbb{Z}_{/2\mathbb{Z}}$. The group is abelian, so (arguing as in the proof of [Pol1], theorem 6.3) $\alpha_* \omega_S^n$ in a sum of line bundles for each $n \in \mathbb{N}$. By proposition 3.1 their smooth minimal models give a subfamily of $\mathscr{M}_{2,3} \cup \mathscr{M}_{4,2} \cup \mathscr{M}_{3,1} \cup \mathscr{M}_{6,1}$.

All Polizzi's surfaces have 4 nodes. Since each of our families contains a (smooth minimal) surface with ample canonical class by proposition 2.2, and Polizzi's family is irreducible, then it gives a proper subfamily of one of the components $\mathscr{M}_{2,3}$, $\mathscr{M}_{4,2}$, $\mathscr{M}_{3,1}$, $\mathscr{M}_{6,1}$. Since it has dimension 4, by proposition 5.2 it has codimension 1 in $\mathscr{M}_{2,3}$.

We can be more precise. The 4 nodes are contained in two fibres of the Albanese morphism (two on each fibre), fibres that are 2-divisible as Weil divisors on the relative canonical model. It follows that the singular conics of $\mathscr{C}$ are two double lines. By the equation of $\mathscr{C}$ in table 3, these are exactly the surfaces for which $b_1 = 0$.

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