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Partial Differential Equations — *Existence of hylomorphic solitary waves in Klein-Gordon and in Klein-Gordon-Maxwell equations*, by VIERI BENCI and DONATO FORTUNATO, communicated on 11 June 2009.

ABSTRACT. — This paper is devoted to the study of solitary waves whose existence is related to the ratio energy/charge. These solitary waves are called hylomorphic. This class includes the Q-balls, which are spherically symmetric solutions of the nonlinear Klein-Gordon equation (NKG), as well as solitary waves and vortices which occur, by the same mechanism, in the nonlinear Schroedinger equation and in gauge theories. It is proved an abstract theorem which allows to show the existence of hylomorphic solitary waves and vortices in the (NKG) and in the nonlinear Klein-Gordon-Maxwell equations (NKGM).

KEY WORDS: Q-balls, Hylomorphic solitons, vortices, Abelian gauge theories.

AMS SUBJECT CLASSIFICATION: 47J30, 35J50, 81V10, 74M15.

Dedicated to the memory of Guido Stampacchia

1. INTRODUCTION

Roughly speaking a *solitary wave* is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. A solitary wave which has a non-vanishing angular momentum is called *vortex*. A *soliton* is a solitary wave which exhibits some strong form of stability so that it has a particle-like behavior (see e.g. [3], [12], [29], [38]).

To day, we know (at least) three mechanisms which might produce solitary waves, vortices and solitons:

- Complete integrability, (e.g. Kortewg-de Vries equation);
- Topological constraints, (e.g. Sine-Gordon equation);
- Ratio energy/charge: (e.g. the nonlinear Klein-Gordon equation).

Following [7], the third type of solitary waves or solitons will be called *hylomorphic*. This class includes the *Q-balls* which are spherically symmetric solutions of the nonlinear Klein-Gordon equation (NKG) (see [20], [27]) as well as solitary waves and vortices which occur, by the same mechanism, in the nonlinear Schrödinger equation and in gauge theories ([11], [15]).

This paper is devoted to an abstract theorem which allows to prove the existence of hylomorphic solitary waves, solitons and vortices in the (NKG) and in

the nonlinear Klein-Gordon-Maxwell equations (NKGM). In this case we prove a multiplicity result in terms of different charges (see theorem 22).

2. Hylomorphic solitons

In this section we will sketch the main ideas relative to hylomorphic solitons. They can be considered as particular states of a system modelled by a field equation.

We assume that the state of the system is described by one or more fields which mathematically are represented by a function

$$\Psi : \mathbb{R}^N \to V \tag{1}$$

where V is a finite dimensional vector space with norm $|\cdot|_V$ and it is called the internal parameters space. We will denote by $\mathscr{X}$ the set of all the states.

A state $\Psi_0 \in \mathscr{X}$ is called solitary wave if its evolution $\Psi(t)$ has the following form:

$$\Psi(t) = h_t \Psi_0(g_t x)$$

where h_t and g_t are transformations on V and $\mathbb{R}^N$ respectively and which depend continuously on t. A solitary wave $\Psi_0 \in \mathscr{X}$ is called soliton if it is *orbitally stable* i.e. if $\Psi_0 \in \Gamma$, where Γ is a finite dimensional manifold which is invariant and stable (see e.g. [6]).

In this paper, we shall consider two cases:

– Equation (NKG) (see section 4.1) where

$$\Psi = (\psi, \psi_t) \in \mathbb{C}^2.$$

– Equation (NKGM) (see section 6.1) where

$$\Psi = (\psi, \psi_t, \phi, \phi_t, \mathbf{A}, \mathbf{A}_t) \in \mathbb{C}^2 \times \mathbb{R}^8.$$

The existence and the properties of hylomorphic solitons are guaranteed by the interplay between *energy* E and another integral of motion which, in the general case, is called *hylenic charge* and it will be denoted by H.

Thus, the most general equations for which it is possible to have hylomorphic solitons need to have the following features:

- **A-1.** *The equations are variational namely they are the Euler-Lagrange equations relative to a Lagrangian density* $\mathscr{L}[\Psi]$.
- **A-2.** *The equations are invariant for time and space translations, namely* $\mathscr{L}$ *does not depend explicitly on* t *and* x.
- **A-3.** *The equations are invariant for a* S^1 *action, namely* $\mathscr{L}$ *does not depend explicitly on the phase of the field* Ψ *which is supposed to be complex valued* (*or at lest to have some complex valued component*).

More exactly, in (NKG), we have the following S^1 action

$$T_\theta \Psi = T_\theta(\psi, \psi_t) = (e^{i\theta}\psi, e^{i\theta}\psi_t), \quad \theta \in \mathbb{R}/(2\pi\mathbb{Z}) = S^1$$

and in (NKGM) we have

$$T_\theta \Psi = T_\theta(\psi, \psi_t, \phi, \phi_t, \mathbf{A}, \mathbf{A}_t) = (e^{i\theta}\psi, e^{i\theta}\psi_t, \phi, \phi_t, \mathbf{A}, \mathbf{A}_t).$$

Solitary waves or solitons for equations satisfying **A-1** and **A-2** and having null momentum are called *stationary waves* or *stationary solitons*.

By Noether theorem assumptions **A-1** and **A-2** guarantee the conservation of the energy $E(\Psi)$ and of the momentum $\mathbf{P}(\Psi)$ (see e.g. [12]), while **A-1** and **A-3** guarantee the conservation of another integral of motion which we call *hylenic charge* $H(\Psi)$ (see [7]).

The quantity

$$\Lambda(\Psi) = \frac{E(\Psi)}{|H(\Psi)|}, \tag{2}$$

which is an invariant of the motion having the dimension of energy, is called *hylomorphy ratio*.

We now set

$$m = \lim_{\varepsilon \to 0} \inf_{\Psi \in \mathscr{X}_\varepsilon} \frac{E(\Psi)}{|H(\Psi)|} \tag{3}$$

where

$$\mathscr{X}_\varepsilon = \{\Psi \in \mathscr{X} : \forall x, |\Psi(x)|_V < \varepsilon\}. \tag{4}$$

Now let $\Psi(t)$ be the evolution of a state such that $\Lambda(\Psi(0)) = \lambda < m$; then, $\Lambda(\Psi(t)) = \lambda$ for all t, and, by definition of m, we have that

$$\liminf_{t \to \infty} |\Psi(t)|_V > 0.$$

Thus it it possible that $\Psi(t)$ tends to a nontrivial stable configuration.

Now let σ be a real number and Ψ be a state such that

$$H(\Psi) = \sigma \quad \text{and} \quad E(\Psi) = \min\{E(v) : H(v) = \sigma\} \tag{5}$$

and denote by Γ_σ the set of such minimizers Ψ, namely

$$\Gamma_\sigma = \{\Psi : \Psi \text{ satisfies (5)}\}.$$

Observe that by **A-2** the energy is a constant of the motion, then Γ_σ is an invariant set.

Now we give the following definition

DEFINITION 1. *A stationary wave Ψ_0 is called hylomorphic wave if*

$$\Psi_0 \in \Gamma_\sigma \quad \text{for some } \sigma. \tag{6}$$

Moreover Ψ_0 is called hylomorphic soliton if it satisfies (6) *and if Γ_σ is a manifold with*

$$dim(\Gamma_\sigma) < \infty \quad \text{and} \quad \Gamma_\sigma \text{ is stable}$$

REMARK 2. *In the examples considered in this paper, the Lagrangian $\mathscr{L}[\Psi]$ is invariant for an action of the Poincaré group. In particular, if the Lagrangian is invariant for the action of a Lorentz boost, then the existence of stationary waves and stationary solitons implies the existence of travelling* (*with velocity* $\mathbf{v}$, $|\mathbf{v}| < c$) *waves and travelling solitons respectively* (*see e.g.* [12]).

3. AN ABSTRACT THEOREM

In many situations the energy E and the charge H have the following form

$$E(u, \omega) = J(u) + \omega^2 K(u), \tag{7}$$

$$H(u, \omega) = 2\omega K(u). \tag{8}$$

where $\omega \in \mathbb{R}$ and J and K are as follows:

$$J(u) = \frac{1}{2}\langle L_1 u, u\rangle + N_1(u)$$

$$K(u) = \frac{1}{2}\langle L_0 u, u\rangle + N_0(u)$$

where $L_i : X \to X'$ $(i = 0, 1)$ are linear continuous operators and N_i $(i = 0, 1)$ are differentiable functionals defined on a Hilbert space X with a norm equivalent to the following one

$$\|u\|^2 = \langle L_1 u, u\rangle.$$

Here $\langle\, , \rangle$ denotes the duality between X and X'.

The existence of solitary waves for the field equations we are interested in lead to study the following abstract eigenvalue problem:

$$J'(u) = \omega^2 K'(u). \tag{9}$$

where J' and K' denote the differentials of J and K.

The most natural way to solve this problem consists in minimizing $J(u)$ on the manifold $\{u : K(u) = const.\}$. However the assumptions which allow such a minima to exist are not adequate for the problems which we want to consider. For this reason we adopt a different variational principle, which permits also to get the existence of particular solitary waves, namely of hylomorphic waves (see Definition 1).

We set for $\sigma > 0$

$$M_\sigma = \{(u, \omega) \in X \times \mathbb{R}^+ : H(u, \omega) = \sigma\}.$$

The variational principle is contained in the following simple result:

THEOREM 3. *The critical points* (u, ω) *of* E *on* M_σ *solve the problem* (9).

PROOF. Let $(u, \omega) \in M_\sigma$ be a critical point of E on M_σ. Then there exists λ real such that

$$\begin{cases} \partial_u E(u, \omega) = \lambda \partial_u H(u, \omega) \\ \partial_\omega E(u, \omega) = \lambda \partial_\omega H(u, \omega) \end{cases}$$

These equations can be written more explicitly

$$\begin{cases} J'(u) + \omega^2 K'(u) = \lambda \omega K'(u) \\ 2\omega K(u) = \lambda K(u) \end{cases}$$

From the second equation we have $\lambda = 2\omega$ and substituting in the first one, we get that (u, ω) solves problem (9). □

The utility of Theorem 3 relies on the fact that the existence of critical points of E on M_σ is guaranteed by an assumption (see assumption (11)), which in many physical problems is the natural one. Moreover in some cases this assumption guarantees the stability of the solutions.

We make the following assumptions:

- (H1) $J \geq 0$ and J is coercive on M_σ, namely for any sequence $(u_n, \omega_n) \in M_\sigma$ we have that ($J(u_n)$ bounded) $\Rightarrow$ (u_n bounded).
- (H2) The differentials N_0', N_1' of N_0, N_1 satisfy the following compactness properties: $N_0' : X \to X'$ is compact. Moreover, if u_n converges weakly in X, then

$$\langle N_1'(u_n) - N_1'(u_m), u_n - u_m \rangle \to 0 \quad \text{as } n, m \to \infty \tag{10}$$

- (H3) $K(u) \geq 0$ for all u and $K(u) \neq 0$ for some $u \in X$.

We shall prove the following theorem:

THEOREM 4. *Assume* (H1,2,3) *and that there is* $\bar{u}$, *such that*

$$0 < \frac{J(\bar{u})}{K(\bar{u})} < m^2 \tag{11}$$

where

$$m^2 = \inf \frac{\langle L_1 u, u \rangle}{\langle L_0 u, u \rangle} > 0. \tag{12}$$

Then there exists a non empty, open set $\Sigma \subset \mathbb{R}$ *such that, for any* $\sigma \in \Sigma$, E *has a minimizer* (u_0, ω_0) *on* M_σ *with* $0 < \omega_0^2 < m^2$.

As an immediate consequence of Theorem 3 and Theorem 4 we get

THEOREM 5. *Under the assumptions of Theorem* 4 *there exists a non empty, open set* $\Sigma \subset \mathbb{R}$ *such that, for any* $\sigma \in \Sigma$ *problem* (9) *has a solution* (u, ω), *such that* $0 < \omega^2 < m^2$, $H(u, \omega) = \sigma$ *and which is a minimizer of* E *on* M_σ.

We set, for $\omega > 0$ and $K(u) > 0$,

$$\Lambda(u, \omega) = \frac{E(u, \omega)}{H(u, \omega)} = \frac{1}{2}\Big(\frac{J(u)}{K(u)} \cdot \frac{1}{\omega} + \omega\Big).$$

REMARK 6. *In this paper we will apply theorem* 5 *in three cases. In these cases,* E *and* H *will represent respectively the energy and the hylenic charge,* Λ *is the hylomorphy ratio and* m *in* (12) *coincides with the constant defined by* (3).

In order to prove Theorem 4, we need several lemmas.

LEMMA 7. *If* $J, K \geq 0$, *then the following assertions are equivalent:*

- (a) *there is* $\bar{u} \in X$, *such that*

$$0 < \frac{J(\bar{u})}{K(\bar{u})} < m^2. \tag{13}$$

- (b) *there exist* $\bar{u} \in X$, $\bar{\omega} > 0$ *such that*

$$\Lambda(\bar{u}, \bar{\omega}) < m. \tag{14}$$

PROOF. (a) $\Rightarrow$ (b) If we take $\bar{\omega} = \sqrt{\frac{J(\bar{u})}{K(\bar{u})}}$, we have that

$$\Lambda(\bar{u}, \bar{\omega}) = \frac{1}{2}\Big(\frac{J(\bar{u})}{K(\bar{u})} \cdot \frac{1}{\bar{\omega}} + \bar{\omega}\Big) = \sqrt{\frac{J(\bar{u})}{K(\bar{u})}} < m.$$

(b) $\Rightarrow$ (a) If $\frac{1}{2}\big(\frac{J(\bar{u})}{K(\bar{u})} \cdot \frac{1}{\bar{\omega}} + \bar{\omega}\big) < m$, then

$$\frac{J(\bar{u})}{K(\bar{u})} < 2m\bar{\omega} - \bar{\omega}^2 \leq \max_{\omega \geq 0}(2m\omega - \omega^2) = m^2. \qquad \square$$

LEMMA 8. *Assume* $J, K \geq 0$ *and let* (u_n, ω_n) *be a sequence in* M_σ, $\sigma > 0$, *with* $\Lambda(u_n, \omega_n)$ *bounded. Then the sequences* ω_n *and* $J(u_n)$ *are bounded.*

The proof is trivial.

We now set

$$\hat{c} = \inf_{\omega \geq m,\, u \in X} \Lambda(u, \omega).$$

LEMMA 9. *Assume that $J, K \geq 0$ and let (u_n, ω_n) be a sequence in M_σ, $\sigma > 0$, such that $\Lambda(u_n, \omega_n) \to c < \hat{c}$. Then (up to a subsequence).*

$$\omega_n \to \omega_0 < m.$$

PROOF. Let (u_n, ω_n) be a sequence in M_σ, $\sigma > 0$, such that $\Lambda(u_n, \omega_n) \to c < \hat{c}$. Since $\Lambda(u_n, \omega_n)$ is bounded, by Lemma 8, ω_n is bounded and hence, up to a subsequence, $\omega_n \to \omega_0$. We have to prove that $\omega_0 < m$. We argue indirectly and assume that $\omega_n = m_1 + \delta_n$ with $\delta_n \to 0$ and $m_1 \geq m$. Since ω_n and $\Lambda(u_n, \omega_n)$ are bounded, also $\frac{J(u_n)}{K(u_n)}$ is bounded, then easy calculations give

$$\begin{aligned}\Lambda(u_n, m_1 + \delta_n) &= \frac{1}{2}\left(\frac{J(u_n)}{K(u_n)} \cdot \frac{1}{m_1 + \delta_n} + m_1 + \delta_n\right) \\ &= \frac{1}{2}\left(\frac{J(u_n)}{m_1 K(u_n)}\left(1 + \frac{\delta_n}{m_1}\right)^{-1} + m_1 + \delta_n\right) \\ &= \Lambda(u_n, m_1) + O(\delta_n).\end{aligned}$$

Then

$$\begin{aligned}c = \lim_{n\to\infty} \Lambda(u_n, \omega_n) &= \lim_{n\to\infty} \Lambda(u_n, m_1 + \delta_n) = \lim_{n\to\infty} (\Lambda(u_n, m_1) + O(\delta_n)) \\ &\geq \inf_{\omega \geq m, u \in X} \Lambda(u, \omega) = \hat{c},\end{aligned}$$

contradicting our assumption. □

LEMMA 10. *Assume* (H1,2,3). *Then for any $\sigma > 0$, Λ satisfies PS in M_σ under the level $\hat{c}$, namely, if (u_n, ω_n) is a sequence in M_σ such that*

$$\Lambda(u_n, \omega_n) \to c < \hat{c} \tag{15}$$

$$d\Lambda|_{M_\sigma}(u_n, \omega_n) \to 0, \tag{16}$$

then (u_n, ω_n) has a converging subsequence.

PROOF. Let (u_n, ω_n) be a sequence in M_σ satisfying (15) and (16). By Lemma 8 $J(u_n)$ is bounded. Then, by the coercivity of J on M_σ, we deduce that u_n weakly converges (up to a subsequence) to $u_0 \in X$. Using Lemma 9, up to a subsequence, we get that

$$\omega_n \to \omega_0 < m. \tag{17}$$

Now we prove that u_n converges strongly to u_0.

By (16) we have that there exists a sequence of real numbers λ_n such that

$$\begin{cases}\partial_u E(u_n, \omega_n) = \lambda_n \partial_u H(u_n, \omega_n) + \varepsilon_n \\ \partial_\omega E(u_n, \omega_n) = \lambda_n \partial_\omega H(u_n, \omega_n) + \eta_n\end{cases}$$

where $\varepsilon_n \to 0$ in X' and $\eta_n \to 0$ in $\mathbb{R}$. These equations can be written more explicitely as follows:

$$\begin{cases} J'(u_n) + \omega_n^2 K'(u_n) = \lambda_n \omega_n K'(u_n) + \varepsilon_n \\ 2\omega_n K(u_n) = \lambda_n K(u_n) + \eta_n \end{cases}. \tag{18}$$

By the second equation we get

$$\lambda_n = 2\omega_n - \frac{\eta_n}{K(u_n)} = 2\omega_n - \frac{2\eta_n \omega_n}{\sigma};$$

replacing λ_n in the first equation, we get

$$J'(u_n) - \omega_n^2 K'(u_n) = -\frac{2\eta_n \omega_n^2}{\sigma} K'(u_n) + \varepsilon_n.$$

This equation can be rewritten as follows

$$L_1 u_n - \omega_0^2 L_0 u_n = -N_1'(u_n) + \omega_n^2 N_0'(u_n) + \delta_n \tag{19}$$

where

$$\delta_n = -(\omega_0^2 - \omega_n^2) L_0 u_n - \frac{2\eta_n \omega_n^2}{\sigma} K'(u_n) + \varepsilon_n.$$

Since u_n is bounded, $L_0 u_n$ and $K'(u_n)$ are bounded; then $\delta_n \to 0$.

Replacing in (19) n with m

$$L_1(u_m) - \omega_0^2 L_0(u_m) = -N_1'(u_m) + \omega_m^2 N_0'(u_m) + \delta_m \tag{20}$$

and, subtracting (20) from (19), we get

$$\begin{aligned} L_1(u_n - u_m) - \omega_0^2 L_0(u_n - u_m) \\ = N_1'(u_m) - N_1'(u_n) + \omega_n^2 N_0'(u_n) - \omega_m^2 N_0'(u_m) + \delta_n - \delta_m. \end{aligned} \tag{21}$$

By (H2) and since u_n is bounded, we easily get

$$\langle N_1'(u_m) - N_1'(u_n) + \omega_n^2 N_0'(u_n) - \omega_m^2 N_0'(u_m), u_n - u_m \rangle \to 0. \tag{22}$$

By (12) we have that

$$\begin{aligned} &\langle L_1(u_n - u_m), u_n - u_m \rangle - \omega_0^2 \langle L_0(u_n - u_m), u_n - u_m \rangle \\ &\quad \geq \langle L_1(u_n - u_m), u_n - u_m \rangle - \frac{\omega_0^2}{m^2} \langle L_1(u_n - u_m), u_n - u_m \rangle \\ &\quad \geq \left(1 - \frac{\omega_0^2}{m^2}\right) \|u_n - u_m\|^2. \end{aligned} \tag{23}$$

Thus, multiplying both sides of (21) by $u_n - u_m$ and using (22), (23), we get

$$\varepsilon_{n,m} \geq \left(1 - \frac{\omega_0^2}{m^2}\right)\|u_n - u_m\|^2 \quad \text{where } \varepsilon_{n,m} \to 0. \tag{24}$$

Since $\omega_0 < m$ (see (17)), by (24) u_n is a Cauchy sequence in X. □

LEMMA 11. *If assertion* (a) (*or* (b)) *in lemma* 7 *holds, then*

$$\hat{c} < m.$$

PROOF. By (a) in lemma 7 we have, for a suitable $\bar{u} \in X$, $\frac{J(\bar{u})}{K(\bar{u})} < m^2$. Then, by definition of $\hat{c}$,

$$\hat{c} \leq \Lambda(\bar{u}, m) = \frac{1}{2}\left(\frac{J(\bar{u})}{K(\bar{u})} \cdot \frac{1}{m} + m\right) < m \qquad \square$$

Now we set

$$\Sigma = \left\{\sigma > 0 : \inf_{(u,\omega)\in M_\sigma} \Lambda(u,\omega) < \hat{c}\right\}. \tag{25}$$

The following Lemma guarantees that the set Σ is not empty.

LEMMA 12. *If assertion* (a) (*or* (b)) *in lemma* 7 *holds, then*

$$\inf_{(u,\omega)\in X\times\mathbb{R}^+} \Lambda(u,\omega) < \hat{c}.$$

PROOF. By definition of $\hat{c}$ there exists a sequence (u_n, ω_n) in $X \times \mathbb{R}^+$ with $\omega_n \geq m$ and such that

$$\Lambda(u_n, \omega_n) \to \hat{c}.$$

Clearly ω_n is bounded and consequently also $\frac{J(u_n)}{K(u_n)}$ is bounded. So, up to a subsequence, we have

$$\omega_n \to \bar{\omega} \geq m \quad \text{and} \quad a_n \to \bar{a}, \quad a_n = \frac{J(u_n)}{K(u_n)}.$$

Then

$$\hat{c} = \frac{1}{2}\left(\frac{\bar{a}}{\bar{\omega}} + \bar{\omega}\right).$$

We claim that

$$\bar{a} < m^2. \tag{26}$$

In fact

$$\hat{c} = \frac{1}{2}\left(\bar{a}\frac{1}{\bar{\omega}} + \bar{\omega}\right) \tag{27}$$
$$= \frac{1}{2}\left(\frac{m^2}{\bar{\omega}} + \bar{\omega}\right) - \frac{1}{2}(m^2 - \bar{a})\frac{1}{\bar{\omega}}.$$

Then, by Lemma 11 and (27), we get

$$m > \frac{1}{2}\left(\frac{m^2}{\bar{\omega}} + \bar{\omega}\right) - \frac{1}{2}(m^2 - \bar{a})\frac{1}{\bar{\omega}}. \tag{28}$$

On the other hand

$$\frac{1}{2}\left(\frac{m^2}{\bar{\omega}} + \bar{\omega}\right) \geq m, \tag{29}$$

then (28) and (29) imply that

$$-\frac{1}{2}(m^2 - \bar{a})\frac{1}{\bar{\omega}} < 0.$$

So (26) is proved.

Now by (26) we can take $\hat{\omega}$ such that

$$m > \hat{\omega} > \sqrt{\bar{a}},$$

and, since $\bar{\omega} \geq m$, we have

$$\bar{\omega} > \hat{\omega} > \sqrt{\bar{a}}.$$

So it can be easily deduced that

$$\frac{1}{2}\left(\frac{\bar{a}}{\hat{\omega}} + \hat{\omega}\right) < \frac{1}{2}\left(\frac{\bar{a}}{\bar{\omega}} + \bar{\omega}\right).$$

Then

$$\lim \Lambda(u_n, \hat{\omega}) = \frac{1}{2}\left(\frac{\bar{a}}{\hat{\omega}} + \hat{\omega}\right) < \frac{1}{2}\left(\frac{\bar{a}}{\bar{\omega}} + \bar{\omega}\right) = \hat{c}.$$

So, for n large, we have $\Lambda(u_n, \hat{\omega}) < \hat{c}$ and the conclusion follows. □

Now we are ready to prove Theorem 4.

PROOF OF TH. 4. By Lemma 12 the set Σ defined in (25) is not empty. Let $\sigma \in \Sigma$ and (u_n, ω_n) be a minimizing sequence for E on M_σ. By standard variational

arguments (see e.g. [2], [37]) we can assume that (u_n, ω_n) is also a P.S. sequence. Since $\sigma \in \Sigma$, we have

$$c = \lim \Lambda(u_n, \omega_n) = \inf\{\Lambda(u, \omega) : (u, \omega) \in M_\sigma\} < \hat{c}.$$

Then, by the lemma 10, (u_n, ω_n) possess a strongly convergent subsequence and hence E has a minimizer on M_σ. Let us finally show that Σ is open. Take $\sigma \in \Sigma$; we have to prove that, for ε small, $\sigma + \varepsilon \in \Sigma$. Let (u_0, ω_0) be a minimizer of E on M_σ, then, since $\sigma \in \Sigma$, we have

$$\Lambda(u_0, \omega_0) < \hat{c}. \tag{30}$$

Since $2\omega_0 K(u_0) = \sigma$, by definition of $M_{\sigma+\varepsilon}$, we have

$$\left(u_0, \omega_0 + \frac{\varepsilon}{2K(u_0)}\right) \in M_{\sigma+\varepsilon}. \tag{31}$$

Then

$$\inf_{(u,\omega)\in M_{\sigma+\varepsilon}} \Lambda(u, \omega) \leq \Lambda\left(u_0, \omega_0 + \frac{\varepsilon}{2K(u_0)}\right). \tag{32}$$

By (30) and by (32) we easily deduce that for ε small we have

$$\inf_{(u,\omega)\in M_{\sigma+\varepsilon}} \Lambda(u, \omega) < \hat{c}. \qquad \square$$

4. *Q*-BALLS

4.1 The Nonlinear Klein-Gordon Equation

In this section we will apply the abstract Theorem 4 to the existence of hylomorphic solitons of the nonlinear Klein-Gordon equation (NKG):

$$\Box\psi + W'(\psi) = 0 \tag{NKG}$$

where $\Box = \partial_t^2 - \nabla^2$, $\psi : \mathbb{R}^N \to \mathbb{C}$ $(N \geq 3)$ and $W : \mathbb{C} \to \mathbb{R}$ with

$$W(\psi) = F(|\psi|) \tag{33}$$

for some smooth function $F : \mathbb{R}^+ \to \mathbb{R}$ and

$$W'(\psi) = F'(|\psi|)\frac{\psi}{|\psi|}.$$

In particular we are interested in the existence of *Q-balls*. Coleman called *Q-balls* ([20]) those solitary waves of (NKG) which are spherically symmetric and this is the name generally used in Physics literature. From now on, we always will assume that

$$W(0) = W'(0) = 0. \tag{34}$$

Eq. (NKG) is the Euler-Lagrange equation of the action functional

$$\int \left(\frac{1}{2} |\partial_t \psi|^2 - \frac{1}{2} |\nabla \psi|^2 - W(\psi) \right) dx\, dt. \tag{35}$$

Sometimes it will be useful to write ψ in polar form, namely

$$\psi(t,x) = u(t,x) e^{iS(t,x)} \tag{36}$$

where $u(t,x) \in \mathbb{R}^+$ and $S(t,x) \in \mathbb{R}/(2\pi\mathbb{Z})$; if we set $u_t = \partial_t u$,

$$\mathbf{k}(t,x) = \nabla S(t,x) \tag{37}$$

and

$$\omega(t,x) = -\partial_t S(t,x), \tag{38}$$

the state Ψ is uniquely defined by the quadruple $(u, u_t, \omega, \mathbf{k})$. Using these variables, the action $\mathscr{S} = \int \mathscr{L}\, dx\, dt$ takes the form

$$\mathscr{S}(u, u_t, \omega, \mathbf{k}) = \frac{1}{2} \int [u_t^2 - |\nabla u|^2 + (\omega^2 - \mathbf{k}^2) u^2]\, dx\, dt - \int W(u)\, dx\, dt = 0 \tag{39}$$

and equation (NKG) becomes:

$$\Box u + (\mathbf{k}^2 - \omega^2) u + W'(u) = 0 \tag{40}$$

$$\partial_t(\omega u^2) + \nabla \cdot (\mathbf{k} u^2) = 0. \tag{41}$$

The energy and the charge take the following form:

$$E(\Psi) = \int \left[\frac{1}{2} |\partial_t \psi|^2 + \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right] dx \tag{42}$$

$$H(\Psi) = -\mathrm{Im} \int \partial_t \psi \bar{\psi}\, dx. \tag{43}$$

(the sign "minus" in front of the integral is a useful convention).

Using (36) we get:

$$E(u, u_t, \omega, \mathbf{k}) = \int \left[\frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} [\omega^2 + \mathbf{k}^2] u^2 + W(u) \right] dx \tag{44}$$

$$H(u, \omega) = \int \omega u^2\, dx. \tag{45}$$

A particular type of solutions of eq. (NKG) are the *standing waves*. A *standing wave* is a finite energy solution of (NKG) having the following form

$$\psi_0(t,x) = u(x)e^{-i\omega t}, \quad u \geq 0,\ \omega \in \mathbb{R} \tag{46}$$

Substituting (46) in eq. (NKG), we get

$$-\Delta u + W'(u) = \omega^2 u, \quad u \geq 0. \tag{47}$$

Let $N = 3$. Since the action functional (35) is invariant for the Lorentz group, we can obtain other solutions $\psi_{\mathbf{v}}(t,x)$ just making a Lorentz transformation on it. Namely, if we take the velocity $\mathbf{v} = (v,0,0)$, $|v| < 1$, and set

$$t' = \gamma(t - vx_1), \quad x_1' = \gamma(x_1 - vt), \quad x_2' = x_2, \quad x_3' = x_3 \quad \text{with } \gamma = \frac{1}{\sqrt{1-v^2}},$$

it turns out that

$$\psi_{\mathbf{v}}(t,x) = \psi(t',x')$$

is a solution of (NKG).

More exactly, given a standing wave $\psi(t,x) = u(x)e^{-i\omega t}$, the function $\psi_{\mathbf{v}}(t,x) := \psi(t',x')$ is a solitary wave which travels with velocity $\mathbf{v}$. Thus, if $u(x) = u(x_1,x_2,x_3)$ is any solution of Eq. (47), then

$$\psi_{\mathbf{v}}(t,x_1,x_2,x_3) = u(\gamma(x_1 - vt),x_2,x_3)e^{i(\mathbf{k}_{\mathbf{v}}\cdot\mathbf{x} - \omega_{\mathbf{v}}t)} \tag{48}$$

is a solution of Eq. (NKG) provided that

$$\omega_{\mathbf{v}} = \gamma\omega \quad \text{and} \quad \mathbf{k}_{\mathbf{v}} = \gamma\omega\mathbf{v}. \tag{49}$$

4.2 Existence Results for Q-balls

We write W as follows

$$W(s) = \frac{m^2}{2}s^2 + N(s), \quad s \geq 0; \tag{50}$$

and we will identify $W(s)$ with $F(s)$. We make the following assumptions:

- (W-i) (**Positivity**) $W(s) \geq 0$
- (W-ii) (**Nondegeneracy**) $W = W(s)$ $(s \geq 0)$ is C^2 near the origin with $W(0) = W'(0) = 0$; $W''(0) = m^2 > 0$
- (W-iii) (**Hylomorphy**) $\exists s_0 : N(s_0) < 0$
- (W-iiii) (**Growth condition**) Al least one of the following assumptions holds:
  - (a) there are constants $a, b > 0$, $2 < p < 2N/(N-2)$ such that for any $s > 0$:

$$|N'(s)| \leq as^{p-1} + bs^{2-2/p}.$$

  - (b) $\exists s_1 > s_0 : N'(s_1) \geq 0$.

Here there are some comments on assumptions (W-i), (W-ii), (W-iii), (W-iiii).

(W-i) As we shall see (W-i) implies that the energy is positive; if this condition does not hold, it is possible to have solitary waves, but not hylomorphic waves (cf. Proposition 16).

(W-ii) In order to have solitary waves it is necessary to have $W''(0) \geq 0$. There are some results also when $W''(0) = 0$ (null-mass case, see e.g. [17] and [4]), however the most interesting situation occurs when $W''(0) > 0$.

(W-iii) This is the crucial assumption which characterizes the potentials which might produce hylomorphic solitons. As we will see, this assumption permits to have states Ψ with hylomorphy ratio $\Lambda(\Psi) < m$.

(W-iiii)(a) This assumption contains the usual growth condition at infinity which guarantees the C^1 regularity of the functional. Moreover it implies that $|N'(s)| = O(s^{2-2/p})$ for s small.

If we assume alternatively (W-iiii)(b), the growth condition (W-iiii)(a) can be avoided by using standard tricks (see Appendix).

We have the following result:

THEOREM 13. *If* (W-i), (W-ii), (W-iii), (W-iiii) *hold, then there exists an open set* Σ *such that for any* $\sigma \in \Sigma$, (NKG) *has a hylomorphic soliton* (*see Definition* 1) *of charge* σ *and having the form* (46).

Theorem 13, in the form given here, is a very recent result [6]. In fact in [6] it has been proved the orbital stability of (46) with respect to the standard topology of $\mathscr{X} = H^1(\mathbb{R}^N, \mathbb{C}) \times L^2(\mathbb{R}^N, \mathbb{C})$ and for all the $W's$ which satisfy (W-i), (W-ii), (W-iii), (W-iiii). Nevertheless Theorem 13 has a very long history starting with the pioneering paper of Rosen [30]. Coleman [19] and Strauss [34] gave the first rigorous proofs of existence of solutions of the type (46) for (NKG) and for some particular $W's$. Later very general existence conditions have been found by Berestycki and Lions [17]. In particular, if W satisfies (W-i), (W-ii), (W-iii), (W-iiii), from their paper we can deduce (see [12]) the existence of Q-balls of type (46) for any $\omega \in (\omega_0, m)$ where

$$\omega_0 := \inf\left\{\lambda > 0 : W(u) < \frac{1}{2}\lambda^2 u^2 \text{ for some } u > 0\right\}.$$

Notice that the hylomorphy condition (W-iii) guarantees that $\omega_0 < m$, and hence that $(\omega_0, m) \neq \emptyset$.

The first orbital stability results are due to Shatah: in [33] a condition for orbital stability is given; however this condition is difficult to be verified in concrete situations. More recently [6] a sufficient and (essentially) necessary condition for the orbital stability has been proved. This condition is given directly on W and it permits to deduce immediately Theorem 13.

Here we study the equation (47) with $0 < \omega^2 < m^2$ by using theorem 4 and prove a weaker version of Theorem 13, namely we do not prove the orbital stability but we confine ourselves to show the existence of hylomorphic waves (see Definition 1) for (NKG).

In this case we set:

$$X = H_r^1 = \{u \in H^1(\mathbb{R}^N) : u \text{ is radially symmetric}\},$$

$$\langle L_1 u, u\rangle = \int (|\nabla u|^2 + m^2 u^2)\, dx; \quad N_1(u) = \int N(u)\, dx, \tag{51}$$

$$\begin{aligned} J(u) &= \frac{1}{2}\langle L_1 u, u\rangle + N_1(u) \\ &= \frac{1}{2}\int (|\nabla u|^2 + m^2 u^2)\, dx + \int N(u)\, dx, \end{aligned} \tag{52}$$

$$\langle L_0 u, u\rangle = K(u) = \frac{1}{2}\int u^2\, dx; \quad N_0(u) = 0. \tag{53}$$

First of all we observe that by (W-iiii)(a) the functional J is C^1. Whereas, if assumption (W-iiii)(b) holds, our problem can be transformed in an equivalent one for which the functional J is C^1 (see Appendix). Now in order to use Theorem 4, we need to prove that assumptions (H1,2,3) and (11) are satisfied.

LEMMA 14. *The functionals J, N_i $(i = 0, 1)$ and K defined in* (51), (52) *and* (53) *satisfy the assumptions* (H1,2,3).

PROOF. Clearly (H3) holds. Let us now prove that (H1) holds. Let u_n be a sequence in X such that $J(u_n)$ is bounded. Then, since $W \geq 0$, we have that

$$\int W(u_n) \text{ and } \int |\nabla u_n|^2 \text{ are bounded.} \tag{54}$$

So in order to show that u_n is bounded in X we need to prove that

$$\|u_n\|_{L^2} \text{ is bounded.} \tag{55}$$

Let

$$2^* = \frac{2N}{N-2}$$

denote, as usual, the critical Sobolev exponent.

By (54) we have that

$$\int |u_n|^{2^*} \text{ is bounded.} \tag{56}$$

Let $\varepsilon > 0$ and set

$$\Omega_n = \{x \in \mathbb{R}^N : |u_n(x)| > \varepsilon\} \quad \text{and} \quad \Omega_n^c = \mathbb{R}^N \backslash \Omega_n.$$

By (54) and since $W \geq 0$, we have

$$\int_{\Omega_n^c} W(u_n) \text{ is bounded.} \tag{57}$$

By (W-ii) we can write

$$W(s) = \frac{m}{2} s^2 + \circ(s^2).$$

Then, if ε is small enough, there is a constant $c > 0$ such that

$$\int_{\Omega_n^c} W(u_n) \geq c \int_{\Omega_n^c} u_n^2. \tag{58}$$

By (57) and (58) we get that

$$\int_{\Omega_n^c} u_n^2 \text{ is bounded.} \tag{59}$$

On the other hand

$$\int_{\Omega_n} u_n^2 \leq \Big(\int_{\Omega_n} |u_n|^{2^*} \Big)^{(N-2)/N} meas(\Omega_n)^{2/N}. \tag{60}$$

By (56) we have that

$$meas(\Omega_n) \text{ is bounded.} \tag{61}$$

By (60), (61), (56) we get that

$$\int_{\Omega_n} u_n^2 \text{ is bounded.} \tag{62}$$

So (55) follows from (59) and (62).

Let us finally prove that (H2) is satisfied.

Let $\{u_n\} \subset H_r^1$

$$u_n \rightharpoonup u \quad \text{weakly in } H_r^1.$$

Now we distinguish two cases:

Assume first that (W-iiii)(a) holds.

Since H_r^1 is compactly embedded into $L^p(\mathbb{R}^N)$, $2 < p < 2^*$, (see [17]), we have that

$$\int |u_n - u|^p \, dx \to 0. \tag{63}$$

Now

$$\left|\int (N'(u_n) - N'(u))(u_n - u)\,dx\right| \tag{64}$$
$$\leq \Big(\int |N'(u_n) - N'(u)|^{p'}\,dx\Big)^{1/p'} \Big(\int |u_n - u|^p\,dx\Big)^{1/p}, \quad p' = \frac{p}{p-1}$$

The sequence u_n is bounded in $L^p(\mathbb{R}^N)$ and in $L^2(\mathbb{R}^N)$. So, by using (W-iiii)a, we deduce that $N'(u_n)$ is bounded in $L^{p'}(\mathbb{R}^N)$. Then, by (63) and (64), we deduce that N' satisfies (10).

Finally we assume that (W-iiii)(b) holds.

Clearly

$$u_n \to u \quad \text{strongly in } L^p(B_R) \tag{65}$$

where $R > 0$ and

$$B_R = \{x \in \mathbb{R}^N : |x| < R\}.$$

Since we can assume $N'(s)$ linear for large s (see Appendix), we have

$$N'(u_n) \to N'(u) \quad \text{in } L^2(B_R). \tag{66}$$

Now

$$\int |N'(u_n) - N'(u)|^2\,dx = \int_{B_R} |N'(u_n) - N'(u)|^2\,dx + \int_{B_R^c} |N'(u_n) - N'(u)|^2\,dx \tag{67}$$

and

$$\int_{B_R^c} |N'(u_n) - N'(u)|^2\,dx = \int_{B_R^c} |N''(\xi_n)|^2 |u_n - u|^2\,dx \tag{68}$$

where

$$B_R^c = \mathbb{R}^N - B_R$$
$$\xi_n(x) = tu_n(x) + (1-t)u(x), \quad 0 \leq t \leq 1.$$

In the following c_1, c_2, c_3 will denote positive constants. By a well known radial lemma [17] and since $\|u_n\|_X$ is bounded, we have that for $|x|$ large

$$|\xi_n(x)| \leq |u(x)| + |u_n(x)| \leq c_1 \frac{\|u\|_X + \|u_n\|_X}{|x|^{(N-1)/2}} \leq \frac{c_2}{|x|^{(N-1)/2}}. \tag{69}$$

Let $\varepsilon > 0$, since N'' is continuous in 0 and $N''(0) = 0$, we have, by using (69), that

$$|N''(\xi_n(x))|^2 < \varepsilon \quad \text{for } |x| > R,\ R \text{ large.} \tag{70}$$

So, by (68) and (70) and since $\|u_n\|_{L^2}$ is bounded, we get

$$\int_{B_R^c} |N'(u_n) - N'(u)|^2\, dx < \varepsilon \int_{B_R^c} |u_n - u|^2\, dx \le \varepsilon c_3. \tag{71}$$

Then by (67), (71) we have

$$\int |N'(u_n) - N'(u)|^2\, dx \le \varepsilon c_3 + \int_{B_R} |N'(u_n) - N'(u)|^2\, dx. \tag{72}$$

So by (66) and (72) we get

$$N'(u_n) \to N'(u) \quad \text{strongly in } L^2(\mathbb{R}^N).$$

Then N satisfies (10). □

LEMMA 15. *Assumption* (11) *is satisfied.*

PROOF. Let $R > 0$ and consider the map u_R defined as follows

$$u_R(x) = \begin{cases} s_0 & \text{if } |x| < R \\ 0 & \text{if } |x| > R + 1 \\ s_0(1 + R - |x|) & \text{if } R \le |x| \le R + 1 \end{cases} \tag{73}$$

where s_0 is a such that $N(s_0) < 0$.

Clearly

$$\frac{J(u_R)}{K(u_R)} = \frac{\int |\nabla u_R|^2}{\frac{1}{2}\int u_R^2} + m^2 + \frac{\int N(u_R)}{\frac{1}{2}\int u_R^2}.$$

Easy estimates show that for R large

$$\int |\nabla u_R|^2 \le c_0 R^{N-1} \tag{74}$$

$$c_2 R^N \le \frac{1}{2}\int u_R^2\, dx \le c_1 R^N \tag{75}$$

$$\int N(u_R)\, dr \le N(s_0) R^N + c_3 R^{N-1} \tag{76}$$

where $c_0, \ldots, c_3$ are positive constants.

Then for R large, since $N(s_0) < 0$, we have

$$\frac{J(u_R)}{K(u_R)} \leq \frac{c_0}{c_2}\frac{1}{R} + m^2 + \frac{N(s_0)R^N}{c_1R^N} + \frac{c_3R^{N-1}}{c_2R^N} < m^2. \qquad \square$$

Assumption (W-i) is a necessary condition for the existence of hylomorphic waves (Definition 1), in fact the following proposition holds:

PROPOSITION 16. *If* (W-i) *does not hold, then for any* $\sigma > 0$, $E(u)$ *is not bounded from below on* M_σ.

PROOF. Let $\sigma > 0$ and assume that there exists s_0 such that $W(s_0) < 0$. We set $\Psi_R = (u_R, -i\omega_R u_R)$ where u_R is defined in (73) and

$$\omega_R = \frac{\sigma}{\int u_R^2\,dx}.$$

Clearly

$$\omega_R = \frac{\sigma}{\int u_R^2\,dx} \leq c_4R^{-N}. \tag{77}$$

Then by (74), (75), (76) (where W replaces N) we have

$$\begin{aligned} E(\Psi_R) &= \int \left[\frac{1}{2}|\nabla u_R|^2 + W(u_R)\right] dx + \frac{1}{2}\omega_R^2 \int u_R^2\,dx \\ &= \int \left[\frac{1}{2}|\nabla u_R|^2 + W(u_R)\right] dx + \frac{1}{2}\omega_R\sigma \\ &\leq \frac{1}{2}c_0R^{N-1} + W(s_0)R^N + c_3R^{N-1} + c_5R^{-N}. \end{aligned}$$

Hence

$$\lim_{R\to\infty} E(\Psi_R) = -\infty \qquad \square$$

REMARK 17. *If* (W-i) *is violated, it is still possible to have orbitally stable solitary waves* (*see* [33]) *which are only local minimizers. They can be destroyed by a perturbation which send them out of the basin of attraction and are not considered solitons according to Def.* 1.

REMARK 18. *We observe that the constant m defined by* (W-ii) *coincides with the constant m defined by* (3) *and the constant m defined by* (12).

5. Vortices

5.1 Main Features

A (*hylomorphic*) *vortex* is a (hylomorphic) solitary wave with nonvanishing *angular momentum*. The angular momentum, by definition, is the quantity which is preserved by virtue of the invariance under space rotations (with respect to the origin) of the Lagrangian (see e.g. [25]). In this section we shall analyze elementary properties of the angular momentum for (NKG) in three space dimensions; of course, making obvious changes, the analysis includes also the two dimensional case.

The angular momentum for the solutions of (NKG) is given by

$$\mathbf{M}(\Psi) = \mathrm{Re} \int \mathbf{x} \times \nabla\psi(\overline{\partial_t \psi})\, dx. \tag{78}$$

Using the polar form (36), it can be written

$$\mathbf{M}(\Psi) = \int (\mathbf{x} \times \nabla S(\partial_t S u^2) + \mathbf{x} \times \nabla u(\partial_t u))\, dx. \tag{79}$$

where $\times$ denotes the wedge product.

It is immediate to check that standing waves (46) have $\mathbf{M}(\Psi) = \mathbf{0}$. However, if we consider:

$$\psi(t, x) = \psi_0(x) e^{-i\omega t}, \quad \omega > 0 \tag{80}$$

where $\psi_0(x)$ is allowed to have complex values, it is possible to have $\mathbf{M}(\Psi) \neq \mathbf{0}$. Thus, we are led to make an ansaz of the following form:

$$\psi(t, x) = u(x) e^{i(\ell\theta(x) - \omega t)}, \quad u(x) \geq 0,\ \omega \in \mathbb{R},\ \ell \in \mathbb{Z} - \{0\} \tag{81}$$

and

$$\theta(x) = \mathrm{Im} \log(x_1 + i x_2) \in \mathbb{R}/2\pi\mathbb{Z}; \quad x = (x_1, x_2, x_3).$$

Moreover, we assume that

$$u(x) = u(r, x_3), \quad \text{where } r = \sqrt{x_1^2 + x_2^2}. \tag{82}$$

By this ansaz, equation (NKG) (in the form (40), (41)) is equivalent to the system

$$\begin{cases} -\triangle u + \ell^2 |\nabla\theta|^2 u + W'(u) = \omega^2 u \\ u\triangle\theta + 2\nabla u \cdot \nabla\theta = 0. \end{cases}$$

By the definition of θ and (82) we have

$$\triangle\theta = 0, \quad \nabla\theta \cdot \nabla u = 0, \quad |\nabla\theta|^2 = \frac{1}{r^2}.$$

where the dot $\cdot$ denotes the euclidean scalar product.

So the above system reduces to

$$-\triangle u + \frac{\ell^2}{r^2}u + W'(u) = \omega^2 u \quad \text{in } \mathbb{R}^3. \tag{83}$$

Direct computations show that the energy (42), the angular momentum (79) and the hylenic charge (43) become

$$E(u(x)e^{i(\ell\theta(x)-\omega t)}) = \int_{\mathbb{R}^3}\left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\Big(\frac{\ell^2}{r^2} + \omega^2\Big)u^2 + W(u)\right]dx \tag{84}$$

$$\mathbf{M}(u(x)e^{i(\ell\theta(x)-\omega t)}) = -\Big(0, 0, \omega\ell\int_{\mathbb{R}^3} u^2\,dx\Big). \tag{85}$$

$$H(u(x)e^{i(\ell\theta(x)-\omega t)}) = \int \omega u^2\,dx. \tag{86}$$

The existence of vortices is an interesting and old issue in many questions of mathematical physics as superconductivity, classical and quantum field theory, string and elementary particle theory (see the pioneering papers [1], [28] and e.g. the more recent ones [26], [35], [36], [38], [21] with their references).

From mathematical viewpoint, the existence of vortices for (NKG) and for (NKGM) has been studied in some recent papers ([16], [4], [5], [13], [14], [15], [8], [9]).

5.2 Existence of Two Dimensional Vortices

In this paper we want to apply theorem 4 to the study of vortices; this is possible for $N = 2$. We get the following theorem:

THEOREM 19. *Let* $W : \mathbb{C} \to \mathbb{R}$ *satisfy* (W-i), (W-ii), (W-iii), (W-iiii) *and fix* $\ell \in \mathbb{Z} - \{0\}$*; then there exists an open set* Σ *such that for any* $\sigma \in \Sigma$*, equation* NKG *has a hylomorphic vortex of the form* (81).

In this case we set:

$$\langle L_1 u, u\rangle = \int\left[|\nabla u|^2 + \Big(\frac{\ell^2}{r^2} + m^2\Big)u^2\right]dx; \quad N_1(u) = \int N(u)\,dx$$

$$X = \{u \in H^1(\mathbb{R}^2)\text{: } u \text{ is radially symmetric and } \langle L_1 u, u\rangle < \infty\}$$

$$J(u) = \frac{1}{2}\langle L_1 u, u\rangle + N_1(u)$$
$$= \frac{1}{2}\int \left[|\nabla u|^2 + \left(\frac{\ell^2}{r^2} + m^2\right) u^2 \right] dx + \int N(u)\, dx$$
$$\langle L_0 u, u\rangle = K(u) = \frac{1}{2}\int u^2\, dx; \quad N_0(u) = 0.$$

LEMMA 20. *Assumptions* (H1), (H2), (H3) *are satisfied.*

PROOF. Clearly assumption (H3) is satisfied. Let us prove that assumption (H1) is satisfied.

Let u_n be a sequence in X such that $J(u_n)$ is bounded. Then clearly also the sequences

$$\int |\nabla u_n|^2, \quad \int \frac{u_n^2}{r^2}, \quad \int W(u_n) \tag{87}$$

are bounded. We have to show that u_n is bounded in L^2. Let us first show that there exists M_1 such that for all n

$$\|u_n\|_{L^\infty} \le M_1. \tag{88}$$

In fact for $u \in C_0^\infty(\mathbb{R}^2\backslash 0)$, u radially symmetric, we set $u(x) = v(r)$ $r = |x|$, then

$$\frac{1}{2}u^2(x) = \frac{1}{2}v(r)^2 = \int_{+\infty}^{r} v(r)v'(r)\, dr \le \left(\int_0^{+\infty} \frac{v(r)^2}{r} dr \int_0^{+\infty} v'(r)^2 r\, dr \right)^{1/2} \tag{89}$$
$$\le c_1 \left(\int_{\mathbb{R}^2} \frac{u^2}{r^2} dx \int_{\mathbb{R}^2} |\nabla u|^2\, dx \right)^{1/2}$$

Then, since the sequences (87) are bounded, by (89) we get (88).

Let $\varepsilon > 0$ and set

$$\Omega_n = \{x \in \mathbb{R}^2 : |u_n(x)| > \varepsilon\} \quad \text{and} \quad \Omega_n^c = \mathbb{R}^2\backslash\Omega_n.$$

Then, by (88), we have

$$\int_{\Omega_n} u_n^2 \le \left(\int_{\Omega_n} u_n^6 \right)^{1/3} (meas(\Omega_n))^{2/3} \tag{90}$$
$$\le \|u_n\|_{L^\infty}^2\, meas(\Omega_n) \le M_1^2\, meas(\Omega_n).$$

On the other hand, if ε is small enough we have (see (58) in the proof of Lemma 14)

$$\int_{\Omega_n^c} W(u_n) \ge c_2 \int_{\Omega_n^c} u_n^2. \tag{91}$$

Since

$$\int W(u_n) \leq M_2,$$

by (90) and (91) we deduce

$$\int u_n^2 = \int_{\Omega_n} u_n^2 + \int_{\Omega_n^c} u_n^2 \leq M_1^2 \, meas(\Omega_n) + \frac{M_2}{c_2}. \tag{92}$$

Then it remains to prove that

$$meas(\Omega_n) \text{ is bounded.} \tag{93}$$

Arguing by contradiction assume that, up to a subsequence

$$meas(\Omega_n) \to \infty. \tag{94}$$

By a Trudingher-Moser type inequality (see [32] and its references) on all $\mathbb{R}^2$, we have for $\alpha < 4\pi$

$$\int e^{\alpha u_n^2} \leq c_3 \int |\nabla u_n|^2. \tag{95}$$

Then, taking $\alpha = 1$ and since $\int |\nabla u_n|^2$ is bounded, we have

$$e^{\varepsilon^2} \, meas(\Omega_n) \leq \int_{\Omega_n} e^{u_n^2} \leq \int e^{u_n^2} \leq c_3 \int |\nabla u_n|^2 \leq M_3$$

which contradicts (94).

Finally, following the same arguments used in the proof of Lemma 14, it can be proved that also assumption (H2) is satisfied. □

LEMMA 21. *Assumption* (11) *is satisfied.*

PROOF. Let $R > 1$ and consider the map u_R defined as follows

$$u_R(x) = \begin{cases} 0 & \text{if } |x| \leq R-1 \text{ or } |x| \geq 2R+1 \\ s_0(|x| - R + 1) & \text{if } R \geq |x| > R-1 \\ s_0 & 2R \geq |x| > R \\ s_0(1 + 2R - |x|) & \text{if } 2R+1 \geq |x| > 2R \end{cases}$$

where s_0 is a such that $N(s_0) < 0$.

Clearly

$$\frac{J(u_R)}{K(u_R)} = \frac{\int |\nabla u_R|^2}{\int u_R^2} + m^2 + \frac{\int \frac{\ell^2 u_R^2}{r^2}}{\int u_R^2} + \frac{\int N(u_R)}{\frac{1}{2}\int u_R^2}. \tag{96}$$

Easy estimates show that for R large

$$\int |\nabla u_R|^2 \le c_0 R$$

$$\int \frac{\ell^2 u_R^2}{r^2} \le \frac{c_1}{R} + c_2$$

$$\int N(u_R)\, dr \le c_3 N(s_0) R^2 + c_4 R$$

$$c_6 R^2 \ge \int u_R^2\, dx \ge c_5 R^2$$

where $c_0, \dots, c_6$ are positive constants.

Then for R large, since $N(s_0) < 0$, we have

$$\frac{J(u_R)}{K(u_R)} < m^2. \qquad \square$$

6. The Nonlinear Klein-Gordon-Maxwell equations

6.1 General Features of NKGM

The Nonlinear Klein-Gordon-Maxwell equations (NKGM) are (see e.g. [12], [11])

$$\text{(NKGM-1)} \qquad (\partial_t + iq\varphi)^2\psi - (\nabla - iq\mathbf{A})^2\psi + W'(\psi) = 0$$

$$\text{(NKGM-2)} \qquad \nabla \cdot (\partial_t \mathbf{A} + \nabla\varphi) = q\,\mathrm{Im}(\partial_t \psi \bar{\psi}) + q^2\varphi|\psi|^2$$

$$\text{(NKGM-3)} \quad \nabla \times (\nabla \times \mathbf{A}) + \partial_t(\partial_t \mathbf{A} + \nabla\varphi) = q\,\mathrm{Im}(\nabla\psi\bar{\psi}) - q^2\mathbf{A}|\psi|^2$$

where q is a parameter which, in some models, is interpreted as the electron charge and W satisfies (33). They are the Euler-Lagrange equations of the action:

$$\mathscr{S} = \int \mathscr{L}\, dx\, dt, \quad \mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 - W(\psi), \tag{97}$$

where

$$\mathscr{L}_0 = \frac{1}{2}[|(\partial_t + iq\varphi)\psi|^2 - |(\nabla - iq\mathbf{A})\psi|^2] \tag{98}$$

$$\mathscr{L}_1 = \frac{1}{2}\left[|\partial_t \mathbf{A} + \nabla\varphi|^2 - \frac{1}{2}|\nabla \times \mathbf{A}|^2\right]. \tag{99}$$

In this case, the state of the system is given by

$$\Psi = (\psi, \psi_t, \varphi, \varphi_t, \mathbf{A}, \mathbf{A}_t).$$

If we use the notation (36, 37, 38) and if we set

$$\mathbf{E} = -(\partial_t \mathbf{A} + \nabla\varphi) \tag{100}$$

$$\mathbf{H} = \nabla \times \mathbf{A} \tag{101}$$

$$\Omega = -(\partial_t S + q\varphi) = \omega - q\varphi \tag{102}$$

$$\rho = q\Omega u^2 \tag{103}$$

$$\mathbf{K} = \nabla S - q\mathbf{A} = \mathbf{k} - q\mathbf{A} \tag{104}$$

$$\mathbf{J} = q\mathbf{K}u^2. \tag{105}$$

Equations (NKGM-1), (NKGM-2), (NKGM-3) can be written as follows (see e.g. [12]):

$$\Box u + (\mathbf{K}^2 + \Omega^2)u + W'(u) = 0 \quad (\text{MATTER})$$

$$\nabla \cdot \mathbf{E} = \rho \quad (\text{GAUSS})$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \quad (\text{AMPERE})$$

Moreover, by the positions (100) and (101), $\mathbf{E}$ and $\mathbf{H}$ satisfy also the equations

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0 \quad (\text{FARADAY})$$

$$\nabla \cdot \mathbf{H} = 0. \quad (\text{NOMONOPOLE})$$

The equations (GAUSS), (AMPERE), (FARADAY), (NOMONOPOLE) are the Maxwell's equations and equation (MATTER) represents a model of interaction of matter with the elecromagnetic field (see for example [12], [24] ch. 3, [31] ch. 2 in Part 1, and [39] ch. 1).

The energy takes the following form (see [12]):

$$E(\Psi) = \int \left[\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(\mathbf{K}^2 + \Omega^2)u^2 + W(u) + \frac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2)\right] dx$$

and the hylenic charge takes the form:

$$H(\Psi) = \int \Omega u^2 \, dx = \int (\omega - q\varphi)u^2.$$

In some models, $H(\Psi)$, if positive, represents the number of particles contained in the state Ψ, otherwise, $-H(\Psi)$ represents the number of antiparticles. The global electric charge is given by

$$Q(\Psi) = qH(\Psi) = \int (q\omega - q^2\varphi)u^2.$$

Thus, if ψ is rescaled in such a way to have $q = 1$, the hylenic charge $H(\Psi)$ and the electric charge $Q(\Psi)$ coincide.

6.2 Existence Results for the NKGM

In this paper we are interested to apply Theorem 4 to find electrostatic standing waves, namely solutions of (NKGM-1), (NKGM-2), (NKGM-3), having the form

$$\psi(t,x) = u(x)e^{-i\omega t}, \quad u \in \mathbb{R}^+, \quad \omega \in \mathbb{R}, \quad s \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \tag{106}$$

$$\mathbf{A} = 0, \quad \partial_t \varphi = 0. \tag{107}$$

The existence of solitary waves for (NKGM) depends on the constant q; more exactly we have the following theorem:

THEOREM 22. *Assume that* (W-i), (W-ii), (W-iii), (W-iiii) *hold. Then there exists a set* $\Sigma_{NKGM} \subset \mathbb{R}^2$ *such that for any* $(\sigma, q) \in \Sigma_{NKGM}$, *the nonlinear Klein-Gordon-Maxwell equations* (*NKGM*) *have an hylomorphic, electrostatic* (*see* (106), (107)) *wave of charge* σ. *Moreover* Σ_{NKGM} *has the following form*

$$\Sigma_{NKGM} = \{(\sigma, q) \in \mathbb{R}^2 : \sigma \in \Sigma_q, 0 < q < q^*\}$$

where $q^* > 0$ *and* Σ_q *is an open set which is not empty for* $0 < q < q^*$.

REMARK 23. *The existence of electrostatic standing waves has been first analyzed when* $W(s) = s^2 - s^p$ $(s > 0, p > 2)$ ([10], [18], [22], [23]). *More recently also cases in which* $W \geq 0$ *have been considered* ([11], [15]). *However, the proof in* [11] *contains a gap, even if the result is correct. In fact, the main result can also be deduced by th.* 22.

If (106) and (107) hold, equation (NKGM-3) is identically satisfied, while (NKGM-1) and (NKGM-2) become

$$-\Delta u + W'(u) = (\omega - q\varphi)^2 u \tag{108}$$

$$-\Delta \varphi = q(\omega - q\varphi)u^2. \tag{109}$$

We set

$$\begin{aligned} \mathcal{X}_0 = \{&\Psi = (u(x), -i\omega u(x), \varphi(x), 0, \mathbf{0}, \mathbf{0}), \\ &u \in H^1(\mathbb{R}^N), \varphi \in \mathscr{D}^{1,2}(\mathbb{R}^3), \omega \in \mathbb{R}\}. \end{aligned} \tag{110}$$

Clearly $\mathcal{X}_0$ is a subset of the phase space which contains the electro-static standing waves. To any state $\Psi \in \mathcal{X}_0$, we can associate a triple

$$(u, \varphi, \omega) \in H^1(\mathbb{R}^3) \times \mathscr{D}^{1,2}(\mathbb{R}^3) \times \mathbb{R};$$

the corresponding energy and charge take the following form:

$$E_q(u,\varphi,\omega) = \int \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\Omega^2 u^2 + W(u)\right] dx \tag{111}$$

$$H_q(u,\varphi,\omega) = \int \Omega u^2\, dx$$

where, according to (102),

$$\Omega = \omega - q\varphi.$$

Now we would like to apply theorem 4. Unfortunately, we cannot do it directly, since E_q and H_q do not satisfy the required properties, namely they do not have the form (7) and (8). However, we can transform this problem in such a way that Theorem 4 can be used. To do this, we introduce a smaller space $\mathscr{Z}_0 \subset \mathscr{X}_0$ which contains the states which satisfy equation (109), namely

$$\mathscr{Z}_0 = \{\Psi \in \mathscr{X}_0 : -\Delta\varphi = q(\omega - q\varphi)u^2\}. \tag{112}$$

We remark that for $u \in H^1(\mathbb{R}^3)$ and $\omega \in \mathbb{R}$ given, equation (109) has a unique solution $\varphi_u \in \mathscr{D}^{1,2}(\mathbb{R}^3)$ (see [10]); then

$$\mathscr{Z}_0 \cong H^1(\mathbb{R}^3) \times \mathbb{R}.$$

Now we want to find a nice and useful way to write E_q, H_q and Λ_q restricted to $\mathscr{Z}_0$. First, we divide the energy in two parts:

$$E_q(u,\varphi,\omega) = J(u) + F_q(u,\varphi,\omega) \tag{113}$$

where

$$J(u) = \int \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] dx \tag{114}$$

$$F_q(u,\varphi,\omega) = \frac{1}{2}\int [|\nabla\varphi|^2 + \Omega^2 u^2]\, dx \tag{115}$$

Now let $u \in H^1(\mathbb{R}^3)$ and consider the solution φ_u of (109).

Multiplying both sides of equation (109) by φ_u and integrating, we get

$$\int |\nabla\varphi_u|^2\, dx = \int q\varphi_u \Omega u^2.$$

Then

$$F_q(u,\varphi_u,\omega) = \frac{1}{2}\int [q\varphi_u\Omega u^2 + \Omega^2 u^2]\, dx$$

$$= \frac{1}{2}\omega^2 \int \left(1 - q\frac{\varphi_u}{\omega}\right) u^2\, dx.$$

So we have

$$F_q(u, \varphi_u, \omega) = \frac{1}{2}\omega^2 \int \Big(1 - q\frac{\varphi_u}{\omega}\Big)u^2\, dx. \tag{116}$$

For $u \in H^1(\mathbb{R}^3)$, let $\Phi = \Phi_u$ be the solution of the equation

$$-\Delta\Phi_u + q^2u^2\Phi_u = qu^2. \tag{117}$$

Clearly

$$\varphi_u = \omega\Phi_u \tag{118}$$

solves eq. (109) and we have that

$$F_q(u, \varphi_u, \omega) = F_q(u, \omega\Phi_u, \omega) = \frac{1}{2}\omega^2 \int (1 - q\Phi_u)u^2\, dx = \omega^2 K_q(u), \tag{119}$$

where

$$K_q(u) := \frac{1}{2}\int (1 - q\Phi_u)u^2\, dx. \tag{120}$$

By (113) and (119) the energy on the states contained in $\mathscr{Z}_0$ (see (112)) can be written as a functional of the two variables ω and u and having the form (7):

$$\tilde{E}_q(u, \omega) = E_q(u, \varphi_u, \omega) = J(u) + \omega^2 K_q(u). \tag{121}$$

Analogously, also the hylenic charge can be expressed via the variables u and ω and having the form (8):

$$\begin{aligned}\tilde{H}_q(u, \omega) &= H_q(u, \varphi_u, \omega) = H_q(u, \omega\Phi_u, \omega)\\ &= \omega \int (1 - q\Phi_u)u^2\, dx\\ &= 2\omega K_q(u).\end{aligned}$$

Notice that, for $q = 0$, all these functionals reduce to the analogous ones for the equation (NKG).

By the following proposition the study of the equations (108) and (109) is reduced to an eigevalue problem of the type (9).

PROPOSITION 24. *Let $q > 0$ and $(u, \omega) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ be a solution of the eigenvalue problem*

$$J'(u) = \omega^2 K_q'(u). \tag{122}$$

Then u, φ_u, ω solve (108) *and* (109).

PROOF. First observe that u, φ, ω solve (108), (109) if and only if (u, φ) is a critical point of the functional

$$I_\omega(u, \varphi) = J(u) - F_q(u, \varphi, \omega) \tag{123}$$

namely if

$$\frac{\partial I_\omega(u, \varphi)}{\partial u} = 0, \quad \frac{\partial I_\omega(u, \varphi)}{\partial \varphi} = 0. \tag{124}$$

Now let (u, ω) be a solution of the eigenvalue problem (122). Then clearly u is a critical point of the functional $u \to J(u) - \omega^2 K_q(u)$ or equivalently, by (119) and (123), a critical point of the functional

$$u \to I_\omega(u, \varphi_u) = J(u) - F_q(u, \varphi_u, \omega). \tag{125}$$

This means that

$$\frac{\partial I_\omega(u, \varphi_u)}{\partial u} + \frac{\partial I_\omega(u, \varphi_u)}{\partial \varphi} \varphi_u' = 0. \tag{126}$$

Since φ_u solves (109), we have

$$\frac{\partial I_\omega(u, \varphi_u)}{\partial \varphi} = 0. \tag{127}$$

Then from (126) and (127) we get

$$\frac{\partial I_\omega(u, \varphi_u)}{\partial u} = 0, \quad \frac{\partial I_\omega(u, \varphi_u)}{\partial \varphi} = 0. \tag{128}$$

So by (128) we have that u, φ_u solve (124). □

We shall show that if q is small enough the eigenvalue problem (122) satisfies all the assumptions of the abstract theorem 4. More precisely in this case we shall set

$$X = \{u \in H^1(\mathbb{R}^3) : u \text{ is radially symmetric}\},$$

$$\langle L_1 u, u\rangle = \int (|\nabla u|^2 + m^2 u^2)\, dx; \quad N_1(u) = \int N(u)\, dx,$$

$$J(u) = \frac{1}{2}\langle L_1 u, u\rangle + N_1(u)$$

$$= \frac{1}{2}\int (|\nabla u|^2 + m^2 u^2)\, dx + \int N(u)\, dx,$$

$$\langle L_0 u, u\rangle = \int u^2\, dx,$$

$$K_q(u) = \frac{1}{2}\langle L_0 u, u\rangle + N_0(u), \quad N_0(u) = -\frac{q}{2}\int \Phi_u u^2\, dx.$$

LEMMA 25. *Assumptions* (H1), (H2), (H3) *are satisfied.*

PROOF. Arguing as in the proof of Lemma 14 it can be proved that assumption (H1) is satisfied and that N_1' satisfies (10).

Then, in order to complete the proof of (H2), we need to show that N_0' is compact. First of all we look for a suitable expression for N_0'.

Observe that

$$K_q'(u) = u + N_0'(u). \tag{129}$$

On the other hand by (116) and (120)

$$K_q(u) = F_q(u, \Phi_u, 1).$$

Then

$$K_q'(u) = \frac{\partial F_q(u, \Phi_u, 1)}{\partial u} + \frac{\partial F_q(u, \Phi_u, 1)}{\partial \varphi}\Phi_u'. \tag{130}$$

Since Φ_u solves (117) and taking into account the definition (115) of F_q, we have

$$\frac{\partial F_q(u, \Phi_u, 1)}{\partial \varphi} = 0, \quad \frac{\partial F_q(u, \Phi_u, 1)}{\partial u} = (1 - q\Phi_u)^2 u. \tag{131}$$

So, comparing (130), (131), we have

$$K_q'(u) = (1 - q\Phi_u)^2 u. \tag{132}$$

By (129), (132) we get the following expression for $N_0'(u)$

$$N_0'(u) = (1 - q\Phi_u)^2 u - u = q^2\Phi_u^2 u - 2q\Phi_u u.$$

Then in order to show that N_0' is compact it is enough to prove that the maps

$$u \to \Phi_u u \quad \text{and} \quad u \to \Phi_u^2 u \tag{133}$$

are compact from X to X'.

Let

$$u_n \rightharpoonup u_0 \quad \text{weakly in } X.$$

We shall prove first that Φ_{u_n} is bounded in $\mathscr{D}^{1,2}(\mathbb{R}^3)$ and that, up to a subsequence,

$$\Phi_{u_n} \rightharpoonup \Phi_{u_0} \quad \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^3). \tag{134}$$

Since Φ_{u_n} solves

$$-\Delta\Phi_{u_n} + q^2 u_n^2 \Phi_{u_n} = q u_n^2, \tag{135}$$

we have

$$\int |\nabla \Phi_{u_n}|^2 + q^2 \int \Phi_{u_n}^2 u_n^2 = q \int \Phi_{u_n} u_n^2. \tag{136}$$

On the other hand

$$\int \Phi_{u_n} u_n^2 \le \|\Phi_{u_n}\|_{L^6} \|u_n\|_{L^{12/5}}^2. \tag{137}$$

Since u_n is bounded in X, it is also bounded in $L^{12/5}$, then by (137) we have

$$\int \Phi_{u_n} u_n^2 \le c_1 \|\Phi_{u_n}\|_{L^6}. \tag{138}$$

From (136), (138) we easily get

$$\|\Phi_{u_n}\|_{\mathscr{D}^{1,2}}^2 \le c_2 \|\Phi_{u_n}\|_{\mathscr{D}^{1,2}},$$

from which we have that, up to a subsequence,

$$\Phi_{u_n} \rightharpoonup \Phi_0 \quad \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^3).$$

In order to prove (134) we have to show that $\Phi_0 = \Phi_{u_0}$ i.e. we show that Φ_0 solves (117) with $u = u_0$.

Let $v \in C_0^\infty$ then, testing (135) on v and passing to the limit, we easily get

$$-\Delta \Phi_0 + q^2 u_0^2 \Phi_0 = q u_0^2.$$

Then (134) is proved.

Now we prove that

$$u_n \Phi_{u_n} \to u_0 \Phi_{u_0} \quad \text{in } L^2. \tag{139}$$

Let $\varepsilon, R > 0$ and set

$$B_R = \{x \in \mathbb{R}^3 : |x| < R\}, \quad B_R^c = \mathbb{R}^3 - B_R.$$

Clearly we have

$$\int_{B_R^c} \Phi_{u_n}^2 u_n^2 \le \Big(\int_{B_R^c} |u_n|^3 \Big)^{2/3} \Big(\int_{B_R^c} \Phi_{u_n}^6 \Big)^{1/3}. \tag{140}$$

Now we have (see [17])

$$|u_n(x)| \le c_1 \frac{\|u_n\|_{H^1}}{|x|} \quad \text{in } B_R^c. \tag{141}$$

From (140) and (141) we get

$$\int_{B_R^c} \Phi_{u_n}^2 u_n^2 \le \Big(c_1 \frac{\|u_n\|_{H^1}}{R}\Big)^{2/3} \Big(\int_{B_R^c} |u_n|^2\Big)^{2/3} \|\Phi_{u_n}\|_{L^6}^2. \tag{142}$$

So, since u_n is bounded in H^1 and Φ_{u_n} is bounded in $\mathscr{D}^{1,2}(\mathbb{R}^3)$ and hence in L^6, if we choose R large enough, we get

$$\int_{B_R^c} \Phi_{u_n}^2 u_n^2 < \varepsilon. \tag{143}$$

Analogously, for R large enough, we have

$$\int_{B_R^c} \Phi_{u_0}^2 u_0^2 < \varepsilon \tag{144}$$

and therefore

$$\int_{B_R^c} |\Phi_{u_n} u_n - \Phi_{u_0} u_0|^2 < 2\varepsilon. \tag{145}$$

On the other hand

$$\begin{aligned} \int_{B_R} |\Phi_{u_n} u_n - \Phi_{u_0} u_0|^2 &= \int_{B_R} (\Phi_{u_n}(u_n - u_0) + u_0(\Phi_{u_n} - \Phi_{u_0}))^2 \\ &\le 2 \int_{B_R} \Phi_{u_n}^2 (u_n - u_0)^2 + u_0^2 (\Phi_{u_n} - \Phi_{u_0})^2 \\ &\le 2\|\Phi_{u_n}\|_{L^6(B_R)}^2 \|u_n - u_0\|_{L^3(B_R)}^2 \qquad (146) \\ &\quad + 2\|u_0\|_{L^6(B_R)}^2 \|\Phi_{u_n} - \Phi_{u_0}\|_{L^3(B_R)}^2. \qquad (147) \end{aligned}$$

The sequence u_n weakly converges to u_0 in H^1, then it strongly converges to u_0 in $L^3(B_R)$. So, since Φ_{u_n} is bounded in L^6, we have

$$\|\Phi_{u_n}\|_{L^6(B_R)} \|u_n - u_0\|_{L^3(B_R)} \to 0. \tag{148}$$

On the other hand $\Phi_{u_n} \rightharpoonup \Phi_{u_0}$ weakly in $\mathscr{D}^{1,2} \subset H^1_{loc} \subset\subset L^3_{loc}$, then we have

$$\|\Phi_{u_n} - \Phi_{u_0}\|_{L^3(B_R)} \to 0. \tag{149}$$

By (146), (148) and (149) we get

$$\int_{B_R} |\Phi_{u_n} u_n - \Phi_{u_0} u_0|^2 \to 0. \tag{150}$$

Finally by (145) and (150) we get (139).

Following analogous arguments it can be shown that also the map $u \to \Phi_u^2 u$ is compact from X to X'.

Finally we prove that assumption (H3) is satisfied i.e. we prove that

$$\int (1 - q\Phi_u)u^2 \, dx \geq 0.$$

Arguing by contradiction assume that there is a region Ω where $q\Phi_u > 1$ and $q\Phi_u = 1$ on $\partial\Omega$. Clearly by (117)

$$-\Delta\Big(\Phi_u - \frac{1}{q}\Big) + q^2u^2\Big(\Phi_u - \frac{1}{q}\Big) = -\Delta\Phi_u + q^2u^2\Phi_u - qu^2 = 0.$$

Then $v = \Phi_u - \frac{1}{q}$ solves the Dirichlet problem

$$-\Delta v + q^2u^2v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

Multiplying by v and integrating in Ω we get

$$\int_\Omega (|\nabla v|^2 + q^2u^2v^2) \, dx = 0.$$

Then $v = \Phi_u - \frac{1}{q} = 0$ in Ω contradicting $q\Phi_u > 1$ in Ω.

Finally observe that, if we take $u \neq 0$ in all $\mathbb{R}^3$, then

$$\int (1 - q\Phi_u)u^2 \, dx > 0.$$

In fact $\int (1 - q\Phi_u)u^2 \, dx = 0$ would imply that $\Phi_u = \frac{1}{q}$ a.e. in $\mathbb{R}^3$, contradicting $\Phi_u \in \mathscr{D}^{1,2}(\mathbb{R}^3)$. □

LEMMA 26. *Assumption* (11) *is satisfied for q sufficiently small.*

PROOF. Let $R > 0$ and consider the map u_R defined in (73). As shown in the proof of Lemma 15, we can choose R be so large that

$$\frac{J(u_R)}{\frac{1}{2}\int u_R^2} < m_0^2. \tag{151}$$

Now consider

$$\frac{J(u_R)}{K_q(u_R)} = \frac{J(u_R)}{\frac{1}{2}\int u_R^2 - \frac{q}{2}\int \Phi_{u_R} u_R^2}. \tag{152}$$

So, by (152), we get that assumption (11) is satisfied if we show that

$$q \int \Phi_{u_R} u_R^2 \to 0 \quad \text{for } q \to 0. \tag{153}$$

Since Φ_{u_R} depends on q a little work is needed to prove (153).

Since Φ_{u_R} solves (117) with $u = u_R$, we have

$$\|\Phi_{u_R}\|^2_{\mathscr{D}^{1,2}} + q^2 \int u_R^2 \Phi_{u_R}^2 = q \int u_R^2 \Phi_{u_R} \leq q \|u_R\|^2_{L^{12/5}} \|\Phi_{u_R}\|_{L^6} \tag{154}$$

and then

$$\frac{\|\Phi_{u_R}\|^2_{\mathscr{D}^{1,2}}}{\|\Phi_{u_R}\|_{L^6}} \leq q \|u_R\|^2_{L^{12/5}}.$$

Then, since $\mathscr{D}^{1,2}$ is continuously embedded into L^6, we easily get

$$\|\Phi_{u_R}\|_{\mathscr{D}^{1,2}} \leq cq \|u_R\|^2_{L^{12/5}}, \tag{155}$$

where c is a positive constant. Then, using again (154), we get

$$q \int u_R^2 \Phi_{u_R} \leq q \|u_R\|^2_{L^{12/5}} \|\Phi_{u_R}\|_{L^6} \leq cq^2 \|u_R\|^4_{L^{12/5}}.$$

From which we get (153). □

Finally we are ready to conclude the proof of Theorem 22.

Proof of Theorem 22.

By Lemma 25 the assumptions (H1), (H2), (H3) of the Theorem 4 are satisfied. Moreover by Lemma 26 there exists $q^* > 0$ such that for $0 < q < q^*$ also assumption (11) is satisfied. Then we can use Theorem 5 and we get that there exists $q^* > 0$ such that for $0 < q < q^*$ there exists a non empty, open subset $\Sigma_q \subset \mathbb{R}$ such that for any $\sigma \in \Sigma_q$ problem (122) has a solution (u, ω) with charge $H_q(u, \omega) = \sigma$. Moreover such a solution minimizes the energy $\tilde{E}_q(u, \omega)$ on the states (u, ω) having charge $H_q(u, \omega) = \sigma$. Then, by Proposition 24, u, ω, $\varphi_u = \omega \Phi_u$ solve (108), (109). □

7. Appendix

Let assumption Wiii) (b) be satisfied i.e. we assume that there exists $s_1 > s_0$ such that $N'(s_1) \geq 0$.

Set

$$\tilde{N}(s) = \begin{cases} N(s) & \text{for } s \leq s_1 \\ N'(s_1)s + c_1 & \text{for } s \geq s_1 \end{cases} \tag{156}$$

where

$$c_1 = N(s_1) - N'(s_1)s_1$$

Set

$$\tilde{W}(s) = \frac{m^2}{2}s^2 + \tilde{N}(s) \tag{157}$$

By the following proposition we can replace in (47) $W'(s)$ with $\tilde{W}'(s)$

PROPOSITION 27. *Let $m^2 \geq \omega^2$. Then for any solution $u \in H^1$ of the equation*

$$-\Delta u + \tilde{W}'(u) = \omega^2 u \tag{158}$$

we have

$$u \leq s_1$$

PROOF. Let $u \in H^1$ be a solution of (158) and set

$$u = s_1 + v.$$

We want to show that $v \leq 0$. Arguing by contradiction, assume that

$$\Omega = \{x : v(x) > 0\} \neq \emptyset.$$

Then, multiplying both members of (158) by v and integrating on Ω, we have

$$\int_\Omega |\nabla v|^2 + \tilde{W}'(s_1 + v)v - \omega^2(s_1 + v)v = 0.$$

So, using (157), we have

$$\int_\Omega |\nabla v|^2 + \tilde{N}'(s_1 + v)v + (m^2 - \omega^2)(s_1 + v)v = 0$$

which, by (156), becomes

$$\int_\Omega |\nabla v|^2 + N'(s_1)v + (m^2 - \omega^2)(s_1 + v)v = 0. \tag{159}$$

Since

$$N'(s_1) \geq 0 \quad \text{and} \quad m^2 \geq \omega^2,$$

expression (159) gives

$$v = 0 \quad \text{in } \Omega,$$

contradicting the definition of Ω. □

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