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Partial Differential Equations — *Surfaces minimizing nonlocal energies*, by Luis Caffarelli[1].

Abstract. — In this lecture, we discuss what we understand by a non local diffusion equation and explain the particular case of surface evolution by non local mean curvature and the corresponding minimal surfaces.

Key words: diffusion, minimal surface, phase field, long range correlations.

AMS Subject Classification (2000): 35J20, 35J60, 35J93, 49Q05, 49Q15.

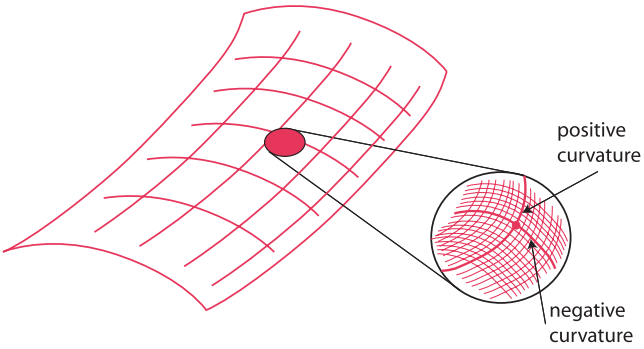
To the memory of Guido Stampacchia

1. What is a diffusion process?

A "diffusion" equation quantifies the idea that the variable under consideration, u, (a temperature, a probability density, the speed of a flow, a body in elastic equilibrium) tries to revert to "an average of itself" in a surrounding infinitesimal neighborhood.

May be the simplest example, is that of a "minimal surface", the surface configuration of soap film or elastic membrane attached to a wire.

For a smooth surface, being minimal implies "the surface has zero mean curvature" (the Euler-Lagrange equation)

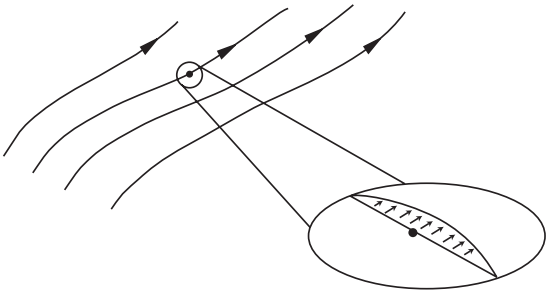


"Laplacian in tangential coordinates" =
"mean curvature" = "sum of principal curvatures"

[1] The author was partially supported by National Science Foundation grant DMS-0654267.

In other words, a point of the surface looks at how the surface bends around itself and it accommodates in such a way that the tensions pulling it upwards compensate with those pulling downwards.

Another example comes from viscous fluids



"... a fluid flow with velocity field $\vec{v}(x,t)$ ..."

Here again, if the particles surrounding x_0 are going faster than x_0, their drag it along, if they are going slower they tend to brake it.

This is a smoothing effect: If the particle at (x_0, t_0) tries to "escape", the surrounding particles will try to "break it".

How do you express that mathematically? In principle this viscosity effect does not have to be linear, it may be very different at low or high speeds but one (linear) possibility is to compare the "average velocity" in a tiny ball B_ε around x_0, with the actual velocity of the particle at x_0, $v(x_0, t_0)$ and take that quantity

$$\fint_{B_\varepsilon(x_0)} v(x,t_0) - v(x_0,t_0)\,dy$$

As a "measure" of how much the speed of the particle deviates from its surrounding average and make it the acceleration (positive or negative) that the particle will suffer

$$D_t v = (v_t + v\nabla v) \sim \frac{1}{\varepsilon^2} \fint_{B_\varepsilon(x)} [v(y,t) - v(x_0,t)]$$

$$+ \text{ other global or external factors}$$

This is clearly a quadratic effect, and as ε goes to zero, the expression

$$\frac{1}{\varepsilon^2} \fint_{B_\varepsilon} v(x,t_0) - v(x_0,t_0)$$

converges to the integral in the unit sphere of

$$\frac{1}{2}\int_S D_{\sigma\sigma} v(x_0, t_0)\, d\sigma = \frac{1}{2}\Delta v$$

and we get the Navier-Stokes equation:

$$\underbrace{D_t v}_{\text{acceleration}} = \underbrace{\Delta v}_{\text{viscosity}} - \underbrace{\nabla p}_{\substack{\text{pressure effect accelerates the flow} \\ \text{from higher to lower pressures due} \\ \text{to incompressibility of the flow}}}$$

Similar considerations can be made for the heat equation where caloric energy is supposed to flow from regions of higher temperature to lower proportionally to $-\nabla T$.

The final result is that the temperature at a point x_0 compares itself with its surrounding "infinitesimal average" and heat will flow away proportionally to the difference.

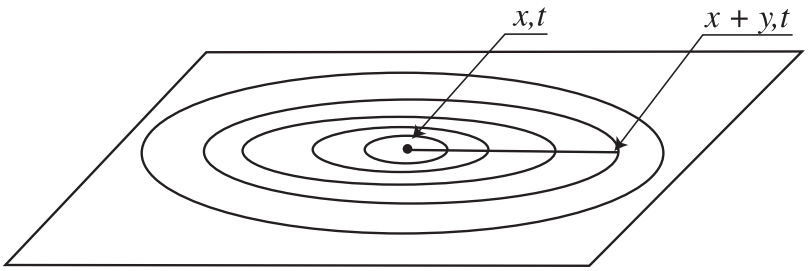
Similar considerations apply to Brownian motion where the probability density $u(x, t)$ of a particle being at the point x; at time t increases in time proportionally to the probability of the particle jumping from somewhere else to x (the surrounding "average") minus the probability of the particle jumping from x to somewhere else, i.e., the gain and the loss of probability density.

2. Nonlocal diffusions as averaging process

2.1. Nonlocal diffusions. By a non local diffusion we mean a problem or evolution equation where the unknown function $u(x)$ is not just reverting to its infinitesimal average, but instead, it is influenced by (it is "aware" of) its values at many scales. It is still a diffusion, but trying to revert now to an integral "average" of its surrounding values.

- In probability, integral diffusions appear when considering jump processes (Levy processes) in optimal control, game theory and finance, where particles jump from and to x_0 discontinuously.
- In continuum mechanics, when considering "surface diffusion" or surface discontinuities, that "perceive" long ranges from the interactions taking place on both sides of the surface:
  the quasigeostrophic equation for ocean atmosphere interaction, semipermeable membranes, planar crack propagation
- In fluid dynamics, in turbulent flow or particles "bouncing" through random media (Bouchaud-Georges, Zaslavsky)

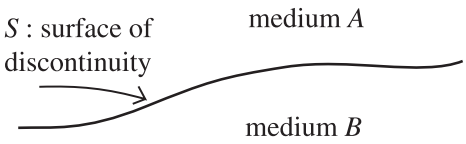
The typical diffusion equation becomes then:



$$u_t(x,t) = \int [u(x+y) - u(x)]K(y)\,dy$$

where the postive kernel $K(y)$, "weights" the averaging process according to the knowledge or importance that the process gives, at the point x, of what is going on at $x + y$.

An important family of examples concern surface diffusion:



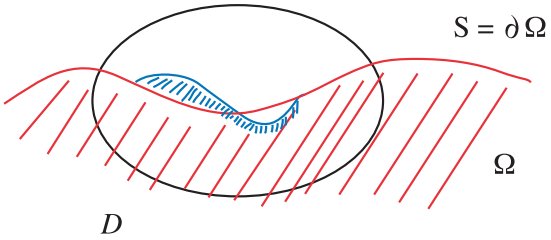
In that case, quantities that "live" on the surface: flow through a membrane, a surface stress, a surface temperature flow, are affected by long range interactions through media A and B at both sides of S.

In fact, flux across a surface can be thought many times as a form of non local diffusion: the flow rate across the surface balances itself with the total flow through boundary potentials.

In this lecture we would like to discuss the nonlocal version of movement by mean curvature and the corresponding integral "minimal surfaces" resulting from this approach.

Since our research follows in many ways the well established theory of movement by mean curvature and the classical theory of minimal surfaces, we review in a few lines the relevant ideas.

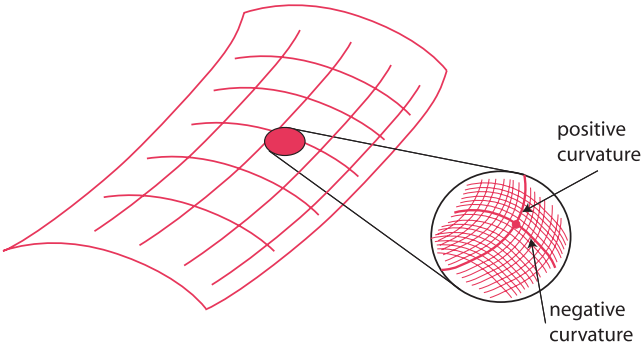
In the context of boundaries of sets of finite perimeter minimal surfaces S are presented the following way: We consider sets $\Omega \in \mathbb{R}^n$, and restrict our attention to the part of Ω inside a fixed domain D. Let S be the boundary of Ω inside D.



"Any local perturbation of the set inside D increases perimeter"

Then $\partial\Omega$ has minimal perimeter restricted to D if any compact perturbation of Ω inside D increases its perimeter. Of course, we only need to prescribe the trace of Ω on ∂D, not what is outside D, since the mean curvature equation is "local".

For a smooth surface, being minimal implies "the surface has zero mean curvature" (the Euler-Lagrange equation)



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We could also express the zero mean curvature condition as a higher order density cancellation on Ω and $C\Omega$ in the ball $B_r(x_0)$

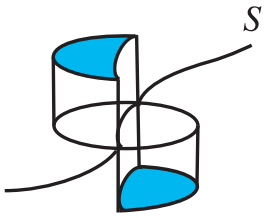
Indeed, if $S = \partial\Omega$ is a smooth surface, for $x_0 \in S$, in general, the integral

$$I(r) = \fint_{B_r(x_0)} \chi_\Omega - \chi_{C\Omega} \sim O(r)$$

If the mean curvature is zero, instead, we have a higher order cancellation

$$I(r) = \fint_{B_r(x_0)} \chi_\Omega - \chi_{C\Omega} \sim o(r)$$

In other words, the Laplacian of $(\chi_\Omega - \chi_{C\Omega})$ along $\partial\Omega$, is still not zero but it has a higher order cancellation



(If S is Lipschitz, all we can say is that)

$$I(r) \sim 1.$$

We make this observation because of its connection with phase transitions and the theory of movement by mean curvature.

Indeed, minimal surfaces appear in phase transitional theory as "stationary interphases":

In some (simple) models, material interphases tend to evolve proportionally to its curvature. (If their evolution is linked to "minus its curvature", i.e., curvature tends to increase, that produces "fingering" or related phenomena.)

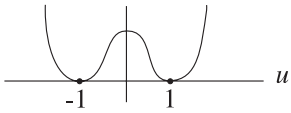
Movement by mean curvature is obtained for instance as limiting process of "phase field" theories for the evolution of interphasial surfaces. We have two phases of a material (solid liquid, opposite magnetization ...) that we try to represent with an indicator function: $u = 1$ in phase A, $u = -1$ in phase B (or $u = \chi_A - \chi_B$).

How is the interphase organized?

We consider instead of u, a "singular perturbation u_ε", a smooth function moving by steepest descent of the energy functional

$$E(u_\varepsilon) = \varepsilon \int (\nabla u_\varepsilon)^2 + \frac{1}{\varepsilon} \int F(u_\varepsilon)$$

with F a "double well potential

$F(u) =$ 

(Landau-Ginzburg)

Observing the energy functional

$$E(u_\varepsilon) = \varepsilon \int (\nabla u_\varepsilon)^2 + \frac{1}{\varepsilon} \int F(u_\varepsilon)$$

we can see that since being different than ± 1 is allowed but heavily penalized, in order to minimize energy, the function in u_ε tries to be very close to one or minus one, reproducing the two phases A and B but with a narrow and steep transition region of "phase change" in between. The organization of this region is dictated by the residual term $\varepsilon \int (\nabla u_\varepsilon)^2$.

In the limit, as ε goes to zero, we will obtain, indeed

$$u_0 = \begin{cases} 1 & \text{in } \Omega_t \\ -1 & \text{in } \mathscr{C}\Omega_t \end{cases}$$

but as a result of the gradient term $\partial\Omega_t$ is now organized to move proportionally to its mean curvature (i.e., evolving towards a zero mean curvature stationary state).

3. A discrete way of generating movement by mean curvature (Merriman, Bence, Osher)

We seek now to build a surface S_t evolving by mean curvature through a discrete sequence of surfaces $S_k = \partial\Omega_k$.

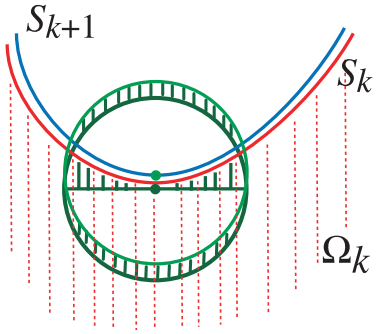
Let S_k be $S_k = \partial\Omega_k$ and $u_k = \chi_{\Omega_k} - \chi_{\mathscr{C}\Omega_k}$

- For epsilon small, we compute

$$w_k(x) = \fint_{B_\varepsilon(x)} u_k$$

As discussed above, the function w_k expresses the imbalance of Ω and $\mathscr{C}\Omega$ from covering exactly half of B_ε, that, infinitesimally is realized by surfaces of zero mean curvature.

Indeed, if the surface S_k is for instance concave towards Ω at x_0, w_k will be negative at x_0 and we will have to shift x_0 to x', inwards of Ω, to have exact volume cancellation, i.e., $w(x') = 0$.



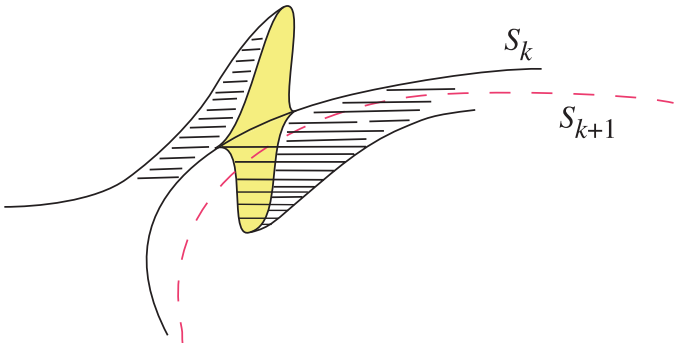
In other words, we define the new surface

$$S_{k+1} = \partial\{w_k > 0\} = \partial\Omega_{k+1}$$

If we choose the time interval by $S_{k+1} = S(t_k)$ and $t_{k+1} - t_k = \Delta t \cdot \sim \varepsilon^2$ we obtain this way a discrete approximation to movement by mean curvature.

In fact, we may replace $B_\varepsilon(x)$ (or more precisely, the probability density $\frac{1}{|B_\varepsilon|}\chi_{B_\varepsilon}$) by any rapidly decaying, radially symmetric probability kernel, K_ε, properly scaled, and proceed the same way.

For instance, we could choose a very concentrated Gaussian: $g_{\varepsilon^2} = \frac{1}{\varepsilon^n} e^{-x^2/\varepsilon^2}$, and "split it" along S_k



This can be viewed as

- Solving the heat equation with initial data $\chi_{\Omega_k} - \chi_{\mathcal{C}\Omega_k}$ and after an interval of time ε^2, look for the zero level surface of the solution. In other words, particles are allowed to diffuse into each other for a small time, and then, forced to segregate again according which one has higher density.

In this lecture, we are interested in those cases in which the kernel K_ε has slow decay.

In phase transition models this corresponds to processes with large scale correlation (like solidification) where information far away from the interphase has a direct influence in it.

In that case, the "free energy" functional that sets the interphase dynamics (F a double well) has the non local form

$$E_\varepsilon(u) = \varepsilon \iint [u(x) - u(y)]^2 K(x, y)\, dx\, dy + \frac{1}{\varepsilon}\int F(u)$$

instead of

$$\int (\nabla u)^2\, dx + \int F(u)$$

Note that for $K(x, y) > 0$ the term

$$\|u\|^2 = \iint [u(x) - u(y)]^2 K(x, y)\, dx\, dy$$

defines a Hilbert space if we factor out constants.

(Many authors worked on non local phase transitions. See Fife, Lebowitz, or Presutti.)

For instance, if $(0 < \sigma < 1)$, an admissible kernel is

$$K(x, y) = (1 - \sigma)|x - y|^{-(n+2\sigma)}$$

Then, the corresponding Hilbert space is H^σ (σ-fractional derivative in L^2)

$$\|u\|^2_{H^\sigma} = (1 - \sigma) \iint \frac{[u(x) - u(y)]^2}{|x - y|^{n+2\alpha}} + \|u\|^2_{L^2}$$

and the elliptic operator, the σ-fractional Laplacian

$$\Delta^\sigma u(x) = (1 - \sigma) \int \frac{[u(x + y) - u(x)]}{|x - y|^{n+2\sigma}} dy$$

3.1. The two questions we would like to consider, are:

1. What is the continuous surface evolution process induced by convolution with kernels K_ε with slow decay: $(s = 2\sigma)$

$$K_1 \sim \frac{1}{(1 + |x|^2)^{(n+s)/2}} \quad 0 < s < 2$$

2. What are the geometric properties of the corresponding stationary (minimal) surfaces?

A relatively simple heuristic calculation suggests the following answer:

- For $s \geq 1$, convolution with $K_\varepsilon = \frac{1}{\varepsilon^n} K\left(\frac{x}{\varepsilon}\right)$ still induces movement by mean curvature, for time scaling $\Delta t = \varepsilon^s$ ($\varepsilon \log \varepsilon$ for $s = 1$)
- For $s < 1$, instead, the normal velocity of the surface s is proportional, with $\Delta t = \varepsilon^s$, to

$$(*) \qquad L(S)(x_0) = (1 - s) \int_{\mathbb{R}^n} [\chi_\Omega(y) - \chi_{\mathscr{C}\Omega}(y)] \cdot |x_0 - y|^{-(n+s)}\, dy$$

for x_0 in S.

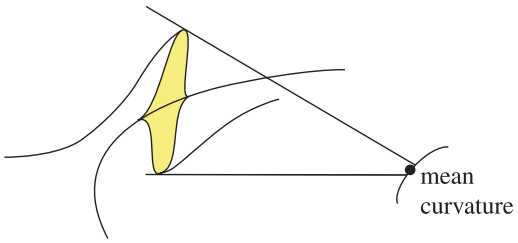
We note that for $s \geq 1$, i.e., as long as the kernel K has finite first moment (and the critical case) as ε goes to zero, $L(s)$ still converges to $\kappa(x_0)$, the mean curvature of S at x_0.

In other words, the averaging process becomes infinitesimal.

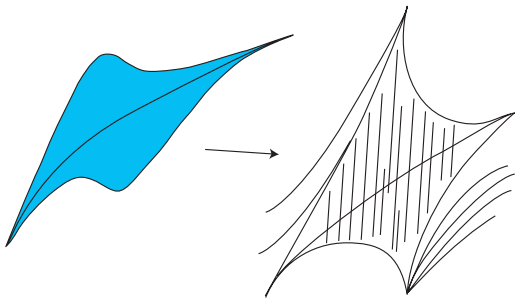
Instead, for $s < 1$, the properly scaled average of $\chi_\Omega - \chi_{\not\Omega}$ in an "infinitesimal ball", i.e., mean curvature, is replaced by a long range weighted average.

Geometrically, in

Case 1). For kernels with decay faster than $|y|^{-(n+1)}$, under "infinitesimal" rescaling the tail of the kernel disappears



Case 2). For kernel decay slower than $|y|^{-(n+1)}$, the tail of the kernel is what remains



THEOREM [C-Souganidis—Arxiv.org, to appear in ARMA].

- *The above is correct in the viscosity sense, although:*
- *"Fattening" may occur, i.e.,* $\partial\Omega$ *may become a thick region between* $u = 1$ *and* $u = -1$

(*More precisely, if the* lim sup *and the* lim inf *coincide the "equation" for the limit holds.*)

REMARK. As with the heat kernel above, this (discrete) process corresponds to "solving the fractional heat equation" (i.e., mixing both phases)

$$u_t = \Delta^{s/2} u$$

with initial data $\chi_{\Omega_k} - \chi_{\mathscr{C}\Omega_k}$ and after a short time, ε^s, defining S_{k+1} to be the zero level surface of the solution (segregating the two phases by higher density).

4. PROPERTIES OF STATIONARY SOLUTIONS TO THE INTEGRAL EQUATION AND "INTEGRAL MINIMAL SURFACES" (work with J. M. Roquejoffre and O. Savin)

The same way that minimal surfaces are the stationary interphases for the classical infinitesimal case, we want to study the properties of stationary surfaces for nonlocal interactions The stationary equation for $(*)$ is obviously

$$L(S)(x) = 0 = \int [\chi_\Omega(y) - \chi_{\mathscr{C}\Omega}(y)]|x-y|^{-(n+s)}\, dy$$

In other words the fractional Laplacian of $\chi_\Omega - \chi_{\mathscr{C}\Omega}$ must be zero at the interphase between Ω and $\mathscr{C}\Omega$. (Compare with the discussion on zero mean curvature above.)

In polar coordinates around x_0 this condition reads

$$0 = \int \frac{1}{r^{(1+s)}} \int [\chi_\Omega(r\sigma) - \chi_{\mathscr{C}\Omega}(r\sigma)]\, d\sigma$$

The spherical integral inside

$$I_0 = \int_{\sigma \in S^1} \chi_\Omega(r\sigma) - \chi_{\mathscr{C}\Omega}(r\sigma)\, d\sigma$$

is bounded and measures the average deviation of Ω in the sphere of radius r from covering "in measure" exactly half of the sphere:

$$I_0(r) = \frac{|\Omega \cap S_r| - |\mathscr{C}\Omega \cap S_r|}{|S_r|}$$

being in that sense an "integral" measure of mean curvature.

The exterior integral

$$I_1 = \int \frac{dr}{r^{(1+s)}} I_0(r)\, dr$$

converges at infinity, but in principle diverges near zero, forcing some cancellation in the spherical average deviation.

For instance, if S is a C^2 surface at x_0, $I_0(r) \sim O(r)$, and I_1 converges.

In fact, $I_0(r) = Cr\kappa(0) + o(r)$ (the mean curvature of S at zero).

5. Stationary surfaces as minimizers of a non local area integral

5.1. The variational character of $L(s)$. The equation $L(S) = \Delta^{s/2}(\chi_\Omega - \chi_{\mathscr{C}\Omega}) = 0$, is formally the Euler-Lagrange equation of the "variational integral: $(u = \chi_\Omega - \chi_{\mathscr{C}\Omega})$

$$E(s) = \iint \frac{[u(x) - u(y)]^2}{|x - y|^{n+s}},$$

in the Hilbert space of functions with "$s/2$ derivatives in L^2, denoted by $H_{s/2}$.

(The same way that "mean curvature" is the Euler-Lagrange equation of an area minimizing surface)

In general, indicator functions of sets (functions with jump discontinuities) are *not* in H_σ for $\sigma \geq 1/2$ (the case $H_{1/2}$ is the classical case of "boundary traces of functions in H_1"). But precisely for $\sigma = s/2 < 1/2$, functions with jump discontinuities across smooth surfaces do have finite H_σ norm.

It is curious that this allows for a "Hilbert space" theory of integral minimal surfaces, instead of having to define sets of finite perimeter and BV functions.

5.2. The "Dirichlet minimal surface problem". We may then pose the "Dirichlet minimal surface problem" in the bounded domain D: As before, given the set Ω_0 among all sets Ω with $\Omega \cap \mathscr{C}D = \Omega_0 \cap \mathscr{C}D$, find the one that minimizes integral area.

That is, for $s < 1$, for $u = \chi_\Omega - \chi_{\mathscr{C}\Omega}$, minimize:

$$E(\Omega) = \iint \frac{[u(x) - u(y)]^2}{|x - y|^{n+s}}\, dx\, dy$$

Since $u = \pm 1$, we can rewrite the energy as

$$E(\Omega) = \iint \frac{\chi_\Omega(x)\chi_{\mathscr{C}\Omega}(y)\, dx\, dy}{|x - y|^{n+s}}$$

Note that in this energy formula any point in Ω interacts with any point in $\mathscr{C}\Omega$, contributing to the total energy but the interaction strengthens as x and y get closer and this is the reason why we should expect some "alignment" of Ω and $\mathscr{C}\Omega$ along their common boundary S, to have the "least number" of x's and y's close to each other. This is the effect that we hope will force S to be smooth.

Another interesting remark is that, at least formally, by the divergence theorem

$$E(\Omega) = \int_{\partial\Omega}\int_{\partial\Omega} \frac{v(x)v(y)}{|x - y|^{n-2+s}}\, dA(x)\, dA(y)$$

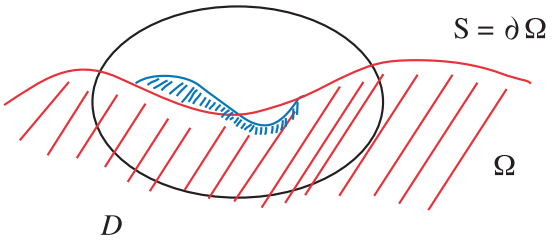
where v is the exterior normal to $\partial\Omega$, and dA is the differential of area on $\partial\Omega$.

We note that as s goes to one, properly scaled $E(\Omega)$ converges to the BV norm of $\partial\Omega$ (the perimeter of Ω).

6. The Dirichlet problem

We have then here the same picture as for minimal surfaces:

We seek a set Ω whose indicator function $\chi_\Omega - \chi_{\mathcal{C}\Omega}$ minimizes H_s norm among possible local perturbations (inside D) of Ω_0



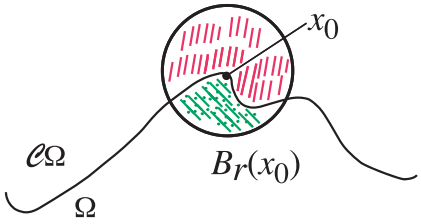
But now, all values of Ω_0 outside D enter into the minimization process, not just its trace on ∂D.

6.1. *Regularity theory of H_σ-minimal surfaces.*

THEOREM ([C-Roquejoffre-Savin]). *H_σ-minimal surfaces are smooth ($C^{1,\beta}$), except on a closed singular set S^* of Hausdorff dimensions $\mathcal{H}^{n-2}$.*

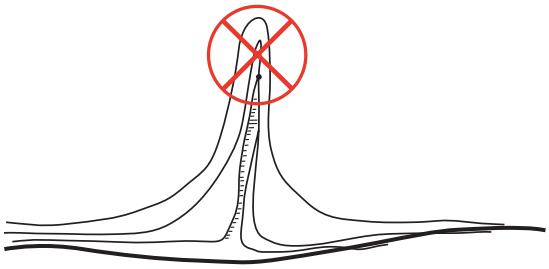
Main steps.

a) Minimal surfaces have no cusps in measure: If $x_0 \in S = \partial\Omega$



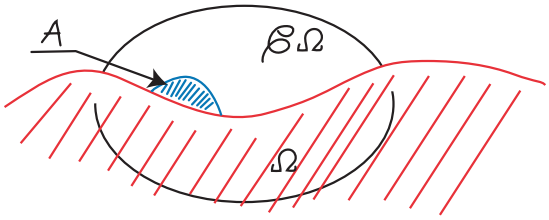
$$\frac{|B_r \cap \Omega|}{|B_r|} \quad \text{and} \quad \frac{|B_r \cap \mathcal{C}\Omega|}{|B_r|} \geq \mu_0 > 0$$

This property transforms L^1 convergence of the sets Ω into uniform convergence of the "minimal surfaces", $S = \partial\Omega$, providing very strong compactness.



b) Minimizers satisfy the expected Euler Lagrange equations in the viscosity sense:

The first variation of the energy integral under a small perturbation of Ω to $\Omega \cup A$ is



$$0 \leq \iint \chi_A(x)[\chi_{\mathcal{C}(\Omega\cup A)}(y) - \chi_\Omega(y)]|x-y|^{-(n+s)}\,dy\,dx$$

We may then define the concept of *Viscosity super (sub) solution*

$$\int [\chi_{\mathcal{C}\Omega}(y) - \chi_\Omega(y)]|x_0 - y|^{-(n+s)}\,dy \geq 0,$$

$(\leq)$ if we have a tangent smooth surface by below (above) at x_0.

Notice that if we have a tangent smooth surface by below at x_0, the positive part of the "area excess" spherical integral must converge, but the supersolution condition implies that the negative contribution is controlled by the positive one, rendering the radial integral absolutely convergent.

Existence of tangent cones and monotonicity of energy average. For classical minimal surfaces we have the monotonicity formula

$$J(r) = \frac{\text{Area}(S \cap B_r)}{r^{n-1}} \quad \text{is increasing in } r$$

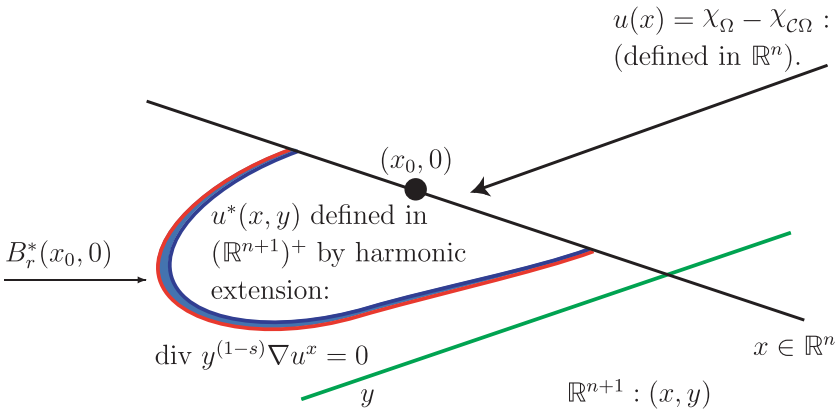
This quotient is invariant under dilations and an important consequence of this monotonicity that "blow ups" (limits by sequences of dilations) are minimal cones.

For the non local case, we have a similar formula, where the "energy" or "area" A^* has to do with an "extension theorem" for the $H_{s/2}$ Hilbert space ([C-Silvestre, Arxiv.org]):

u is extended to $u^*(x, y)$, in one extra variable, satisfying the equation

$$\frac{1}{y^{(1-s)}} \operatorname{div} y^{(1-s)} \nabla u^* = 0$$

This is valid for any $0 < s < 2$.



The proof of the monotonicity formula follows the classical idea:
We perturb u^* by

$$u_\varepsilon^* = \begin{cases} u^*((1+\varepsilon)x, (1+\varepsilon)y) \text{ for } (x, y) \text{ in } B_{r/(1+\varepsilon)}(x_0, 0) \\ u^*(x, y) \text{ outside } B_r(x_0, 0) \\ \text{constant on rays in between} \end{cases}$$

Going back to the statement c_1) "$J(r) = \dfrac{A^*(B_r \cap \Omega)}{r^{n-s}}$ is monotone," where A_r^* is now the "local energy" of the extension:

$$A_r^* = \iint_{B_r^*(x_0, 0)} y^{(1-s)} (\nabla u^*)^2 \, dx \, dy$$

We still have the important conclusion: The quotient

$$A_r^* / r^{n-s}$$

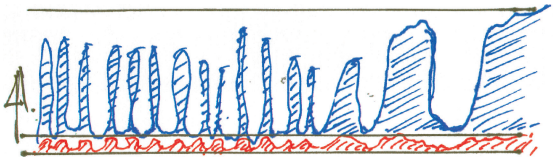
is constant if and only if χ_Ω is a cone with center at x_0. In particular, infinite dilations of a minimal surface are still cones as in the classical theory.

d) "Flat minimal surfaces are smooth": Next we prove that

If the tangent cone is a plane, i.e., if S can be trapped in a flat enough cylinder, then S is a $C^{1,\alpha}$ graph in a neighborhood of the point.

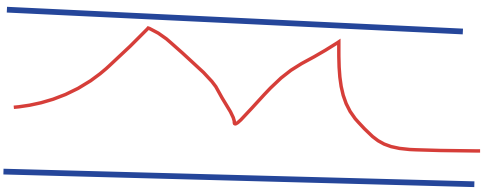
The main steps of the proof are based on an "improvement of flatness" theorem, suggested by the fact that the "linearization" of our "minimal surface" are formally solutions of a fractional Laplace equation, above the critical $1/2$ exponent.

Step 1: Non homogeneous blow up



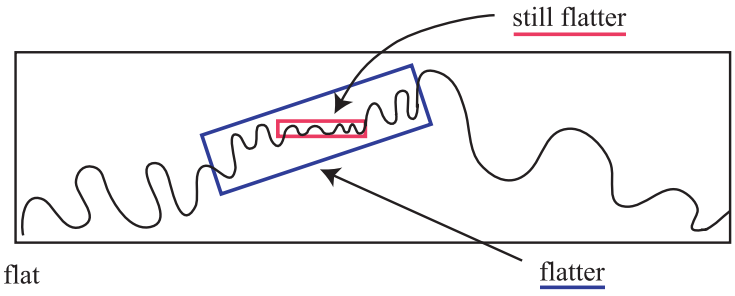
Step 2: + partial Harnack like inequality (as the surface gets flatter, goes through a partial "geometric regularization" by which oscillatin decays for a larger and larger number of steps.

Step 3: As a consequence of this "partial Harnack", the sequence of sets converges to a Hölder graph with control at infinity



Step 4: ... but the Euler equation for the surfaces implies that this graph is a viscosity solution of $\Delta^{1/2+s}u = 0$. (The same way that minimal surfaces linearize to the Laplacian, fractional minimal surfaces linearize to the fractional Laplacian.)

In particular, u is $C^{1,\alpha}$, and near the origin, u becomes "flatter"



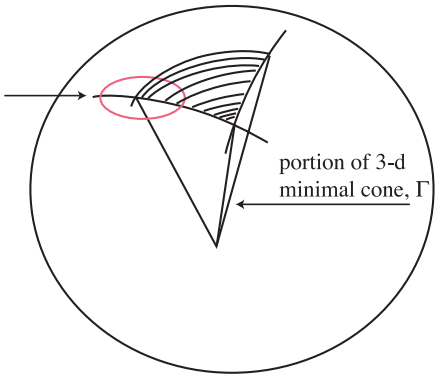
Valid at any point near x_0, this implies that the approximating surfaces, $\partial\Omega_k$ are a "$C^{1,\alpha}$ graph".

Notice that, as a consequence of the flatness theorem there is a gap between cones and planes: cones cannot be "too flat" neither have energy too close to that of a plane.

e) To complete the theorem we prove: The singular set has dimension $n-2$.

This is just an application of Federer's "dimension reduction" argument, plus the existence of tangent cones with a "flatness gap", plus the fact that in 2 dimensions, minimal cones have finitely many rays.

We recall Federers reduction argument for $n=3$: Given a minimal cone, Γ, in $\mathbb{R}^3$,



On the surface ∂B_1, Γ looks like a 2-d minimal cone, that is a finite number of rays, hence singular sets in $\mathbb{R}^3$ consist of, at most, one dimensional rays.

Since 2-d minimal cones have isolated singularities, 3-d minimal cones have isolated ray's (Hausdorff dimension one).

Some applications of this work are to plane like phase transitions for large scale correlations, and to image reconstruction.

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