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Mechanics — *Stability-Instability criteria for nonautonomous systems*, by SALVATORE RIONERO.

To Renato Caccioppoli, unforgettable past master of the academic years 1951–1952, 1952–1953

ABSTRACT. — Nonautonomous binary systems of O.D.Es are considered. Apart from a critical case, it is shown that a temporal uniform validity of the Hurwitz conditions appear to be a basic condition to require for guaranteing the stability. Stability-instability criteria are obtained. Applications to the equation $\ddot{x} + p(t)\dot{x} + q(t)x = 0$ and in particular to the Hill equation, are furnished. The Hill equation associated to the (linear) stability of the nonautonomous Lotka-Volterra system is considered.

KEY WORDS: nonautonomous systems, Liapunov Direct Method, Stability.

1991 MATHEMATICS SUBJECT CLASSIFICATION: 34DXX, 34D20; 37C60, 37C75; 93DXX.

1. INTRODUCTION

As it is well known the equation governing the one dimensional motion of a (punctiform) body about the rest state under the contemporary action of an elastic force and a viscous drag, with time depending elasticity and viscosity coefficients respectively, can be reduced to

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{1}$$

and, in particular, disregarding the viscous drag, to the so called *Hill equation*

$$\ddot{x} + q(t)x = 0 \tag{2}$$

By setting $\dot{x} = y$, the stability of the rest state is reduced to the stability of the null solution of the *nonautonomous* system

$$\begin{cases} \dot{x} = a(t)x + b(t)y \\ \dot{y} = c(t)x + d(t)y \end{cases} \tag{3}$$

with

$$a \equiv 0, \quad b = 1, \quad c = -q, \quad d = -p \tag{4}$$

Equation (1)–(3) have been studied by a large number of Authors [1]–[9]. In particular in [2] R. Caccioppoli obtains that if $q(t)$, $t \in \mathbb{R}^+$, is an *only continuous* positive real function, then the null solution of (2) may be unstable. Precisely—by means of an example—he obtains that (2) admits unbounded solutions. We refer to [3]–[4] and the references therein for further details concerned with the unboundedness of solutions of (1)–(2) and for the foundations of the stability state of art of the nonautonomous system (3). We confine ourselves to recalling that

i) the stability of the null solution of (3) has been deeply studied [1]–[9] under various assumptions on the coefficients a, b, c, d and, in particular, when the coefficients are periodic functions of t of the same period {cfr. [1]–[9] and specially chapter II of [3]};
ii) if q is a positive continuous function of bounded variations, then all solutions of (2) are bounded {[2] and pp. 80–90 of [3]}.

We remark that

1) *Boundedness does not imply Liapunov stability.*

2) *For* $p \neq 0$ *the boundedness of the solutions of* (1) *and the stability of the null solution of* (3) *are not guaranteed when the coefficients are only continuous functions of bounded variations as the equation*

$$\ddot{x} - \frac{2}{t}\dot{x} + x = 0, \quad \forall t \in \left[\frac{\pi}{2}, \infty\right[\tag{5}$$

shows. In fact, denoting by c_i, $(i = 1, 2)$ *two constants, one easily verifies that*

$$x = c_1(\sin t - t\cos t) + c_2(\cos t + t\sin t)$$

is the solution of (5) *associated to the initial data*

$$t_0 = \frac{\pi}{2}, \quad x\left(\frac{\pi}{2}\right) = c_1 + c_2, \quad \dot{x}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}c_1,$$

and is unbounded for any nonzero values of the initial data.

3) *When* (3) *is autonomous, the Hurwitz conditions guaranteing the stability of the null solution are*

$$I = a + d < 0, \quad A = ad - bc > 0 \tag{6}$$

hence in the case of (5)—*in view of* $\left\{I(t) = \frac{2}{t}, A = 1\right\}$—$(6)_1$ *is disregarded at any instant.*

4) *The coefficients of the equation* [3]

$$\ddot{x} + \frac{2}{t}\dot{x} + x = 0, \quad t \in \left[\frac{\pi}{2}, \infty\right[\tag{7}$$

have the same properties of the coefficients of (5) *but—in view of* $\left\{I(t) = -\frac{2}{t}, A = 1\right\}$, *the Hurwitz conditions* (6) *are verified uniformly with respect to t according to*

$$I^* = \sup_{\mathbb{R}^+} I < 0, \quad A_* = \inf_{\mathbb{R}_*} A > 0. \tag{8}$$

5) *The null solution of* (7) *is asymptotically stable. In fact denoting by* c_i, $(i = 1, 2)$, *two constants, one easily verifies that*

$$x = c_1 t^{-1} \sin t + c_2 t^{-1} \cos t$$

is the solution of (7) *associated to the initial data*

$$x\left(\frac{\pi}{2}\right) = \frac{2}{\pi} c_1, \quad \dot{x}\left(\frac{\pi}{2}\right) = -\frac{2}{\pi}\left(\frac{2}{\pi} c_1 + c_2\right).$$

6) *The uniform Hurwitz conditions* (8)*—at least when* $\{I = const. < 0, A = const. > 0\}$*—are not sufficient to guarantee the stability of the null solution of* (3). *In fact one easily verifies that*

$$x = -ce^{t/2} \cos t, \quad y = ce^{t/2} \sin t$$

with $c = const.$, *is an unbounded solution of* [16]

$$\begin{cases} \dot{x} = \left(-1 + \frac{3}{2}\cos^2 t\right)x + \left(1 - \frac{3}{4}\sin 2t\right)y \\ \dot{y} = -\left(1 + \frac{3}{4}\sin 2t\right)x + \left(-1 + \frac{3}{2}\sin^2 t\right)y \end{cases}$$

$\forall c \in \mathbb{R}$, *although* $\left\{I = -\frac{1}{2}, A = \frac{1}{2}, \forall t \in \mathbb{R}^+\right\}$.

In view of 2)–6), the following main questions arise

MAIN QUESTIONS. *Are the uniform Hurwitz conditions* (8) *necessary for guaranteing the stability of the null solution of* (3)*? Which are the conditions that one has to couple to* (3) *for guaranteing the stability?*

In the present paper we assume derivable the coefficients a, b, c, d and introduce the polynomials

$$P = (2IA + \dot{A})(x^2 + y^2), \quad Q = \frac{\partial}{\partial t}[(c^2 + d^2)x^2 + (a^2 + b^2)y^2 - 2(ac + bd)xy].$$

Then, apart from the (critical) case

$$P(P + Q) \le 0, \quad \text{on subsets of } \mathbb{R}^+,$$

we determine the conditions on the time derivatives appearing in P and Q, able to guarantee—together with (8)—the stability of the null solution of (3).

In the framework of the Liapunov Direct Method for *nonautonomous* systems {[8], pp. 226–227}, in Section 2 we introduce either the classic Liapunov function $E = \frac{1}{2}[\mu_1(t)x^2 + \mu_2(t)y^2]$ (with μ_1, μ_2 functions to be chosen suitably) or a *peculiar* Liapunov function V linked in a simple direct way to the eigenvalues of the coefficients matrix

$$L = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \tag{9}$$

through A and I.

Further we study some definiteness properties of E and V and obtain their temporal derivatives along the solutions of (3). Section 3 is devoted to obtaining estimates guaranteing that either $\dot{E}$ or $\dot{V}$ are definite or semidefinite. The stability criteria of the null solution of (3) are given in Section 4 while Section 5 is devoted to the instability criteria. The stability-instability of the zero solution of (1) and in particular of the Hill equation, are studied in Section 6. Section 7 is devoted to the nonautonomous Lotka-Volterra system. The paper ends with some final remarks (Section 8).

2. A peculiar Liapunov function

We call *peculiar* (or *eigenvalues depending*) the Liapunov functions linked in a simple direct way (together with their temporal derivative along the solutions) to the eigenvalues of the coefficients matrix (9), through A and I. Our aim now is to introduce a such function for (3). We denote by $\lambda_i(t)$, $(i = 1, 2)$, the eigenvalues of (9) and observe that the parameters I, A introduced in (6) can be written

$$I(t) = \lambda_1 + \lambda_2 = a + d, \quad A(t) = \lambda_1 \lambda_2 = ad - bc \tag{10}$$

I and A being the invariants of L. Here and in the sequel *we assume that a, b, c, d are derivable in $\mathbb{R}^+$ and bounded there together with the derivatives $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{d}$*. Further we introduce the function

$$V = \frac{1}{2}[A(x^2 + y^2) + (ay - cx)^2 + (by - dx)^2]. \tag{11}$$

This function is the O.D.Es "adaptation" of a peculiar Liapunov function introduced by the author, in the context of L^2-stability analysis for binary reaction-diffusion systems of P.D.Es {cfr [10]–[14] and the appendix of [15]}

Remark 1. *For any function $f : \mathbb{R}^+ \to \mathbb{R}$, we set*

$$f_* = \inf_{\mathbb{R}^+} f, \quad f^* = \sup_{\mathbb{R}^+} f$$

Lemma 1. *By virtue of the assumptions on a, b, c, d it follows that:*

i) *at any instant $\bar{t} \in \mathbb{R}^+$, in any circle centered at $(x = y = 0)$, exists a domain that verifies the inequality $V(\bar{t}, x, y) > 0$;*

ii) *under the condition*

$$A_* > 0, \tag{12}$$

V is positive definite;

iii) *the temporal derivative of* V *along the solutions of* (3) *is given by*

$$\dot{V} = \frac{1}{2}\sum_{i=1}^{3} P_i(t, x, y) \tag{13}$$

with

$$\begin{cases} P_1 = P = (2IA + \dot{A})(x^2 + y^2) \\ P_2 = \dfrac{dc^2}{dt}x^2 + \dfrac{da^2}{dt}y^2 - 2\dfrac{d(ac)}{dt}xy \\ P_3 = \dfrac{dd^2}{dt}x^2 + \dfrac{db^2}{dt}y^2 - 2\dfrac{d(bd)}{dt}xy. \end{cases} \tag{14}$$

Proof. As concerns i) it is enough to remark that at each instant $\bar{t} \in \mathbb{R}^+$,

$$V(\bar{t}) = 0 \tag{15}$$

is the equation of a conic passing through $O = (0, 0)$, $\forall A(\bar{t})$. Passing to ii), when (12) holds, it immediately follows that

$$V \geq A_* W(x, y) \tag{16}$$

with $W = x^2 + y^2$, positive definite function independent of t. Hence V is positive definite [8] and, moreover, since a, b, c, d are bounded, in view of (11), it follows that

$$\begin{cases} V \leq M(x^2 + y^2), \\ M \geq \dfrac{1}{2}|A|^* + (a^2 + b^2 + c^2 + d^2)^* \end{cases} \tag{17}$$

and hence V admits an upper bound which is infinitely small at the origin.

Finally, passing to iii), we recall that *when* (3) *is autonomous*, in [10]–[15] has been shown that

$$\dot{V} = IA(x^2 + y^2). \tag{18}$$

Therefore, in the *nonautonomous* case, one obtains

$$\begin{aligned} \dot{V} = {} & IA(x^2 + y^2) + \frac{1}{2}\dot{A}(x^2 + y^2) \\ & + (ay - cx)(\dot{a}y - \dot{c}x) + (by - dx)(\dot{b}y - \dot{d}x) \end{aligned} \tag{19}$$

and hence (13) easily follows.

Although (11) will appear to be the more appropriate Liapunov function, also the *nonautonomous* generalized "energy" [6]

(20) $$E = \frac{1}{2}\left[\mu_1(t)x^2 + \mu_2(t)y^2\right]$$

with μ_i, $(i = 1, 2)$, suitable positive derivable functions in $\mathbb{R}^+$ and bounded there together with the derivatives $\dot{\mu}_1$, $\dot{\mu}_2$, allows to obtain stability (instability) criteria of the zero solution of (3). Since, along the solutions of (3), it turns out that

(21) $$\dot{E} = \frac{1}{2}\left[(\dot{\mu}_1 + 2a\mu_1)x^2 + (\dot{\mu}_2 + 2d\mu_2)y^2 + 2(\mu_1 b + \mu_2 c)xy\right],$$

setting

(22) $$m_* = \frac{1}{2}\inf(\mu_{1*}, \mu_{2*}), \quad m^* = \frac{1}{2}\sup(\mu_{1*}, \mu_{2*})$$

it follows that

(23) $$E \geq m_* W, \quad E \leq m^* W$$

and hence E is *positive definite* for $m_* > 0$ and admits an upper bound which is infinitely small at the origin for $m^* > 0$.

Finally—in connection with the instability properties—it can be useful to consider the function (20) with one of the function μ_i, negative in $\mathbb{R}^+$. In this case E is indefinite but in any circle centered at the origin exists a domain in which E is positive. Hence—if along the solutions of (3)—$\dot{E}$ is positive definite, then the null solution of (3) is (Chetaev) unstable {cfr. [8], p. 227, theorem 7.4}.

3. Preliminary Lemmas

Lemma 2. *The quadratic polynomial $P_2 + P_3 = Q$ is*

i) *positive semidefinite either for*

(24) $$ac + bd = const., \quad \frac{d}{dt}(c^2 + d^2) \geq k_1, \quad \frac{d}{dt}(a^2 + b^2) \geq k_2, \quad \forall t \in \mathbb{R}^+$$

or for

(25) $$\frac{a}{c} = const., \quad \frac{b}{d} = const., \quad \frac{dc^2}{dt} \geq k_3, \quad \frac{dd^2}{dt} \geq k_4, \quad \forall t \in \mathbb{R}^+$$

or for

(26) $$\frac{a}{c} = const., \quad d = 0, \quad \frac{dc^2}{dt} \geq k_5, \quad \frac{db^2}{dt} \geq k_6, \quad \forall t \in \mathbb{R}^+$$

or for

$$c = 0, \quad \frac{b}{d} = const., \quad \frac{da^2}{dt} \geq k_7, \quad \frac{dd^2}{dt} \geq k_8, \quad \forall t \in \mathbb{R}^+ \tag{27}$$

with k_i, $(i = 1, \ldots, 8)$ *non negative constants;*

ii) *positive definite if* i) *holds and the constants* k_i *appearing—either in* (24) *or* (25) *or* (26) *or in* (27)*—are positive;*

iii) *negative semidefinite either for*

$$ac + bd = const., \quad \frac{d}{dt}(c^2 + d^2) \leq -k_1, \quad \frac{d}{dt}(a^2 + b^2) \leq -k_2, \quad \forall t \in \mathbb{R}^+ \tag{28}$$

or for

$$\frac{a}{c} = const., \quad \frac{b}{d} = const., \quad \frac{dc^2}{dt} \leq -k_3, \quad \frac{dd^2}{dt} \leq -k_4, \quad \forall t \in \mathbb{R}^+ \tag{29}$$

or for

$$\frac{a}{c} = const., \quad d = 0, \quad \frac{dc^2}{dt} \leq -k_5, \quad \frac{db^2}{dt} \leq -k_6, \quad \forall t \in \mathbb{R}^+ \tag{30}$$

or for

$$c = 0, \quad \frac{b}{d} = const., \quad \frac{da^2}{dt} \leq -k_7, \quad \frac{dd^2}{dt} \leq -k_8, \quad \forall t \in \mathbb{R}^+ \tag{31}$$

with k_i, $(i = 1, \ldots, 8)$ *non negative constants;*

iv) *negative definite if* iii) *hold and the constants* k_i *appearing—either in* (28) *or* (29) *or* (30) *or in* (31)*—are positive.*

Proof. In view of

$$P_2 + P_3 = x^2 \frac{d}{dt}(c^2 + d^2) + y^2 \frac{d}{dt}(a^2 + b^2) - 2xy \frac{d}{dt}(ac + bd) \tag{32}$$

in the case $ac + bd = \text{const.}$, i)–iv) immediately follow. In the other cases i)–iv) are implied by

$$\begin{cases} \left[\dfrac{d(ac)}{dt}\right]^2 - \dfrac{da^2}{dt}\dfrac{dc^2}{dt} = (\dot{a}c - a\dot{c})^2 \\ \dfrac{a}{c} = \text{const.} = k \Rightarrow P_2 = \dfrac{dc^2}{dt}(x - ky)^2 \\ a \equiv 0 \Rightarrow P_2 = \dfrac{dc^2}{dt}x^2; \; c \equiv 0 \Rightarrow P_2 = \dfrac{da^2}{dt}y^2 \end{cases} \tag{33}$$

concerned with P_2 and by

$$\begin{cases} \left[\dfrac{d(bd)}{dt}\right]^2 - \dfrac{db^2}{dt}\dfrac{dd^2}{dt} = (\dot{b}d - b\dot{d})^2 \\ \dfrac{b}{d} = \text{const.} = \tilde{k} \Rightarrow P_3 = \dfrac{dd^2}{dt}(x - \tilde{k}y)^2 \\ b \equiv 0 \Rightarrow P_3 = \dfrac{dd^2}{dt}x^2;\ d \equiv 0 \Rightarrow P_3 = \dfrac{db^2}{dt}y^2 \end{cases} \tag{34}$$

concerned with P_3.

LEMMA 3. *Let*

$$A_* > 0, \quad I_* > 0. \tag{35}$$

Then does not exist a positive constant h such that

$$P_1 \leq -h(x^2 + y^2), \quad \forall t \in \mathbb{R}^+ \tag{36}$$

and P_1 is semidefinite positive for

$$A \geq A_0 e^{-2I_* t}, \quad A_0 = A(0) \tag{37}$$

and definite positive, according to

$$P_1 \geq A_* I_* (x^2 + y^2), \tag{38}$$

for

$$\dot{A} \geq 0, \quad \forall t \in \mathbb{R}^+ \tag{39}$$

PROOF. Assume by contradiction that exists a positive constant h such that along the solutions of (3), (36) holds. Since (36) is equivalent to

$$2IA + \dot{A} \leq -h, \quad \forall t \in \mathbb{R}^+ \tag{40}$$

in view of (35), one obtains

$$2I_* A + \dot{A} \leq -h \tag{41}$$

and hence

$$A \leq \left[A_0 - \frac{h}{2I_*}(e^{2I_* t} - 1)\right] e^{-2I_* t} \tag{42}$$

But

$$t > \bar{t} = \frac{1}{2I_*}\log\left(1 + \frac{2I_* A_0}{h}\right) \Rightarrow A_0 < \frac{h}{2I_*}(e^{2I_* t} - 1) \tag{43}$$

i.e. $A(t) < 0$, $\forall t > \bar{t}$, which is not admissible for $(35)_1$. Passing to the semidefiniteness,

$$P_1 \geq 0, \quad \forall t \in \mathbb{R}^+ \Leftrightarrow 2IA + \dot{A} \geq 0 \tag{44}$$

and hence, in view of (35), one obtains

$$2I_* A + \dot{A} \geq 0 \Rightarrow P_1 \geq 0 \quad \forall t \in \mathbb{R}^+ \tag{45}$$

Obviously (39)—by virtue of (35)—implies (38).

Lemma 4. *Let*

$$A_* > 0, \quad A^* < \infty, \quad I^* < 0. \tag{46}$$

Then does not exist a positive constant h such that

$$P_1 \geq h(x^2 + y^2) \tag{47}$$

and P_1 is semidefinite negative for

$$A \leq A_0 e^{-2I_* t} \tag{48}$$

and negative definite, either according to

$$P_1 \leq -A_* |I_*| (x^2 + y^2) \tag{49}$$

for

$$\dot{A} \leq 0 \tag{50}$$

or according to

$$P_1 \leq -2\varepsilon A_* |I_*| (x^2 + y^2), \quad 0 < \varepsilon = const. < 1 \tag{51}$$

for

$$A \leq A_0 (1 - \varepsilon) e^{2|I_*|t}, \quad \forall t > 0. \tag{52}$$

Proof. Assume by contradiction that exists a positive h such that (47) holds, i.e.

$$2IA + \dot{A} \geq h \tag{53}$$

Then—in view of $(46)_3$—one obtains

$$-2|I^*|A + \dot{A} \geq h \tag{54}$$

and hence

$$A \geq \left(A_0 + \frac{h}{2|I^*|}\right) e^{2|I^*|t} - \frac{h}{2|I^*|} \tag{55}$$

which implies

$$\lim_{t\to\infty} A = +\infty \tag{56}$$

in contradiction with $(46)_2$.

Since

$$P_1 \leq 0, \quad \forall t \in \mathbb{R}^+ \Leftrightarrow 2IA + \dot{A} \leq 0, \quad \forall t \in \mathbb{R}^+ \tag{57}$$

and—in view of $(46)_3$—one obtains that (57) is implied by

$$2I^*A + \dot{A} \leq 0, \quad \forall t \in \mathbb{R}^+ \tag{58}$$

and hence by (48).

Finally (49) is immediately implied by (46) and (50). Observing that, in view of (46), one obtains

$$-2|I^*|A + \dot{A} < -h \Rightarrow 2IA + \dot{A} < -h, \tag{59}$$

integrating (59), with $h = 2\varepsilon A_0|I^*|$, one obtains (49) under the condition (52).

Lemma 5. *Let*

$$A^* < 0, \quad I^* < 0, \quad |A|^* < \infty. \tag{60}$$

Then does not exist a positive constant h such that (36) *holds. Further* P_1 *is semidefinite positive for*

$$A \geq A_0 e^{-2I^*t} \tag{61}$$

and positive definite according to

$$P_1 \geq A_* I_* (x^2 + y^2), \tag{62}$$

when (39) *holds.*

Proof. Assume by contradiction that exists a positive constant h such that (36) holds. In view of $(60)_1$, one obtains

$$2I|A| + \frac{d}{dt}|A| \geq h. \tag{63}$$

Then—by virtue of Lemma 4 with $|A|$ at the place of A—one obtains $\lim_{t\to\infty} |A| = \infty$ in contradiction with $(60)_3$. On the other hand in view of

$$2I^*A + \dot{A} \geq 0 \quad \forall t \in \mathbb{R}^+ \Rightarrow P_1 \geq 0 \quad \forall t \in \mathbb{R}^+ \tag{64}$$

either the positive semidefiniteness or the positive definiteness immediately follow by virtue either of (48) or (50).

4. Stability criteria

The main stability theorems of the Direct Method for nonautonomous systems {cfr [8], p. 226} guarantee that the existence of a positive definite function implies

i) stability if the temporal derivative along the solutions is semidefinite negative;
ii) asymptotic stability if admits an upper bound which is infinitely small at the origin and its temporal derivative along the solutions is negative definite.

Then by means of Lemmas 1–5, the following stability criteria immediately follow.

Theorem 1. *Let* (46) *and* (48) *hold together with the conditions guaranteing that* iii) *of Lemma* 2 *hold. Then the null solution of* (3) *is stable.*

Proof. Then in fact, V is positive definite and $\dot{V}$ semidefinite negative.

Theorem 2. *Let* (46) *and either* (50) *or* (52) *hold together with the conditions guaranteing that* iii) *of Lemma* 2 *hold. Then the null solution of* (3) *is asymptotically stable.*

Proof. Then in fact, V is positive definite while $\dot{V}$ is definite negative. Precisely, either by virtue of (50) or by virtue of (52), it turns out that

$$\dot{V} \leq -m(x^2 + y^2) \tag{65}$$

with

$$m = \begin{cases} A_*|I_*| & \text{in the case (50)} \\ 2\varepsilon A_*|I|_* & \text{in the case (52)} \end{cases} \tag{66}$$

Theorem 3. *Let*

$$A_* > 0, \quad I = 0, \quad \forall t \in \mathbb{R}^+, \tag{67}$$

hold together with the conditions guaranteing that iii) *of Lemma* 2 *hold. Then the null solution of* (3) *is stable for*

$$\dot{A} \leq 0, \quad \forall t \in \mathbb{R}^+ \tag{68}$$

and asymptotically stable for

$$(\dot{A})^* = -\tilde{k}, \quad \tilde{k} = const. > 0. \tag{69}$$

Proof. In fact then V is positive definite and

$$\dot{V} \leq \frac{1}{2}\dot{A}(x^2 + y^2). \tag{70}$$

Remark 2. *By virtue of Lemmas* 3–5 *and Theorems* 1–3, *apart from the case* $IA \equiv 0$, *the conditions* (8) *or the equivalent*

$$A_* > 0, \quad (AI)^* < 0, \tag{71}$$

appear to be basic conditions to require for guaranteing the stability of the null solution of (3) {*cfr also theorems* 6–11}. *This is also supported by the following two theorems.*

Theorem 4. *Let* (71) *hold by virtue of*

$$\begin{cases} (bc)^* < 0,\ a^* \le -h_1,\ d^* \le -h_2 \quad \forall t \in \mathbb{R}^+ \\ \inf(|b|_*, |c|_*) > 0 \end{cases} \tag{72}$$

h_i, $(i = 1, 2)$, *being positive constants. Then*

$$\begin{cases} |b| \le |b_0| e^{2h_1 t} \\ |c| \le |c_0| e^{2h_2 t} \end{cases} \tag{73}$$

guarantee the stability of the null solution of (3) *while*

$$\begin{cases} |b| \le |b_0| e^{2(h_1-\varepsilon)t} \\ |c| \le |c_0| e^{2(h_2-\varepsilon)t} \end{cases} \tag{74}$$

with $0 < \varepsilon = const. < \inf(h_1, h_2)$ *guarantee the exponential asymptotic stability.*

Proof. We give the proof in the case $\{b_* > 0, c^* < 0\}$.
Choosing

$$\mu_1 = |c|, \quad \mu_2 = |b| \tag{75}$$

it turns out that

$$\mu_1 b + \mu_2 c = -bc + bc = 0 \quad \forall t \in \mathbb{R}^+ \tag{76}$$

and (21) reduces to

$$\dot{E} = \frac{1}{2}\left[(\dot{\mu}_1 + 2a\mu_1)x^2 + (\dot{\mu}_2 + 2d\mu_2)y^2\right]. \tag{77}$$

On the other hand (73) guarantee that

$$\begin{cases} \dot{\mu}_1 + 2a\mu_1 \le 0 \\ \dot{\mu}_2 + 2d\mu_2 \le 0 \end{cases} \tag{78}$$

while (74) guarantee

$$\begin{cases} \dot{\mu}_1 + 2a\mu_1 \le \varepsilon a\mu_1 \le -\varepsilon h_1 |c|_* \\ \dot{\mu}_2 + 2d\mu_2 \le \varepsilon d\mu_2 \le -\varepsilon h_2 |b|_*. \end{cases} \tag{79}$$

Therefore $\dot{E} \leq 0$ in the case (73) and $\dot{E}$ is negative definite in the case (74). In view of

$$E \geq \frac{1}{2}\inf(|b|_*, |c|_*)(x^2 + y^2) \tag{80}$$

E is positive definite. Further—by virtue of (79), in the case (74) exists a positive constant m such that

$$\dot{E} \leq -mE \Leftrightarrow E \leq E_0 e^{-mt}. \tag{81}$$

Theorem 5. *Let* (71) *hold together with*

$$\begin{cases} (bc)_* > 0,\ a^* \leq -h_1,\ d^* \leq -h_2 \\ \inf(|b|_*, |c|_*) > 0 \end{cases} \tag{82}$$

h_i, $(i = 1, 2)$ *being positive constants. Then*

$$|b| \leq |b_0| e^{h_1 t}, \quad |c| \leq |c_0| e^{h_2 t} \tag{83}$$

guarantee the stability of the null solution of (3) *while*

$$|b| \leq |b_0| e^{-(h_1 - \varepsilon)t}, \quad |c| \leq |c_0| e^{-(h_2 - \varepsilon)t} \tag{84}$$

with $0 < \varepsilon = const. < \inf(h_1, h_2)$, *guarantee the asymptotic exponential stability.*

Proof. We begin by observing that (71)—in view of $(82)_1$—imply

$$a < 0, \quad d < 0, \quad bc < ad - A_*, \quad \forall t \in \mathbb{R}^+. \tag{85}$$

In view of (75) one obtains

$$\dot{E} = \frac{1}{2}[(\dot{\mu}_1 + 2a\mu_1)x^2 + (\dot{\mu}_2 + 2d\mu_2)y^2 + 2bc|xy|]. \tag{86}$$

By virtue of

$$2bcxy \leq 2\sqrt{\mu_1\mu_2}\sqrt{ad}|xy| \leq \mu_1|a|x^2 + \mu_2|d|y^2, \tag{87}$$

$\dot{E}$ reduces to

$$\dot{E} = \frac{1}{2}\left[(\dot{\mu}_1 + a\mu_1)x^2 + (\dot{\mu}_2 + d\mu_2)y^2\right]. \tag{88}$$

Hence (83) guarantee

$$\begin{cases} \dot{\mu}_1 + a\mu_1 \leq 0 \\ \dot{\mu}_2 + d\mu_2 \leq 0 \end{cases} \tag{89}$$

and (84) guarantee

$$\begin{cases} \dot{\mu}_1 + a\mu_1 \leq \varepsilon a\mu_1 \leq -\varepsilon h_1 \inf_{\mathbb{R}^+}|c| \\ \dot{\mu}_2 + d\mu_2 \leq \varepsilon d\mu_2 \leq -\varepsilon h_2 \inf_{\mathbb{R}^+}|b|. \end{cases} \tag{90}$$

Since (88)–(90) are completely analogous to (77)–(79), the proof can be completed by following—step by step—the proof of theorem 4.

Remark 3. *Also in the case* $\{a = 0, b = 1, (IA)^* < 0\}$ *can be convenient to introduce a suitable Liapunov function of type* (20). *This choice will be done in Section* 6.

Remark 4. *Obviously* (3) *cannot admit periodic solutions when the conditions guaranteing the asymptotic stability of the null solution hold.*

5. Instability criteria

The main instability theorems of the Direct Method for nonautonomous system {cfr [8], pp. 226–227, theorems 7.3–7.4} are

i) (*Liapunov instability theorem*). *If exists a function V such that it has an infinitely small upper limit and its derivative* $\dot{V}$ *along the solutions is definite and also if for* $t \geq t_0$ *with arbitrarily large to the function V can have the same sign as* $\dot{V}$ *in a neighborhood of* $x_1 = x_2 = 0$, *then the null solution is unstable.*
ii) (*Chetaiev instability theorem*). *If exists a function V taking positive values in any circle centered at* $(x = y = 0)$ *and if for all* $t \geq t_0$, *in which V is bounded and its derivative* $\dot{V}$ *along the solutions is positive definite, then the null solution is unstable.*

Then by means of the Lemmas 1–5 and the assumptions made on the coefficients a, b, c, d, the following instability criteria can be immediately obtained.

Theorem 6. *Let* (35) *and* (39) *hold together with the conditions guaranteing that* i) *of Lemma* 2 *hold. Then the null solution of* (3) *is unstable.*

Proof. V—in view of $(35)_1$—is positive definite. By virtue of (19), (39) and i) of Lemma 2 it follows that

$$\dot{V} \geq \frac{1}{2} A_* I_* (x^2 + y^2) \tag{91}$$

and—in view of (12)—

$$\dot{V} \geq mV \Leftrightarrow V \geq V_0 e^{mt} \tag{92}$$

with $m =$ positive const. In view of (16) and the Liapunov instability theorem, the instability immediately follows.

THEOREM 7. *Let* (60) *and* (39) *hold together with the conditions guaranteing that* i) *of Lemma* 2 *hold. Then the null solution of* (3) *is unstable.*

PROOF. V in this case is not positive definite but i) of Lemma 1 holds. Further (62), i) of Lemma 2 and (39) imply that $\dot{V}$ is positive definite. Hence—in view of $(17)_1$—the null solution of (3) is unstable by virtue of the Chetaiev instability theorem.

THEOREM 8. *Let*

$$I = 0, \quad \forall t \in \mathbb{R}^+ \tag{93}$$

hold together with the conditions guaranteing that i) *of Lemma* 2 *hold. Then the null solution of* (3) *is unstable if*

$$(\dot{A})_* \geq \tilde{k} \tag{94}$$

with $\tilde{k}$ = *positive constant.*

PROOF. In fact—by virtue of i) of Lemma 2, (13) implies

$$\dot{V} \geq \tilde{k}(x^2 + y^2) \tag{95}$$

i.e. $\dot{V}$ is positive definite. Then—in view of i) of Lemma 1 and the Chetaiev instability theorem—the null solution of (3) is unstable.

THEOREM 9. *Let*

$$A = 0 \quad \forall t \in \mathbb{R}^+ \tag{96}$$

hold together with ii) *of Lemma* 2*. Then the null solution of* (3) *is unstable.*

PROOF. By virtue of (96) V and $\dot{V}$ reduce respectively to

$$\begin{cases} V = \dfrac{1}{2}[(ay - cx)^2 + (by - dx)^2] \\ \dot{V} = \dfrac{1}{2}(P_2 + P_3) > \tilde{k}(x^2 + y^2) \end{cases} \tag{97}$$

with $\tilde{k}$ positive constant. By virtue of i) of Lemma 1 and $(17)_1$, the proof immediately follows by virtue of the Chetaiev instability theorem.

THEOREM 10. *Let* (60) *hold by virtue of*

$$\begin{cases} h_1 < a < h_2, \ b_* > 0, \ c_* > 0, \ d^* < -h_3, \ h_2 < h_3; \quad \forall t \in \mathbb{R}^+ \\ \inf(b_*, c_*) > 0 \end{cases} \tag{98}$$

with h_i, $(i = 1, 2, 3)$, positive constants. Then

$$\begin{cases} c > c_0 e^{-2(h_1-\varepsilon)t} \\ b < b_0 e^{2(h_3-\varepsilon)t} \end{cases} \tag{99}$$

with $0 < \varepsilon = const < \inf(h_1, h_3)$, guarantee the instability of the null solution of (3).

PROOF. Choosing

$$\mu_1 = c, \quad \mu_2 = -b \tag{100}$$

(20)–(21) reduce respectively to

$$E = \frac{1}{2}(cx^2 - by^2) \tag{101}$$

and

$$\dot{E} = \frac{1}{2}[(\dot{c} + 2ca)x^2 - (\dot{b} + 2bd)y^2]. \tag{102}$$

Since $(99)_1$–$(99)_2$ guarantee respectively

$$\begin{cases} \dot{c} + 2ca > 2\varepsilon c > 2\varepsilon c_* > 0 \\ \dot{b} + 2bd < -2\varepsilon b < -2\varepsilon b_* < 0, \end{cases} \tag{103}$$

all the conditions of the Chetaiev instability theorem are verified.

THEOREM 11. *Let* (35) *hold by virtue of*

$$\begin{cases} h_1 < a_*, \; b_* > 0, \; c^* < 0, \; h_2 < d_* \\ b_* > 0, \; c^* < 0 \end{cases} \tag{104}$$

with h_i, $(i = 1, 2)$, positive constants. Then

$$\begin{cases} b > b_0 e^{-2(h_1-\varepsilon)t} \\ c < c_0 e^{2(h_2-\varepsilon)t} \end{cases} \tag{105}$$

with $0 < \varepsilon = const < \inf(h_1, h_2)$, guarantee the instability of the null solution of (3).

PROOF. Choosing

$$\mu_1 = -c, \quad \mu_2 = b \tag{106}$$

(20)–(21) reduce respectively to

$$E = \frac{1}{2}(by^2 - cx^2) \tag{107}$$

and

$$\dot{E} = \frac{1}{2}[-(\dot{c} + 2ca)x^2 + (\dot{b} + 2bd)y^2]. \tag{108}$$

Since (104)–(105) guarantee

$$\begin{cases} \dot{c} + 2ca < 2\varepsilon c < 2\varepsilon c^* < 0 \\ \dot{b} + 2bd > 2\varepsilon b > 2\varepsilon b_* > 0, \end{cases} \tag{109}$$

E and $\dot{E}$ are both positive definite and all the conditions of the Liapunov instability theorem are verified.

Remark 5. *Obviously* (3) *cannot admit periodic solutions when conditions guaranteing either*

$$\lim_{t \to \infty} E = \infty \tag{110}$$

or

$$\lim_{t \to \infty} V = \infty \tag{111}$$

hold. For instance, when the assumption of theorem 10 *hold, E is positive definite and exists a positive number m such that, along the solution, it turns out that*

$$\dot{E} \geq mE. \tag{112}$$

Hence $E \geq E_0 e^{mt}$ and (3) *cannot admit periodic solutions.*

6. Stability-instability of the null solution of (1)

In view of (4), equation (1) is equivalent to

$$\begin{cases} \dot{x} = y \\ \dot{y} = -qx - py \end{cases} \tag{113}$$

It follows that

$$I = -p, \quad I_* = -p^*, \quad I^* = -p_*, \quad A = q \tag{114}$$

Choosing $\{\mu_1 = q, \mu_2 = 1\}$, (20)–(21) reduce respectively to

$$E = \frac{1}{2}(qx^2 + y^2) \tag{115}$$

and

$$\dot{E} = \frac{1}{2}(\dot{q}x^2 - 2py^2). \tag{116}$$

The following theorems hold

Theorem 12. *Let*

$$q_* > 0, \tag{117}$$

then the null solution of (1) *is stable for*

$$(\dot{q})^* \leq 0, \quad p_* \geq 0 \tag{118}$$

and asymptotically stable for

$$(\dot{q})^* < 0, \quad p_* > 0. \tag{119}$$

Proof. In fact, when (117) holds, E is definite positive. $\dot{E}$ is negative semidefinite when (118) holds and negative definite when (119) holds.

Theorem 13. *Let*

$$(\dot{q})_* > 0, \quad p^* < 0. \tag{120}$$

Then the null solution of (1) *is unstable.*

Proof. In fact E takes positive values for $(x = 0, y \neq 0)$ and $\dot{E}$—by virtue of (120)—is positive definite. Hence $(x = y = 0)$ is (Chetaiev) unstable.

Remark 6. *The Hill equation* $(p \equiv 0)$ *belongs to the case* $(I \equiv 0)$. *The stability-instability conditions for the zero solution can be easily obtained via the Liapunov function* (11). *In fact—in the case of the Hill equation—one easily obtain that* (11) *and* (13) *reduce respectively to*

$$V = \frac{1}{2}(1 + q)(qx^2 + y^2) \tag{121}$$

and

$$\dot{V} = \frac{1}{2}\dot{q}\big[(1 + 2q)x^2 + y^2\big]. \tag{122}$$

In view of (121)–(122) *it follows that*

i) *the stability is guaranteed by*

$$q_* > 0, \quad (\dot{q})^* \leq 0 \tag{123}$$

while the asymptotic stability is guaranteed by

$$q_* > 0, \quad (\dot{q})^* < 0; \tag{124}$$

ii) *the instability is guaranteed by*

$$(\dot{q})_* > 0. \tag{125}$$

7. Stability-instability criteria for the nonautonomous Lotka-Volterra system

The nonautonomous Lotka-Volterra system can be written

$$\begin{cases} \dot{x} = \alpha(t)x - \beta(t)xy \\ \dot{y} = -\gamma(t)y + \delta(t)xy \end{cases} \tag{126}$$

with α, β, γ, δ positive function of $t \in \mathbb{R}^+$. The system (126) admits the solution

$$\bar{x} = \frac{\gamma(t)}{\delta(t)}, \quad \bar{y} = \frac{\alpha(t)}{\beta(t)}. \tag{127}$$

Setting

$$x = \bar{x} + \xi, \quad y = \bar{y} + \eta \tag{128}$$

one obtains

$$\begin{cases} \dfrac{d\xi}{dt} = -\dfrac{\beta}{\delta}\gamma\eta - \beta\xi\eta \\[2ex] \dfrac{d\eta}{dt} = \dfrac{\delta}{\beta}\alpha\xi + \delta\xi\eta. \end{cases} \tag{129}$$

The (linear) stability of (129) is then governed by $\ddot{\xi} + \alpha\gamma\xi = 0$ i.e. by

$$\begin{cases} \dfrac{d\xi}{dt} = b\eta \\[2ex] \dfrac{d\eta}{dt} = c\xi \end{cases} \tag{130}$$

with

$$b = -\frac{\beta}{\delta}\gamma, \quad c = \frac{\delta}{\beta}\alpha, \quad I = 0, \quad A = \gamma\alpha \tag{131}$$

i.e. by $\ddot{\xi} + \alpha\gamma\xi = 0$. Then the (linear) stability-instability of $(\bar{x}, \bar{y})$ can be studied by means of theorem 3, 8 and remark 6.

We end by observing that, in the case at hand, one can also use the function of type (20)

$$E = \frac{1}{2}(c\xi^2 - b\eta^2) \tag{132}$$

either for the linear or for the nonlinear stability.

8. Final remarks

i) By means of the functions (11) and (20) criteria guaranteing either the stability or the instability of the null solution of the general nonautonomous system (3) have been obtained. As far as we know these criteria appear to be new in the existing literature.

ii) Apart from the case $IA \equiv 0$, the uniform Hurwitz conditions (8)—by virtue of the Lemmas 3–5 and Theorems 1–2, 4–5—appear to be the basic necessary conditions that one has to require for obtaining the "best" stability conditions.

iii) In the autonomous case, the Hurwitz conditions are, without any other condition, also sufficient for the stability of the null solution of (3). Apart from the case $IA \equiv 0$, in the nonautonomous case the problem of showing if exists a general class of binary differential system for which (8)—by alone—are not only necessary but also sufficient for guaranteing the stability, arises.

iv) More general stability-instability criteria can be obtained by requiring that $P_1 + P_2 + P_3$ is either negative or positive definite respectively. This will be done in a forthcoming paper.

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