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Functional Analysis — *Recognizing the Farey-Stern-Brocot AF algebra*, by DANIELE MUNDICI.

Dedicated to the memory of Renato Caccioppoli

ABSTRACT. — In his 2008 paper published in the Canadian Journal of Mathematics, F. Boca investigates an AF algebra $\mathfrak{A}$, whose Bratteli diagram arises from the Farey-Stern-Brocot sequence. It turns out that $\mathfrak{A}$ coincides with the AF algebra $\mathfrak{M}_1$ introduced in 1988 by the present author in a paper published in Advances in Mathematics. We give a procedure to recognize $\mathfrak{A}$ among all finitely presented AF algebras whose Murray-von Neumann order of projections is a lattice. Further: (i) $\mathfrak{A}$ is a *-subalgebra of Glimm universal algebra; (ii) tracial states of $\mathfrak{A}$ correspond to Borel probability measures on the unit real interval; (iii) all primitive ideals of $\mathfrak{A}$ are essential; (iv) the automorphism group of $\mathfrak{A}$ has exactly two connected components.

KEY WORDS: Approximately finite dimensional algebra; Elliott classification; Grothendieck group; lattice-ordered group; Farey sequence.

AMS SUBJECT CLASSIFICATION: Primary: 46L05. Secondary: 46L80, 46M40, 06F20, 11A55, 11B57, 37E05, 52C45, 52B20, 14E05, 14M25.

1. INTRODUCTION

For each $m = 1, 2, \dots$ let $\Phi(m) = 1 + 2^{m-1}$. The sequence $M_1, M_2, \dots$ of $\{0, 1\}$-matrices with $\Phi(m+1)$ rows and $\Phi(m)$ columns is defined by:

$$M_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdots$$

where for each $j = 1, \ldots, 2^{m-1}$ the $2j$th row of M_m has a 1 in positions j, $j+1$, the $(2j-1)$th row has a 1 in position j, and the last row has all 0 except a final 1. The map $x \in \mathbb{Z}^{\Phi(m)} \mapsto M_m x \in \mathbb{Z}^{\Phi(m+1)}$ is a positive one-one homomorphism of the simplicial group $\mathbb{Z}^{\Phi(m)}$ into $\mathbb{Z}^{\Phi(m+1)}$ (see [15], [9, p. 15] for this terminology). Equipping $\mathbb{Z}^{\Phi(1)}$ with the order unit $u_1 = (1,1)$ it follows that the element $u_t = M_t M_{t-1} \cdots M_1 u$ is an order unit in $\mathbb{Z}^{\Phi(t)}$, $t = 1, 2, \ldots$. (Note that [2] writes "unité forte" for "order unit"). Thus the limit of the direct system $\{((\mathbb{Z}^{\Phi(m)}, u_m), M_m), \mid m = 1, 2, \ldots\}$ of unital simplicial groups is a unital dimension group [9], [15]. From the unital positive homomorphisms M_m one obtains the Bratteli diagram $\mathscr{D}$ of a unital AF algebra by the following familiar construction, [9, Chapter 2], [14]: (i) the number of vertices of $\mathscr{D}$ at depth $d = 0, 1, 2, \ldots$ equals the number of columns in M_{d+1}; (ii) the ith vertex of $\mathscr{D}$ at depth $d+1$ is connected to the jth vertex at depth d iff the entry of M_d in row j and column i is 1; (iii) the two top vertices have multiplicity 1, and the multiplicity of every vertex v at depth $d = 1, 2, \ldots$ is the sum of the multiplicities of the vertices at depth $d-1$ connected to v.

We refer to [2] and [12] for background on ℓ-groups (where "ℓ" is short for "lattice-ordered abelian"). For $n = 1, 2, \ldots$ we let $\mathcal{M}_n$ be the unital ℓ-group of all continuous functions $f : [0,1]^n \to \mathbb{R}$ with the following property: there are (affine) linear polynomials $p_1, \ldots, p_m$ with integer coefficients, such that for all $x \in [0,1]^n$ there is $i \in \{1, \ldots, m\}$ with $f(x) = p_i(x)$. $\mathcal{M}_n$ is equipped with the pointwise operations $+, -, \wedge, \vee$ of $\mathbb{R}$, and with the constant function 1 as the distinguished order unit.

The universal property of $\mathcal{M}_n$ is given by the following characterization:

Lemma 1.1 ([22, 4.16]). *The unital ℓ-group $\mathcal{M}_n$ is generated by the coordinate functions $x_i : [0,1]^n \to [0,1]$ together with the order unit given by the constant function* 1. *For every unital ℓ-group (G, u) and elements $g_1, \ldots, g_n$ in the unit interval $[0, u]$ of G, if $g_1, \ldots, g_n, u$ generate G, then there is a unique unital ℓ-homomorphism ψ of $\mathcal{M}_n$ onto G such that $\psi(x_i) = g_i$ for each $i = 1, \ldots, n$.*

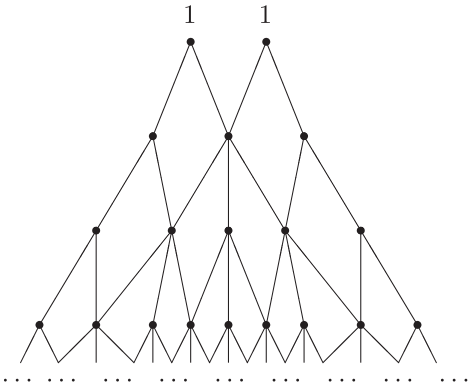
We refer to [14] and [9] for K_0 of AF algebras and Elliott classification. In [24] the present author introduced the unital AF algebra $\mathfrak{M}_1$ by the stipulation

$$(K_0(\mathfrak{M}_1), [1_{\mathfrak{M}_1}]) = (\mathcal{M}_1, 1), \tag{1}$$

and proved ([24, 3.3]) that $\lim((\mathbb{Z}^{\Phi(m)}, u_m), M_m) = (\mathcal{M}_1, 1)$, Therefore, $\mathscr{D}$ is the Bratteli diagram of $\mathfrak{M}_1$.

* * *

In [5, p. 977], F. Boca defines the unital AF algebra $\mathfrak{A}$ by the following Bratteli diagram:



The two depth 0 vertices a and b have a label 1, $L(a) = L(b) = 1$. The label $L(v)$ of any vertex v at depth $d = 1, 2, \ldots$ is the sum of the labels of the vertices at depth $d - 1$ connected to v by an edge. Replacing each vertex v by the matrix algebra $\mathbb{M}_{L(v)}$, the unital finite-dimensional C^*-algebra $\mathfrak{A}_d$ is defined as the direct sum of the matrix algebras lying at depth d, for each $d = 0, 1, 2, \ldots$. As explained in [9, Chapter 2] and [14, §17], the edges of the above diagram now determine a unital *-homomorphism $\phi_d : \mathfrak{A}_d \to \mathfrak{A}_{d+1}$. Finally, the unital AF algebra $\mathfrak{A}$ is defined by $\mathfrak{A} = \lim(\mathfrak{A}_d, \phi_d)$. Direct inspection shows that $\mathscr{D}$ coincides with the above diagram of $\mathfrak{A}$, whence

$$\mathfrak{A} = \mathfrak{M}_1, \tag{2}$$

because Bratteli diagrams completely characterize their associated AF algebras.

By (1), $\mathfrak{A}$ is an *AFℓ algebra*, i.e., an AF algebra whose Murray-von Neumann order of projections is a lattice. These algebras are a generalization of AF algebras with comparability of projections in the sense of Murray-von Neumann, [10]. Concrete examples of AFℓ algebras include the Effros-Shen algebras $\mathfrak{F}_\theta$ [9, p. 65], Blackadar's simple AF algebra B of [3, p. 504], various (Behnke-Leptin) C^*-algebras with finite primitive ideal spaces, including the unital AF algebras $\mathscr{A}_{k,n}$ of [1], $(n = 1, 2, \ldots; k = 0, \ldots, n-1)$, the liminary C^*-algebras considered in [8], and the C^*-algebras of [28]. All primitive ideals I of $\mathfrak{A}$, as well as all primitive quotients $\mathfrak{A}/I$ are further examples of AFℓ algebras.

2. Recognizing $\mathfrak{A}$ among finitely presented AFℓ algebras

We refer to [21] for all unexplained notions involving algorithms and effective decidability. On page 55 of his book [4], Blackadar writes:

> *one major problem restricts the usefulness of the study of AF C^*-algebras by diagrams: many quite different diagrams yield isomorphic algebras, and there is no known reasonable algorithm for determining when two diagrams give isomorphic algebras.*

If we give "reasonable algorithm" its usual mathematical meaning of "Turing computable function", this remark necessarily deals with AF algebras having Turing computable Bratteli diagrams. Then Rice's theorem [21, 5.20] confirms that no Turing machine can decide whether two Turing machines compute the Bratteli diagrams of isomorphic AF algebras.

Decision problems for classes of AF algebras having less general presentations than Bratteli diagrams, are increasingly considered in the literature. Among the possible codings of an AF algebra $\mathfrak{B}$ by a finite string of symbols let us mention

(i) the traditional presentations by generators and relations of the dimension group $K_0(\mathfrak{B})$, [28];
(ii) presentations by integer matrices, [6], [7], by abstract simplicial complexes [26], by formulas in Łukasiewicz logic [23];
(iii) Boca's presentation of $\mathfrak{A}$, [5, §6].

In this section we will prove the recognizability of $\mathfrak{A}$ among all AFℓ algebras having a presentation of type (i). To fix ideas, let us recall that a (*unital ℓ-group*) *term* $\tau = \tau(X_1, \dots, X_n)$ is a string of symbols obtained from the variable symbols X_i and the constant symbols u and 0, by a finite number of applications of the ℓ-group operation symbols $+$, $-$, $\vee$, $\wedge$. The map sending X_i to the ith coordinate function $x_i : [0,1]^n \to [0,1]$ and sending the symbol u to the constant function 1, canonically extends to a map interpreting each term τ as a function $f_\tau \in \mathcal{M}_n$. Denoting by $\langle f_\tau \rangle$ the ideal of $\mathcal{M}_n$ generated by f_τ, and taking the quotient in the usual way [2], we obtain from τ the unital ℓ-group $\mathcal{M}_n/\langle f_\tau \rangle$.

The term τ is said to be a *presentation* of the unital ℓ-group $\mathcal{M}_n/\langle f_\tau \rangle$, (by generators $X_1, \dots, X_n$ and the relation $\tau = 0$). Our restriction to just one relation is immaterial: as a matter of fact, any finite set of relations $\{\tau_1 = \rho_1, \dots, \tau_m = \rho_m\}$ is always reducible to the single relation $|\tau_1 - \rho_1| \vee \cdots \vee |\tau_m - \rho_m| = 0$. Since unital dimension groups are in one-one correspondence with AF algebras via K_0, it is natural to say that τ is a *presentation* of the unital AFℓ algebra $\mathfrak{B}$ defined by $(K_0(\mathfrak{B}), 1_{\mathfrak{B}}) = \mathcal{M}_n/\langle f_\tau \rangle$. An AF$\ell$ algebra is said to be *finitely presented* if it can be presented by a term.

Examples. It is not hard to see that all finite-dimensional C^*-algebras are finitely presented. A large class of nontrivial examples is provided by all unital AFℓ algebras $\mathfrak{P}$ such that $(K_0(\mathfrak{P}), 1_{[\mathfrak{P}]})$ is finitely generated *projective*. This is so because any such unital ℓ-group is finitely presented, [27, Proposition 5].

The universal property of Lemma 1.1 ensures that $\mathcal{M}_1$ is projective in the category of unital ℓ-groups. Here morphisms are *unital ℓ-homomorphisms*, i.e., homomorphisms that also preserve the order unit and the lattice structure. It follows that $\mathfrak{A}$ is finitely presented.

If an AFℓ algebra $\mathfrak{B}$ is finitely presented, then so is the n-fold "free product" AFℓ algebra $\mathfrak{B}^{[[n]]}$, defined by

$$(K_0(\mathfrak{B}^{[[n]]}), [1_{\mathfrak{B}^{[[n]]}}]) = (K_0(\mathfrak{B}), [1_{\mathfrak{B}}]) \amalg \cdots \amalg (K_0(\mathfrak{B}), [1_{\mathfrak{B}}]), \tag{3}$$

where II denotes free product in the category of unital ℓ-groups, [25]. In particular, $\mathfrak{A}^{[[n]]}$ is finitely presented, for each n.

Recalling the initial quotation of this section, our first problem now is to decide if two terms ρ and τ are presentations of the same AFℓ algebra. While terms are very simple strings of symbols, we have the following

THEOREM 2.1. *There is no algorithm to decide if two terms* $\rho(Y_1, \ldots, Y_n)$ *and* $\tau(X_1, \ldots, X_m)$ *are presentations of* stably *isomorphic AFℓ algebras.*

PROOF. Following [31], let us say that a *rational polyhedron* in the n-cube $[0,1]^n$ is a finite union of *rational* simplexes $T_1, \ldots, T_k \subseteq [0,1]^n$, in the sense that the coordinates of the vertices of each T_i are rational. Following [13] we first reduce the PL-homeomorphism problem $\mathscr{P}_1$ of rational polyhedra to the isomorphism problem $\mathscr{P}_2$ for finitely presented (possibly non-unital) ℓ-groups. Markov's celebrated unrecognizability result [20, p. 143–144] shows that $\mathscr{P}_1$ is undecidable, whence so is $\mathscr{P}_2$. The proof is concluded by recalling that $\rho(Y_1, \ldots, Y_n)$ and $\tau(X_1, \ldots, X_m)$ are presentations of *stably* isomorphic AFℓ algebras $\mathfrak{B}$ and $\mathfrak{C}$ iff the (possibly non-unital) ℓ-groups $K_0(\mathfrak{B})$ and $K_0(\mathfrak{C})$ are isomorphic. For more details see [26, 6.1]. □

Despite Markov's undecidability result, the variant of Theorem 2.1 with "isomorphic" in place of "stably isomorphic" is an open problem: as a matter of fact, already the *recognizability* of the unital AFℓ-algebra $\mathfrak{A}^{[[n]]}$ defined as in (3) is open for each $n = 2, 3, \ldots$. By a quirk of fate, for $n = 1$ we have

THEOREM 2.2. $\mathfrak{A}$ *is recognizable among finitely presented AFℓ algebras. In other words, the problem whether a term* $\tau(X_1, \ldots, X_n)$ *is a presentation of* $\mathfrak{A}$*, is Turing-decidable.*

PROOF. For each closed set $\emptyset \neq Y \subseteq [0,1]^n$ we denote by $\mathcal{M}_n \restriction Y$ the unital ℓ-group of restrictions to Y of the functions in $\mathcal{M}_n$, with the order unit given by the constant function 1 on Y. In symbols,

$$\mathcal{M}_n \restriction Y = \{g \restriction Y \mid g \in \mathcal{M}_n\}.$$

For every $f \in \mathcal{M}_n$ let $Z_f = f^{-1}(0)$ denote the zeroset of f.

CLAIM 1. Fix an element f of $\mathcal{M}_n$ and suppose $Z_f \neq \emptyset$. Then, letting g range over all elements of $\mathcal{M}_n/\langle f \rangle$, it follows that the map $g/\langle f \rangle \mapsto g \restriction Z_f$ is an isomorphism of the unital ℓ-groups $\mathcal{M}_n/\langle f \rangle$ and $\mathcal{M}_n \restriction Z_f$.

One argues as in the proof of Baker theorem for finitely presented ℓ-groups (possibly without order unit), [12, 5.2.2]. The appropriate universal property of $\mathcal{M}_n$ is ensured by Lemma 1.1. See the proof of [19, 5.1, 5.2] and [27, Proposition 4] for further details.

CLAIM 2. The term τ is a presentation of $\mathfrak{A}$ iff $\mathcal{M}_n \restriction Z_{f_\tau}$ is unitally ℓ-isomorphic to $\mathcal{M}_1$, in symbols, $\mathcal{M}_n \restriction Z_{f_\tau} \cong \mathcal{M}_1$.

As a matter of fact, by Claim 1 we can identify $\mathcal{M}_n/\langle f_\tau\rangle$ and $\mathcal{M}_n \upharpoonright Z_{f_\tau}$. In the light of (1)–(2), the desired conclusion now follows by combining [19, 5.1, 5.2] with [27, Proposition 4].

CLAIM 3. $\mathcal{M}_1 \cong \mathcal{M}_n \upharpoonright Z_{f_\tau}$ iff the real unit interval $[0,1]$ is $\mathbb{Z}$-*homeomorphic* to Z_{f_τ}, in the sense that there is a homeomorphism $\eta = (\eta_1,\dots,\eta_n) : [0,1] \to Z_{f_\tau}$ such that each η_i belongs to $\mathcal{M}_1$, and η^{-1} belongs to $\mathcal{M}_n \upharpoonright Z_{f_\tau}$.

This is an instance of the duality between finitely presented unital ℓ-groups and rational polyhedra. For details see the proof of [19, 6.5].

Our first three claims are to the effect that a term $\tau(X_1,\dots,X_n)$ is a presentation of $\mathfrak{A}$ iff the rational polyhedron Z_{f_τ} is $\mathbb{Z}$-homeomorphic to $[0,1]$. A routine induction on the number of operation symbols in τ shows that the transformation $\tau \mapsto Z_{f_\tau}$ can be effectively computed: the output is a list of rational simplexes $T_1,\dots,T_r$ with $Z_{f_\tau} = T_1 \cup \cdots \cup T_r$, where each T_i is presented by its vertices. Thus, to conclude the proof, we must give an effective procedure to decide if the rational polyhedron Z_{f_τ} is $\mathbb{Z}$-homeomorphic to $[0,1]$.

To this purpose we prepare some elementary material from polyhedral topology, [11, 31]. For any rational point $x \in \mathbb{R}^n$ we denote by $\mathrm{den}(x)$ the least common denominator of the coordinates of x, and we say for short that $\mathrm{den}(x)$ is the *denominator* of x. The vector $\tilde{x} = \mathrm{den}(x)(x,1) \in \mathbb{Z}^{n+1}$ is called the *homogeneous correspondent* of x. Conversely, for every integer vector $y = (y_1,\dots,y_{n+1}) \in \mathbb{Z}^{n+1}$ with $y_{n+1} > 0$ and $0 \le y_1,\dots,y_n \le y_{n+1}$, the rational point $\left(\frac{y_1}{y_{n+1}},\dots,\frac{y_n}{y_{n+1}}\right) \in [0,1]^n$ is said to be the *affine correspondent* of y. For every rational m-simplex $T = \mathrm{conv}(v_0,\dots,v_m) \subseteq [0,1]^n$, the *cone* $T^\uparrow \subseteq \mathbb{R}^{n+1}$ is defined by $T^\uparrow = \mathbb{R}_{\ge 0}\tilde{v}_0 + \cdots + \mathbb{R}_{\ge 0}\tilde{v}_m$. We say that the simplicial complex Ω is *rational* if all simplexes of Ω are rational: in this case, the complex of cones $\Omega^\uparrow = \{T^\uparrow \mid T \in \Omega\}$ is known as a *simplicial fan* [11, 29]: letting $\mathrm{vert}(\Omega)$ denote the set of vertices (of all simplexes) of Ω, the *primitive generating vectors* of $\Omega^\uparrow$ are precisely the homogeneous correspondents of elements of $\mathrm{vert}(\Omega)$. Following [32] and [11], a simplex $U = \mathrm{conv}(w_0,\dots,w_m) \subseteq \mathbb{R}^n$ is said to be *regular* if it is rational and the set of integer vectors $\{\tilde{w}_0,\dots,\tilde{w}_m\}$ can be extended to a basis of the free abelian group $\mathbb{Z}^{n+1}$. A rational, simplicial complex Ω is said to be *regular* if all simplexes in Ω are regular. In other words, $\Omega^\uparrow$ is a *regular fan*, [11] ("nonsingular fan" in [29]). For every simplicial complex Ω, its *support* $|\Omega|$ is the pointset union of all simplexes of Ω. Given two simplicial complexes Ω' and Ω with the same support, we say that Ω' is a *subdivision* of Ω if every simplex of Ω' is contained in a simplex of Ω. For any $c \in |\Omega| \subseteq \mathbb{R}^n$, the *blow-up of* Ω *at* c is the subdivision of Ω given by replacing every simplex $C \in \Omega$ that contains c by the set of all simplexes of the form $\mathrm{conv}(F \cup \{c\})$, where F is any face of C that does not contain c (see [32, p. 376], [11, III, 2.1]). For any regular m-simplex $T = \mathrm{conv}(v_0,\dots,v_m) \subseteq \mathbb{R}^n$, the *Farey mediant of* T is the rational point c of T whose homogeneous correspondent $\tilde{c}$ coincides with $\tilde{v}_0 + \cdots + \tilde{v}_m$. If T belongs to a regular complex Δ and c is the Farey mediant of T, then the blow-up $\Delta_{(c)}$ of Δ at c is called a *Farey blow-up*. Direct inspection shows that $\Delta_{(c)}$ is regular.

We are now ready to give the promised decision procedure. If the zeroset Z_{f_τ} is *not homeomorphic* to $[0, 1]$ (a property which can be effectively checked by direct inspection of Z_{f_τ}), we declare that τ is not a presentation of $\mathfrak{A}$. Otherwise, Z_{f_τ} has the form of a broken line contained in the n-cube. We choose an orientation of Z_{f_τ}, and denote by α and ω the initial and final vertices of Z_{f_τ}. Let $\alpha = v_0, v_1, \ldots, v_k, v_{k+1} = \omega$ be the list of all *nodes* (= nondifferentiability points) of Z_{f_τ}. By definition of f_τ, all these points are rational, and their list can be effectively computed from the term τ. On the support Z_{f_τ} there is a unique (one-dimensional, rational) simplicial complex Ω_{f_τ} having as its 1-simplexes the segments $\mathrm{conv}(v_i, v_{i+1})$, $i = 0, \ldots, k$. If it is not the case that $\mathrm{den}(\alpha) = \mathrm{den}(\omega) = 1$ we declare that τ is not a presentation of $\mathfrak{A}$, otherwise we proceed to construct the *minimal regular subdivision* ∇_τ of Ω_{f_τ}. ∇_τ is the affine version of the Hirzebruch-Jung [17, 18] continued fraction algorithm. The construction of ∇_τ proceeds through the following steps, [29, p. 24]:

(i) For each $i = 0, \ldots, k$, letting $\tilde{v}_i$, $\tilde{v}_{i+1}$ be the homogeneous correspondents of v_i, v_{i+1}, we compute the *finite* set $X \subseteq \mathbb{Z}^{n+1}$ of nonzero integer points of the triangle $\mathrm{conv}(0, \tilde{v}_i, \tilde{v}_{i+1}) \subseteq \mathbb{R}^{n+1}$; we then let $\Theta = \mathrm{conv}(X) \subseteq \mathbb{R}^{n+1}$.

(ii) Let $l_0 = \tilde{v}_i, l_1, \ldots, l_s, l_{s+1} = \tilde{v}_{i+1}$ in this order be the points of $\mathbb{Z}^{n+1}$ lying in the edges of the boundary polygon of Θ, other than those lying in the relative interior of the edge $\mathrm{conv}(\tilde{v}_i, \tilde{v}_{i+1})$.

(iii) Let $\tilde{\nabla}_{\tau,i}$ be the fan whose primitive generating vectors are the vectors $l_0 = \tilde{v}_i, l_1, \ldots, l_s, l_{s+1} = \tilde{v}_{i+1}$. Let the fan $\tilde{\nabla}_\tau$ be the union of all $\tilde{\nabla}_{\tau,i}$, for $i = 0, \ldots, k$.

(iv) We finally define ∇_τ to be the subdivision of Ω_{f_τ} obtained from $\tilde{\nabla}_\tau$ by taking the affine correspondents of all primitive generating vectors in $\tilde{\nabla}_\tau$.

In [29, 1.19] it is shown that $\tilde{\nabla}_\tau$ is a regular fan, whence ∇_τ is a regular subdivision of Ω_{f_τ}. Direct inspection shows that the map $\tau \mapsto \nabla_\tau$ is effectively computable. Let $\mathrm{den}(\nabla_\tau)$ denote the naturally ordered list of denominators of the vertices of ∇_τ. Let $d = \max(\mathrm{den}(\nabla_\tau))$ be the largest such denominator. One can verify that $d = \max(\mathrm{den}(v_0), \ldots, \mathrm{den}(v_{k+1}))$. If the denominator of the Farey mediant of every 1-simplex of ∇_τ is $> d$ we say that ∇_τ is *d-saturated.* Otherwise, by blowing-up ∇_τ at some Farey mediant c with denominator $\leq d$, we obtain the regular subdivision $\nabla' = \nabla_{\tau_{(c)}}$ of ∇_τ. Similarly, by blowing-up ∇' at some Farey mediant d of ∇' with denominator $\leq d$, we obtain the regular subdivision $\nabla'' = \nabla'_{(c)}$ of ∇'. The sequence ∇_τ, ∇', ∇'', ∇''', ... of Farey blow-ups at Farey mediants with denominator $\leq d$ must terminate, because the n-cube contains only finitely many points with denominator $\leq d$. The final outcome is a d-saturated regular subdivision ∇^*_τ of ∇_τ. Note that ∇^*_τ is uniquely determined by ∇_τ, and the map $\tau \mapsto \nabla^*_\tau$ is effectively computable. Let $\mathrm{den}(\nabla^*_\tau)$ denote the naturally ordered list of denominators of the vertices in ∇^*_τ.

CLAIM 4. Let x, y be two consecutive vertices of ∇^*_τ. Then the denominator of every rational point z lying in the relative interior of the segment $\mathrm{conv}(x, y)$ is $> d$.

As a matter of fact, the regularity of the simplex ∇^*_τ ensures that the homogeneous correspondent $\tilde{z}$ of z is a linear combination of $\tilde{x}$ and $\tilde{y}$ with integer coefficients ≥ 1. Since ∇_τ^* is d-saturated, the denominator of the Farey mediant m of x and y is $> d$. One now observes that $\tilde{m} = \tilde{x} + \tilde{y}$, whence $\mathrm{den}(z) \geq \mathrm{den}(m) > d$, which settles our claim.

Let F_d be the dth *Farey sequence*, i.e., the naturally ordered list of all rationals in $[0, 1]$ having denominator $\leq d$. Then F_d is the set of vertices of a unique subdivision $\mathscr{F}_d$ of $[0, 1]$. As a reformulation of the unimodularity property of F_d, $\mathscr{F}_d$ is regular. Let $\mathrm{den}(\mathscr{F}_d)$ be the naturally ordered list of denominators of the vertices of $\mathscr{F}_d$.

CLAIM 5. If $\mathrm{den}(\mathscr{F}_d)$ coincides with $\mathrm{den}(\nabla_\tau^*)$, then Z_{f_τ} is $\mathbb{Z}$-homeomorphic to $[0, 1]$.

For any two consecutive points $v_1, v_2 \in F_d = \mathrm{vert}(\mathscr{F}_d)$, letting w_1, w_2 be the points in the same position in $\mathrm{vert}(\nabla_\tau^*)$, it is sufficient to exhibit a *linear* $\mathbb{Z}$-homeomorphism of the rational 1-simplex $S = \mathrm{conv}(v_1, v_2)$ onto $T = \mathrm{conv}(w_1, w_2)$ mapping v_1 into w_1 and v_2 into w_2. Writing $e_1 = (1, 0)$ and $e_2 = (0, 1)$ for the standard basis vectors of $\mathbb{R}^2$, let the 1-simplex $E \subseteq \mathbb{R}^2$ be defined by $E = \mathrm{conv}(e_1/\mathrm{den}(v_1), e_2/\mathrm{den}(v_2))$. Then it is easy to exhibit a linear $\mathbb{Z}$-homeomorphism χ of S onto E such that $\chi(v_i) = e_i/\mathrm{den}(v_i)$ for all $i = 1, 2$. We will construct a linear $\mathbb{Z}$-homeomorphism ξ of T onto E such that, for all $i = 1, 2$, $\xi(w_i) = e_i/\mathrm{den}(w_i) = e_i/\mathrm{den}(v_i)$. For $i \in \{1, 2\}$ there are uniquely determined integers $a_{1i}, \ldots, a_{ni}$ such that the homogeneous correspondent $\tilde{w}_i \in \mathbb{Z}^{n+1}$ of w_i can be displayed as $\tilde{w}_i = (a_{1i}, \ldots, a_{ni}, \mathrm{den}(w_i))$. Let Q be the $n \times (n+1)$ integer matrix whose rightmost $n-1$ columns are all zero, and whose ith column coincides with $a_{1i}, \ldots, a_{ni}$ for each $i = 1, 2$. By construction, Q sends $e_i/\mathrm{den}(w_i)$ to w_i, and the restriction $Q \restriction E$ is a one-one linear map of E onto T with integer coefficients. There remains to be proved that the inverse of $Q \restriction E$ has integer coefficients, too. To this purpose, let $\hat{T} \subseteq \mathbb{R}^n$ be a (positively oriented) regular n-simplex such that T is a face of $\hat{T}$. We can write $\hat{T} = \mathrm{conv}(w_1, w_2, \ldots, w_{n+1})$. Let M be the $(n+1) \times (n+1)$ integer matrix whose jth column $(j = 1, \ldots, n+1)$ coincides with the homogeneous correspondent $\tilde{w}_j$ of the jth vertex w_j of $\hat{T}$. The regularity of $\hat{T}$ ensures that the inverse $N = M^{-1}$ is an integer matrix. It follows that $N\tilde{w}_j = e_j$ and $N\tilde{w}_j/\mathrm{den}(w_j) = N(w_j, 1) = e_j/\mathrm{den}(w_j)$. Thus the homogeneous linear map N determines a one-one linear (affine) map, whose restriction ξ to T has the desired properties to settle our claim.

CLAIM 6. If there is a $\mathbb{Z}$-homeomorphism $\eta : [0, 1] \to Z_{f_\tau}$ then $\mathrm{den}(\mathscr{F}_d)$ coincides with $\mathrm{den}(\nabla_\tau^*)$.

By definition of $\mathbb{Z}$-homeomorphism, η preserves the denominator of each rational point of $[0, 1]$. By Claim 4, the set of rational points $\eta(F_d)$ is contained in $\mathrm{vert}(\nabla_\tau^*)$. Actually, $\eta(F_d) = \mathrm{vert}(\nabla_\tau^*)$, for otherwise, letting $x \in \mathrm{vert}(\nabla_\tau^*) \setminus \eta(F_d)$, the point $\eta^{-1}(x)$ would contradict the definition of F_d. This settles our last claim.

Summing up, the term τ is a presentation of $\mathfrak{A}$ iff the two lists of integers $\mathrm{den}(\mathscr{F}_d)$ and $\mathrm{den}(\nabla_\tau^*)$ are identical. This latter property can be easily verified, since the three maps $\tau \mapsto \nabla_\tau^*$, $\nabla_\tau^* \mapsto d$, and $d \mapsto F_d$ are effectively computable. □

3. Further properties of $\mathfrak{A}$

Following [15, p. 310], for every unital AF algebra $\mathfrak{C}$ we let $T(\mathfrak{C}) \subseteq \mathfrak{C}^*$ be the weak*-compact convex set of tracial states (i.e., positive traces taking the value 1 on the unit) of $\mathfrak{C}$, where $\mathfrak{C}^*$ denotes the (Banach) dual space of $\mathfrak{C}$. A (tracial) state ϕ of $\mathfrak{C}$ is *faithful* if $\phi(x) = 0$ implies $x = 0$, for all positive elements of $\mathfrak{C}$.

Theorem 3.1. *Every finitely presented AFℓ algebra $\mathfrak{B}$, whence in particular the AF algebra $\mathfrak{A}$, has a rational-valued faithful (automorphism-)invariant tracial state, whence $\mathfrak{B}$ is *-embeddable into the Glimm universal algebra $\mathfrak{U}$,* [10, *p.* 41], [9].

Proof. By a *state* of a unital ℓ-group (G, u) we mean a unit-preserving homomorphism $s : G \to \mathbb{R}$ such that $0 \le g \in G \Rightarrow 0 \le s(g)$. We say that s is *faithful* if $s(x) = 0$ implies $x = 0$, for all elements $x \ge 0$ of G. The state s is said to be (automorphism-)*invariant* if for every unit-preserving ℓ-group automorphism α of (G, u) and for each $x \in G$ we have $s(x) = s(\alpha(x))$. Following [15, p. 95], we denote by $S(G, u)$ the compact convex set of states of (G, u).

As is well known in AF algebra theory, (see, for instance, [15, p. 310]) there is an affine homeomorphism $\omega_{\mathfrak{B}} : \alpha \mapsto \sigma_\alpha$ of $T(\mathfrak{B})$ onto $S(K_0(\mathfrak{B}), [1_{\mathfrak{B}}])$. Under this map, faithful (resp., invariant) tracial states of $\mathfrak{B}$ correspond to faithful (resp., invariant) states of its unital K_0-group. In [27, Theorem 4.1] it is proved that $(K_0(\mathfrak{B}), [1_{\mathfrak{B}}])$ has a rational-valued faithful invariant state, whence the desired first conclusion follows.

To prove the *-embeddability of $\mathfrak{B}$ into $\mathfrak{U}$, let us recall that, up *-isomorphism, $\mathfrak{U}$ is the only AF algebra such that $(K_0(\mathfrak{U}), [1_{\mathfrak{U}}]) = (\mathbb{Q}, 1)$. The Lebesgue integral $f \in \mathcal{M}_1 \mapsto \int_{[0,1]} f \in \mathbb{R}$ is a positive unit-preserving homomorphism λ of $\mathcal{M}_1$ (as a partially ordered group) into the totally ordered group $\mathbb{R}$. Since f is piecewise linear with integer coefficients, $\lambda(f)$ is a rational number. The preservation properties of the K_0-functor (see, e.g., [14, 20B, p. 172], or [16, 1.1(iv)]) yield a unital *-homomorphism $\mu : \mathfrak{B} \to \mathfrak{U}$ such that $\lambda = K_0(\mu)$. Since $\lambda(f) > 0$ for every nonzero $f \ge 0$, μ is injective, [14, 19J, p. 160]. Combining (1) with (2), we conclude that μ is the desired injective unital *-homomorphism of $\mathfrak{B}$ into $\mathfrak{U}$. □

Theorem 3.2. *Up to *-isomorphism, the infinite-dimensional simple quotients of $\mathfrak{A}$ coincide with the Effros-Shen AF algebras $\mathfrak{F}_\theta$ of* [9, *p.* 65]. *Thus, each irrational rotation C^*-algebra $\mathfrak{A}_\theta$ is *-embeddable into a simple quotient of $\mathfrak{A}$.*

Proof. The first statement is proved in [24, 3.1]. The second statement now immediately follows from the celebrated Pimsner-Voiculescu embedding [30]. □

Theorem 3.3. *Suppose $\mathfrak{B}$ is a unital AF algebra and $(K_0(\mathfrak{B}), [1_{\mathfrak{B}}])$ is a unital ℓ-group, generated (as an ℓ-group) by the order unit $[1_{\mathfrak{B}}]$ together with $[p]$, for some projection $p \in \mathfrak{B}$. Then $\mathfrak{B}$ is *-isomorphic to $\mathfrak{A}/I$ for some ideal I of $\mathfrak{A}$.*

Proof. By Lemma 1.1, because K_0 preserves exact sequences, [10, p. 34], [9, §9]. □

The AF algebra $\mathfrak{A}$ has many other interesting properties, besides those proved in [24], [5], and in the above sections. Here is a sample of new results, whose proofs will appear elsewhere:

THEOREM 3.4. *Let $\mathscr{B}_{[0,1]}$ be the compact convex subset of the dual Banach space $\mathscr{C}([0,1],\mathbb{R})^*$ given by all Borel probability measures on the unit interval $[0,1]$. Then the tracial space $T(\mathfrak{A})$ is affinely homeomorphic to $\mathscr{B}_{[0,1]}$.*

THEOREM 3.5. *Let us equip the group $\mathrm{Aut}(\mathfrak{A})$ of *-automorphisms of $\mathfrak{A}$ with the topology whose basic open sets are all sets of the form $O_{\alpha,a,\varepsilon} = \{\beta \in \mathrm{Aut}(\mathfrak{A}) \mid \|\alpha(a) - \beta(a)\| < \varepsilon\}$, for $\alpha \in \mathrm{Aut}(\mathfrak{A})$, $a \in \mathfrak{A}$ and $\varepsilon > 0$. Then $\mathrm{Aut}(\mathfrak{A})$ has exactly two connected components.*

THEOREM 3.6. *Every primitive ideal I of $\mathfrak{A}$ is essential. In other words, every nonzero ideal K of $\mathfrak{A}$ has nonzero intersection with I.*

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