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Partial Differential Equations — *Alternative Forms of the Harnack Inequality for Non-Negative Solutions to Certain Degenerate and Singular Parabolic Equations*,

by Emmanuele DiBenedetto[1], Ugo Gianazza and Vincenzo Vespri.

Dedicated to the memory of Renato Caccioppoli

Abstract. — Non-negative solutions to quasi-linear, degenerate or singular parabolic partial differential equations, of p-Laplacian type for $p > \frac{2N}{N+1}$, satisfy Harnack-type estimates in some intrinsic geometry ([2, 3]). Some equivalent alternative forms of these Harnack estimates are established, where the supremum and the infimum of the solutions play symmetric roles, within a properly redefined intrinsic geometry. Such equivalent forms hold for the non-degenerate case $p = 2$ following the classical work of Moser ([5, 6]), and are shown to hold in the intrinsic geometry of these degenerate and/or parabolic p.d.e.'s. Some new forms of such an estimate are also established for $1 < p < 2$.

Key words: Degenerate and Singular Parabolic Equations, Harnack Estimates.

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1. Introduction and main results

Let E be an open set in $\mathbb{R}^N$ and for $T > 0$, let E_T denote the cylindrical domain $E \times (0, T]$, and consider quasi-linear, parabolic differential equations of the form

$$\begin{aligned} &u \in C_{\rm loc}(0,T; L^2_{\rm loc}(E)) \cap L^p_{\rm loc}(0,T; W^{1,p}_{\rm loc}(E)) \\ &u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = 0 \quad \text{weakly in } E_T \end{aligned} \tag{1.1}$$

where the function $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ is only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,Du) \cdot Du \ge C_o |Du|^p \\ |\mathbf{A}(x,t,u,Du)| \le C_1 |Du|^{p-1} \end{cases} \quad \text{a.e. in } E_T \tag{1.2}$$

where $p > 1$ and C_o and C_1 are given positive constants. The parameters $\{N, p, C_o, C_1\}$ are the data, and we say that a generic constant $\gamma = \gamma(N, p, C_o, C_1)$ depends upon the data, if it can be quantitatively determined a priori only in terms of the indicated parameters.

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For $\rho > 0$ let B_ρ denote the ball of radius ρ about the origin of $\mathbb{R}^N$ and let $Q_\rho^\pm(\theta)$ denote the "forward" and "backward" parabolic cylinders

$$Q_\rho^-(\theta) = B_\rho \times (-\theta\rho^p, 0], \quad Q_\rho^+(\theta) = B_\rho \times (0, \theta\rho^p) \tag{1.3}$$

where θ is a positive parameter that determines, roughly speaking the relative height of these cylinders. The origin $(0, 0)$ of $\mathbb{R}^{N+1}$ is the "upper vertex" of $Q_\rho^-(\theta)$ and the "lower vertex" of $Q_\rho^+(\theta)$. If $p = 2$ and $\theta = 1$ we write $Q_\rho^\pm(1) = Q_\rho^\pm$. For a fixed $(x_o, t_o) \in \mathbb{R}^{N+1}$ denote by $(x_o, t_o) + Q_\rho^\pm(\theta)$ cylinders congruent to $Q_\rho^\pm(\theta)$ and with "upper vertex" and "lower vertex" respectively at (x_o, t_o).

1.1 Harnack Estimates for the non-Degenerate Case $p = 2$

The classical Harnack estimate of Hadamark–Pini ([4, 7]) for non-negative local solutions of the heat equation, and the Moser Harnack estimate for non-negative solutions of (1.1)–(1.2) for the non-degenerate case $p = 2$, take the equivalent form

$$\gamma^{-1} \sup_{B_\rho(x_o)} u(\cdot, t_o - \rho^2) \le u(x_o, t_o) \le \gamma \inf_{B_\rho(x_o)} u(\cdot, t_o + \rho^2) \tag{1.4}$$

for a constant $\gamma > 0$ depending only upon the data, provided the parabolic cylinder $(x_o, t_o) + Q_{4\rho}^\pm$ is all contained in E_T. It is then natural to ask what forms, if any, the Harnack inequality might take for non-negative solutions of (1.1)–(1.2), for $p \neq 2$.

1.2 Intrinsic, Equivalent Forms of the Harnack Estimates for the Degenerate Case $p > 2$

THEOREM 1.1. *Let u be a non-negative, local, weak solution to* (1.1)–(1.2) *for $p > 2$. There exist constants $c_1 > 1$ and $\gamma_1 > 1$ depending only upon the data, such that for all intrinsic cylinders*

$$(x_o, t_o) + Q_{4\rho}^\pm(\theta_1) \subset E_T, \quad \textit{with} \quad \theta_1 = c_1[u(x_o, t_o)]^{2-p} \tag{1.5}$$

there holds

$$\gamma_1^{-1} \sup_{B_\rho(x_o)} u(x, t_o - \theta_1\rho^p) \le u(x_o, t_o) \le \gamma_1 \inf_{B_\rho(x_o)} u(x, t_o + \theta_1\rho^p). \tag{1.6}$$

Thus the form (1.4) continues to hold for non-negative solutions of the degenerate equations (1.1)–(1.2), although in their own intrinsic geometry, made precise by (1.5). As $p \searrow 2$ the constants c_1 and γ_1 tend to finite, positive constants, thereby recovering the classical form (1.4). The upper estimate of (1.6) was established in [2]. We will show here that the upper estimate implies the lower inequality for all intrinsic cylinders $(x_o, t_o) + Q_{4\rho}^\pm(\theta_1)$ as in (1.5).

1.3 Intrinsic, Equivalent Forms of the Harnack Estimates for the Singular, Super-Critical Case $\frac{2N}{N+1} < p < 2$

THEOREM 1.2. *Let u be a non-negative, local, weak solution to* (1.1)–(1.2), *for* $\frac{2N}{N+1} < p < 2$. *There exist constants* $c_2 \in (0,1)$ *and* $\gamma_2 > 1$ *depending only upon the data, such that for all intrinsic cylinders*

$$(x_o, t_o) + Q_{4\rho}^{\pm}(\theta_2) \subset E_T, \quad \textit{with} \quad \theta_2 = c_2[u(x_o, t_o)]^{2-p} \tag{1.7}$$

and for all $0 \le \tau \le \theta_2 \rho^p$, *there holds*

$$\gamma_2^{-1} \sup_{B_\rho(x_o)} u(x, t_o \pm \tau) \le u(x_o, t_o) \le \gamma_2 \inf_{B_\rho(x_o)} u(x, t_o \pm \tau) \tag{1.8}$$

Thus the form (1.4) continues to hold for non-negative solutions of the singular equations (1.1)–(1.2), for $\frac{2N}{N+1} < p < 2$, although in their own intrinsic geometry. However the constant γ_2 tends to infinity as either $p \nearrow 2$ or $p \searrow \frac{2N}{N+1}$. The validity of (1.8) for all $0 \le \tau \le \theta_2 \rho^p$ implies that these Harnack estimate have a strong elliptic form. Such a form would be false for the non-singular case $p = 2$, and accordingly the constant γ_2 deteriorates as $p \nearrow 2$. The upper estimate of (1.6) was established in [2]. We will show here that the upper estimate implies the lower inequality for all intrinsic cylinders $(x_o, t_o) + Q_{4\rho}^{\pm}(\theta_2)$ as in (1.7).

1.4 A Form of the Harnack Inequality for the Singular Case $1 < p < 2$

It was shown in [3] by explicit counterexamples, that neither (1.5)–(1.6), nor (1.7)–(1.8) hold for p in the critical and sub-critical range $1 < p \le \frac{2N}{N+1}$. This raises the question of what form, if any, a Harnack estimate might take for weak solutions of (1.1)–(1.2) for p in such a critical and sub-critical range.

The next inequality provides a possible weak form of a Harnack estimate valid in the whole singular range $1 < p < 2$.

PROPOSITION 1.1. *Let u be a non-negative, local, weak solution to* (1.1)–(1.2), *for* $1 < p < 2$. *Assume moreover that*

$$u \in L^r_{\mathrm{loc}}(E_T) \textit{ with } r \ge 1 \textit{ such that } \lambda_r \overset{def}{=} N(p-2) + rp > 0. \tag{1.9}$$

Then there exist positive constants c_3 *and* γ_3 *depending only upon the data, such that for all intrinsic cylinders*

$$(x_o, t_o) + Q_{4\rho}^{+}(\theta_3) \subset E_T, \quad \textit{with} \quad \theta_3 = c_3 \Big(\fint_{B_{2\rho}(x_o)} u^r(\cdot, t_o)\, dx \Big)^{(2-p)/r} \tag{1.10}$$

and for all $\frac{1}{2}\theta_3 \rho^p \le \tau \le \theta_3 \rho^p$, *there holds*

$$\sup_{B_\rho(x_o)} u(x, t_o + \tau) \le \gamma_3 \Big(\fint_{B_{2\rho}(x_o)} u^r(\cdot, t_o)\, dx \Big)^{1/r}. \tag{1.11}$$

PROPOSITION 1.2. *Let u be a non-negative, local, weak solution to* (1.1)–(1.2), *for* $1 < p < 2$, *satisfying* (1.9). *Then there exist positive constants* c_4 *and* γ_4 *depending only upon the data, such that for all intrinsic cylinders*

$$(x_o, t_o) + Q^-_{4\rho}(\theta_4) \subset E_T, \quad \textit{with} \quad \theta_4 = c_4[u(x_o, t_o)]^{2-p} \tag{1.12}$$

there holds

$$u(x_o, t_o) \le \gamma_4 \sup_{B_\rho(x_o)} u(\cdot, t_o - \theta_4 \rho^p). \tag{1.13}$$

The constants γ_3 and γ_4 tend to infinity as either $p \searrow 1$ or as $p \nearrow 2$ or as $\lambda_r \searrow 0$. It was shown in [1] that local weak solutions of (1.1)–(1.2) need not be bounded unless they are in $L^r_{\rm loc}(E_T)$ for some $r \ge 1$ satisfying (1.9). The latter then guarantees that the solution is in $L^\infty_{\rm loc}(E_T)$. As $\lambda_r \searrow 0$ weak solutions are not prevented to become unbounded and accordingly (1.11) becomes vacuous.

2. PROOF OF THEOREM 1.1

Fix $(x_o, t_o) \in E_T$ and assume $u(x_o, t_o) > 0$, and let $(x_o, t_o) + Q^{\pm}_{4\rho}(\theta_1)$ as in (1.5). Seek those values of $t < t_o$, if any, for which

$$u(x_o, t) = 2\gamma_1 u(x_o, t_o) \tag{2.1}$$

where γ_1 is as in the right estimate (1.6), which by the results of [2], holds for all such intrinsic cylinders. If such a t does not exist

$$u(x_o, t) < 2\gamma_1 u(x_o, t_o) \quad \text{for all } t \in [t_o - \theta_1(4\rho)^p, t_o]. \tag{2.2}$$

We establish by contradiction that this in turn implies

$$\sup_{B_\rho(x_o)} u(\cdot, \tilde{t}) \le 2\gamma_1^2 u(x_o, t_o), \quad \text{for } \tilde{t} = t_o - \theta_1 \rho^p. \tag{2.3}$$

If not, by continuity there exists $x_* \in B_\rho(x_o)$ such that $u(x_*, \tilde{t}) = 2\gamma_1^2 u(x_o, t_o)$. Applying the Harnack right inequality (1.6) with (x_o, t_o) replaced by $(x_*, \tilde{t})$, gives

$$u(x_*, \tilde{t}) \le \gamma_1 \inf_{B_\rho(x_*)} u(\cdot, \tilde{t} + \tilde{\theta}_1 \rho^p), \quad \text{where } \tilde{\theta}_1 = c_1[u(x_*, \tilde{t})]^{2-p}. \tag{2.4}$$

Now $x_o \in B_\rho(x_*)$ and, since $\gamma_1 > 1$ and $p > 2$,

$$\tilde{t} + \tilde{\theta}_1 \rho^p = t_o - c_1[u(x_o, t_o)]^{2-p} \rho^p + c_1 \frac{[u(x_o, t_o)]^{2-p}}{(2\gamma_1^2)^{p-2}} \rho^p < t_o.$$

Therefore from (2.2) and (2.4)

$$2\gamma_1^2 u(x_o, t_o) = u(x_*, \tilde{t}) \le \gamma_1 u(x_o, \tilde{t} + \tilde{\theta}_1 \rho^p) < 2\gamma_1^2 u(x_o, t_o).$$

The contradiction establishes (2.3).

2.1 There Exists $t < t_o$ Satisfying (2.1)

Let $t_1 < t_o$ be the first time for which (2.1) holds. For such a time

$$t_o - t_1 > c_1[u(x_o, t_1)]^{2-p}\rho^p = c_1 \frac{[u(x_o, t_o)]^{2-p}}{(2\gamma_1)^{p-2}}\rho^p. \tag{2.5}$$

Indeed if such inequality were violated, by applying the Harnack right inequality (1.5)–(1.6) with (x_o, t_o) replaced by (x_o, t_1) would give

$$u(x_o, t_1) \le \gamma_1 u(x_o, t_o) \quad \Leftrightarrow \quad 2\gamma_1 u(x_o, t_o) \le \gamma_1 u(x_o, t_o).$$

Set

$$t_2 = t_o - c_1 \frac{[u(x_o, t_o)]^{2-p}}{(2\gamma_1)^{p-2}}\rho^p.$$

From the definitions, the continuity of u and (2.5)

$$t_1 < t_2 < t_o \quad \text{and} \quad u(x_o, t_o) \le u(x_o, t_2) \le 2\gamma_1 u(x_o, t_o).$$

Let v denote the unit vector in $\mathbb{R}^N$ and for (x_o, t_2) consider points $x_s = x_o + sv$ where s is a positive parameter. Let s_o be the first positive s, if any, such that $u(x_o + s_o v, t_2) = 2\gamma_1 u(x_o, t_o)$. We claim that either such a s_o does not exist or $s_o \ge \rho$. In either case

$$\sup_{B_\rho(x_o)} u\Big(\cdot, t_o - c_1 \frac{[u(x_o, t_o)]^{2-p}}{(2\gamma_1)^{p-2}}\rho^p\Big) \le 2\gamma_1 u(x_o, t_o). \tag{2.6}$$

To establish the claim, assume that s_o exists and $s_o < \rho$. Apply the Harnack right inequality (1.5)–(1.6) with (x_o, t_o) replaced by $x_2 = x_o + s_o v$ and t_2, to get

$$u(x_2, t_2) \le \gamma_1 \inf_{B_\rho(x_2)} u(\cdot, t_2 + \theta'\rho^p), \quad \theta' = c_1[u(x_2, t_2)]^{2-p}.$$

Notice that

$$t_2 + \theta'\rho^p = t_o - c_1 \frac{[u(x_o, t_o)]^{2-p}}{(2\gamma_1)^{p-2}}\rho^p + c_1 \frac{[u(x_o, t_o)]^{2-p}}{(2\gamma_1)^{p-2}}\rho^p = t_o.$$

Therefore, since $x_o \in B_\rho(x_2)$

$$2\gamma_1 u(x_o, t_o) = u(x_2, t_2) \le u(x_2, t_2) \le \gamma_1 \inf_{B_\rho(x_2)} u(\cdot, t_o) \le \gamma_1 u(x_o, t_o).$$

The contradiction implies that (2.6) holds. Thus for all $\rho > 0$, either (2.3) or (2.6) holds true. The proof is now concluded by using the arbitrariness of ρ and by properly redefining γ_1. □

3. Proof of Theorem 1.2

Let c_2 and γ_2 be the constants appearing on the Harnack right inequality (1.7)–(1.8) which, by the results of [3], holds true for all $\rho > 0$. We may assume that $(x_o, t_o) = (0,0)$, and that $Q^{\pm}_{8\rho}(\theta_2) \subset E_T$, where θ_2 is as in (1.7). It suffices to prove that there exists a positive constant α depending only upon the data and independent of u and ρ, such that

$$\sup_{B_{\alpha\rho}} u(\cdot, -\theta_2\rho^p) \leq \gamma_2 u(0,0), \quad \theta_2 = c_2[u(0,0)]^{2-p}. \tag{3.1}$$

Let $\alpha > 0$ to be chosen and consider the set

$$U_\alpha = B_{\alpha\rho} \cap [u(\cdot, -\theta_2\rho^p) \leq \gamma_2 u(0,0)].$$

Since u is continuous such a set is a closed subset of $B_{\alpha\rho}$. The parameter $\alpha > 0$ will be chosen, depending only on the data, such that U_α is also open. Therefore $U_\alpha = B_{\alpha\rho}$ and (3.1) holds for such α.

Fix $z \in U_\alpha$. Since u is continuous there exists a ball $B_\varepsilon(z) \subset B_{\alpha\rho}$, such that

$$u(y, -\theta_2\rho^p) \leq 2\gamma_2 u(0,0) \quad \text{for all } y \in B_\varepsilon(z). \tag{3.2}$$

The parameter α will be chosen to insure that $B_\varepsilon(z) \subset U_\alpha$ thereby establishing that U_α is open. For $y \in B_\varepsilon(z)$ construct the solid p-paraboloid

$$t + \theta_2\rho^p \geq |x - y|^p c_2[u(y, -\theta_2\rho^p)]^{2-p}.$$

If the origin belongs to such a paraboloid, then by the Harnack right inequality (1.7)–(1.8), with (x_o, t_o) replaced by $(y, -\theta_2\rho^p)$, there holds

$$u(y, -\theta_2\rho^p) \leq \gamma_2 u(0,0)$$

and therefore $y \in U_\alpha$. The origin $(0,0)$ belongs to the paraboloid if

$$|y|^p c_2[u(y, -\theta_2\rho^p)]^{2-p} \leq |y|^p c_2(2\gamma_2)^{2-p}[u(0,0)]^{2-p} \leq \theta_2\rho^p.$$

By the definition of θ_2, the last inequality is verified if

$$|y| \leq \alpha\rho \quad \text{where} \quad \alpha = (2\gamma_2)^{(p-2)/p}. \qquad \square$$

4. Proof of Propositions 1.1 and 1.2

The following Proposition follows by a minor adaptation of the arguments of [1] Chapter V, §5, and Chapter VII, §4.

Proposition 4.1. *Let u be a non-negative, local, weak solution to* (1.1)–(1.2) *for $1 < p < 2$, satisfying* (1.9). *There exists a constant $\gamma = \gamma(N, p, r)$ such that for any cylindrical domain*

$$B_{2\rho}(y) \times [s - (t - s), s + (t - s)] \subset E_T$$

there holds

$$\sup_{B_\rho(y)\times[s,t]} u \leq \frac{\gamma}{(t-s)^{N/\lambda_r}} \left(\int_{B_{2\rho}(y)} u^r(x, 2s - t)\, dx \right)^{p/\lambda_r} + \gamma \left(\frac{t-s}{\rho^p} \right)^{1/(2-p)}. \tag{4.1}$$

Fix $(x_o, t_o) \in E_T$ and $\rho > 0$ and θ_3 as in (1.10) with $c_3 > 0$ to be chosen. The estimate (1.11) follows from (4.1) by choosing $t = t_o + \theta_3 \rho^p$ and $2s - t = t_o$, and by properly redefining γ_3 and c_3 in terms of the set of parameters $\{\gamma, N, p, r\}$.

Inequality (1.12)–(1.13) follows from (4.1) by choosing $s = t_o$ and $t - s = \varepsilon[u(x_o, t_o)]^{2-p} \rho^p$, for $\varepsilon > 0$ to be chosen. □

4.1 Further Results Linking Weak and Strong Harnack Inequalities

The strong Harnack estimates (1.7)–(1.8) cease to exist for $1 < p \leq \frac{2N}{N+1}$. Counterexamples are provided in [3]. However the weak form (1.10)–(1.11) continues to hold for all $1 < p < 2$. It would be of interest to understand what form, if any, a Harnack-type estimate might take for p in the sub-critical range $\left(1, \frac{2N}{N+1}\right]$ and in what form it might be connected to the weak form (1.10)–(1.11). While the problem is open, the next Proposition provides some information in this direction.

PROPOSITION 4.2. *Let u be a non-negative function, locally continuous in E_T satisfying the weak Harnack estimate* (1.9)–(1.11) *for some $p \in (1, 2)$ and $r \geq 1$ for which $\lambda_r > 0$, and the left forward strong Harnack estimate in the form*

$$\sup_{B_\rho(x_o)} u(x, t_o - \theta_2 \rho^p) \leq \gamma_2 u(x_o, t_o) \tag{4.2}$$

for all intrinsic cylinders

$$(x_o, t_o) + Q^{\pm}_{4\rho}(\theta_2) \subset E_T, \quad \text{with} \quad \theta_2 = c_2[u(x_o, t_o)]^{2-p}. \tag{4.3}$$

Then u satisfies the elliptic Harnack estimate in the form

$$\sup_{B_\rho(x_o)} u(x, t_o) \leq \gamma_5 u(x_o, t_o) \tag{4.4}$$

for all intrinsic cylinders of the form (4.3), *for a constant γ_5 depending only upon the set of parameters $\{N, p, r, c_2, \gamma_2, c_3, \gamma_3\}$.*

REMARK 4.1. Solutions of (1.1)–(1.2) for $1 < p < 2$ satisfy the weak Harnack estimate (1.9)–(1.11). For p in the super-critical range $\left(\frac{2N}{N+1}, 2\right)$ they also satisfy the strong left forward inequality (4.2)–(4.3) as follows from Theorem 1.2. For this reason in the assumption (4.2)–(4.3) we have used the same symbols c_2, and γ_2. The Proposition however continues to hold for any function satisfying both inequalities with any given but fixed constants.

PROOF. Fix $(x_o, t_o) \in E_T$, let θ_2 be defined by (4.3), and set

$$\theta_\alpha = c_3\Big(\fint_{B_{2\alpha\rho}(x_o)} u^r(\cdot, t_o - \theta_2\rho^p)\,dx\Big)^{(2-p)/r}, \quad t_\alpha = t_o - \theta_2\rho^p + \theta_\alpha(2\alpha\rho)^p$$

where α is a positive parameter to be chosen. Assume momentarily that for such an α,

$$(x_o, t_o) + Q^{\pm}_{4\alpha\rho}(\theta_\alpha) \subset E_T \quad \text{and} \quad (x_o, t_o) + Q^{\pm}_{4\rho}(\theta_2) \subset E_T. \tag{4.5}$$

Apply (1.10)–(1.11) with t_o replaced by $t_o - \theta_2\rho^p$, and ρ replaced by $\alpha\rho$, to get

$$\sup_{B_{\alpha\rho}} u(\cdot, t_\alpha) \le \gamma_3\Big(\fint_{B_{2\alpha\rho}(x_o)} u^r(\cdot, t_o - \theta_2\rho^p)\,dx\Big)^{1/r}.$$

If $t_\alpha = t_o$, by the definition of t_α and (4.2)–(4.3)

$$\sup_{B_{\alpha\rho}} u(\cdot, t_o) \le \gamma_3\gamma_1^{1/r} u(x_o, t_o). \tag{4.6}$$

Since $\lambda_r > 0$, the function $\alpha \to t_\alpha$ is monotone increasing and the equation $t_\alpha = t_o$ has a root. If $\alpha \in (0, 1]$, the equation $t_\alpha = t_o$ and the forward Harnack estimate (4.2)–(4.3) imply

$$\begin{aligned} c_2[u(x_o, t_o)]^{2-p} &= 2^p\alpha^p c_3\Big(\fint_{B_{2\alpha\rho}(x_o)} u^r(\cdot, t_o - \theta_2\rho^p)\,dx\Big)^{(2-p)/r} \\ &\le 2^p\alpha^p c_3\Big[\sup_{B_{2\alpha\rho}(x_o)} u(\cdot, t_o - \theta_2\rho^p)\Big]^{2-p} \\ &\le 2^p\alpha^p c_3\gamma_2^{2-p}[u(x_o, t_o)]^{2-p}. \end{aligned}$$

If $\alpha > 1$, the equation $t_\alpha = t_o$ and the weak Harnack estimate (1.10)–(1.11) with t_o replaced by $t_o - \theta_2\rho^p$ and $\tau = \theta_2\rho^p$, give

$$\begin{aligned} c_2[u(x_o, t_o)]^{2-p} &= 2^p\alpha^p c_3\Big(\fint_{B_{2\alpha\rho}(x_o)} u^r(\cdot, t_o - \theta_2\rho^p)\,dx\Big)^{(2-p)/r} \\ &\ge \frac{2^p\alpha^p c_3}{\gamma_3^{2-p}}[u(x_o, t_o)]^{2-p}. \end{aligned}$$

Thus in either case the root α of $t_\alpha = t_o$ satisfies

$$\min\Big\{1; \frac{1}{2}\Big(\frac{c_2}{c_3}\Big)^{1/p}\gamma_2^{(p-2)/p}\Big\} = \alpha_o \le \alpha \le \alpha_1 = \max\Big\{1; \frac{1}{2}\Big(\frac{c_2}{c_3}\Big)^{1/p}\gamma_3^{(2-p)/p}\Big\}.$$

With α_o and α_1 determined quantitatively only in terms of the set of parameters $\{N, p, c_2, c_3, \gamma_2, \gamma_3\}$ condition (4.5) can be always insured by a proper, quantitative choice of ρ, and thus (4.6) holds in all cases for some α in the indicated range. This implies (4.4) for a proper definition of γ_5.

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Emmanuele DiBenedetto
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville TN 37240 (USA)
em.diben@vanderbilt.edu

Ugo Gianazza
Dipartimento di Matematica "F. Casorati"
Università di Pavia
via Ferrata 1
27100 Pavia (Italy)
gianazza@imati.cnr.it

Vincenzo Vespri
Dipartimento di Matematica "U. Dini"
Università di Firenze
viale Morgagni 67/A
50134 Firenze (Italy)
vespri@math.unifi.it