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Mathematical Analysis — *Orlicz-Sobolev regularity of mappings with subexponentially integrable distortion*, by Albert Clop, Pekka Koskela.

Abstract. — We study regularity properties of mappings of finite distortion. We show that some sort of self-improvement phenomena hold also when only subexponential integrability is assumed for the distortion function. We extend to this setting results by Faraco, Koskela and Zhong [9] and Bildhauer, Fuchs and Zhong [6].

Key words: mappings of finite distortion, higher integrability, subexponential distortion.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain. We say that $f : \Omega \to \mathbb{R}^n$ is a *mapping of finite distortion* if:

(a) f belongs to the Sobolev space $W^{1,1}_{loc}(\Omega; \mathbb{R}^n)$.
(b) The Jacobian determinant $J(\cdot, f)$ belongs to $L^1_{loc}(\Omega; \mathbb{R})$.
(c) There exists a measurable function $K : \Omega \to [1, \infty]$ finite almost everywhere, such that the *distortion inequality*

$$|Df(x)|^n \le K(x)J(x, f) \tag{1.1}$$

holds for almost every $x \in \Omega$.

Above, $|Df(x)|$ stands for the operator norm of the differential matrix of f at the point x. The smallest function K satisfying (1.1) is called the *distortion function* of f. When $K \in L^\infty$ one recovers the well known class of mappings of bounded distortion, or quasiregular mappings (see for instance [25]). More generally, under suitable conditions of K one automatically has that nonconstant mappings of finite distortion are continuous, discrete and open (see [20, 23] and also the monograph [21]).

When K is bounded, the distortion inequality immediately gives us that $|Df| \in L^n_{loc}$. However, by the seminal works of Boyarski [5] (for planar quasiconformal mappings), Gehring [11] (for spatial quasiconformal mappings) and Elcrat and Meyers [8] (for spatial quasiregular mappings), one actually has $|Df| \in L^p_{loc}$ for all $p < p_0$, where $p_0 = p_0(n, \|K\|_\infty)$ depends only on n and $\|K\|_\infty$, and $p_0 > n$. This result has a dual version, which asserts (see [16], [22]) that there exists $q_0 = q_0(n, \|K\|_\infty) < n$ such that if $f \in W^{1,q}_{loc}$ for some $q \in (q_0, n)$ and satisfies (1.1), then automatically $f \in W^{1,n}_{loc}$. By the work of Astala [1], one has $p_0(2,K) = \frac{2K}{K-1}$ and $q_0 = \frac{p_0}{p_0-1}$. Later it was shown by Petermichl and Volberg [27] that in the plane one can even take $q = q_0$.

The improved regularity theorem of Gehring [11] is based on a rather technical lemma, which asserts that the local reverse Hölder inequality is an open-ended property with respect to the exponent. This fact has been shown to be extremely useful in harmonic analysis and partial differential equations. The lemma, known since then as Gehring's Lemma, have been systematically used and several new versions have been formulated. We refer the reader to [6] for one of such versions, which found applications also in fluid mechanics. See also the survey [17] and the monographs [21] and [13].

The regularity theory for mappings of finite distortion has been deeply studied during the last decade. See for instance the papers [2], [7], [9], [18], [19], [20], [23], and also the monograph [21] and the references therein. Special interest has been focused in understanding what is the Orlicz-Sobolev regularity for mappings of finite distortion whose distortion function K is exponentially integrable, that is,

$$e^{pK} \in L^1 \tag{1.2}$$

for some $p > 0$. Under this assumption, the inequality

$$\frac{ab}{\log(e+ab)} \le a + e^b - 1 \tag{1.3}$$

says that locally $|Df| \in \frac{L^n}{\log L}$. However, as in the quasiregular case, a better degree of regularity can be obtained also in this weaker situation, although now the self-improving rate is slower and has to be measured at a logarithmic scale. We refer the reader to [2], [9], [19] and [20]. Basically, one has the implication

$$e^{pK} \in L^1 \quad \text{for some } p > 0 \quad \Rightarrow \quad |Df| \in L^n \log^{\beta-1} L \quad \text{for all } \beta < p_0 \tag{1.4}$$

for some number $p_0 = p_0(p,n) > 0$. In [9] the authors go even further and give quite precise estimates for p_0, showing that in the above implication one can take

$$\beta < c(n)p \tag{1.5}$$

where $c(n) \ge 1$ is a constant depending only on the dimension n. Concerning the dual problem, a similar behavior is also shown in [9]. Namely, if $f : \Omega \to \mathbb{R}^n$ belongs to $W^{1,1}_{loc}$ and satisfies the distortion inequality (1.1) with K as in (1.2), then

$$|Df| \in \frac{L^n}{\log^{\beta+1} L} \quad \text{for some } \beta < q_0 \quad \Rightarrow \quad J(\cdot, f) \in L^1 \tag{1.6}$$

where $q_0 = q_0(p,n) > 0$ depends linearly on p. As far as we know, [9, Theorem 1.3] is the first published self-improvement result concerning Sobolev solutions to the inequality (1.1) with unbounded K and which are not assumed to have locally integrable Jacobian determinant. On the other hand, implications (1.4) and (1.6) can be used to obtain measure distortion estimates and removability results in terms of Orlicz-Sobolev capacities. Very recently, a planar factorization argument has been used in [4] to show that in (1.5) one may take $c(2) = 1$. This value is sharp, as shown by Kovalev's example,

$$f(z) = \frac{z}{|z|}\left(\log\left(e + \frac{1}{|z|}\right)\right)^{-p/2}\left(\log\log\left(e + \frac{1}{|z|}\right)\right)^{-1/2}. \tag{1.7}$$

Unfortunately the tools from [4] are not available in higher dimensions.

In the present paper, we prove implications analogous to (1.4) and (1.6) hold for mappings of finite distortion whose distortion function is only *subexponentially integrable*. This means that instead of (1.2) we only assume that

$$\exp(\mathscr{A}(pK)) \in L^1$$

for some $p > 0$, where $\mathscr{A}$ is slightly below being linear, that is,

$$\int_1^\infty \frac{\mathscr{A}(t)}{t^2}\, dt = \infty. \tag{1.8}$$

Condition (1.8) is critical for mappings of subexponentially integrable distortion to be continuous, either constant or both discrete and open, and to satisfy Lusin's N-condition [23]. See also [15], [24], [26]. We are restricted to the borderline situation in (1.8), so that examples as $\mathscr{A}(t) = \frac{t}{\log^\gamma(e+t)}$, $0 < \gamma < 1$ are excluded from our discussion and will be subject of forthcoming work. The examples we have in mind are

$$\begin{aligned}
&\mathscr{A}(t) = t \\
&\mathscr{A}(t) = \frac{t}{\log(e+t)} \\
&\mathscr{A}(t) = \frac{t}{\log(e+t)\log\log(e^e+t)}
\end{aligned} \tag{1.9}$$

and so on. A convenient way of characterizing our functions $\mathscr{A}$ is by assuming that

$$\int_1^t \frac{\mathscr{A}(s)}{s^2}\, ds \simeq \frac{\mathscr{A}(t)\mathscr{A}^{-1}(\log t)}{t} \tag{1.10}$$

(see Section 2 for details). Indeed, it comes easily from the results in [23] and assumption (1.10) that

$$f \in W^{1,1}_{loc}(\Omega; \mathbb{R}^n),\ e^{\mathscr{A}(K)} \in L^1(\Omega) \text{ and } J(\cdot, f) \in L^1_{loc}(\Omega) \quad \Rightarrow \quad |Df|^n \in L^{P_0}_{loc}(\Omega),$$

where we denote

$$P_\beta(t) = \mathscr{A}(t)\Big(1 + \int_1^t \frac{\mathscr{A}(s)}{s^2}\,ds\Big)^{\beta-1}.$$

for $\beta \in \mathbb{R}$. This is, hence, the starting point for our main result.

Theorem 1.1. *There exist two constants $c_0 = c_0(n, \mathscr{A}) < 0$, $c_1 = c_1(n, \mathscr{A}) > 0$, with the following property. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $f : \Omega \to \mathbb{R}^n$ belong to $W^{1,1}_{loc}(\Omega; \mathbb{R}^n)$, and assume that*

$$|Df(x)|^n \le K(x)J(x, f) \quad \textit{for almost every } x \in \Omega$$

where $K : \Omega \to [1, \infty]$ is a measurable function such that $e^{\mathscr{A}(pK)} \in L^1(\Omega)$. If

$$|Df|^n \in L^{P_\beta}_{loc}(\Omega; \mathbb{R}) \quad \textit{for some } \beta > c_0 p$$

then also

$$|Df|^n \in L^{P_\beta}_{loc}(\Omega; \mathbb{R}) \quad \textit{for all } \beta < c_1 p.$$

In particular, $J(\cdot, f)$ is locally integrable and f is a mapping of finite distortion.

This result extends Theorem 1.1 and Theorem 1.3 in [9] to the setting of subexponentially integrable distortion. As in [9], the proof of Theorem 1.1 is divided into two parts. First, we show that

$$e^{\mathscr{A}(pK)} \in L^1 \text{ and } J(\cdot, f) \in L^1_{loc} \quad \Rightarrow \quad |Df|^n \in L^{P_\beta}_{loc} \text{ for all } \beta < c_1 p. \tag{1.11}$$

The above implication is implicitely included in a much more general result, recently obtained by Gianetti, Greco and Passarelli di Napoli [12, Theorem 2.2]. Here we get (1.11) as an easy corollary of Lemma 3.1 below, which is a sort of Gehring Lemma for scalar functions, in the spirit of [6, Lemma 1.2]. That is, it concerns the improved integrability of real valued functions, with a very slow scale of self-improvement. This lemma has its own interest, and partially motivated the present paper. The second part in the proof of Theorem 1.1 follows by proving

$$e^{\mathscr{A}(pK)} \in L^1 \text{ and } |Df|^n \in L^{P_\beta}_{loc} \text{ for some } \beta > c_0 p \quad \Rightarrow \quad J(\cdot, f) \in L^1_{loc}. \tag{1.12}$$

This is precisely the claim of Theorem 4.1, and provides an extension of [9, Theorem 1.3]. The applications regarding measure distortion and removability theorems will be reported elsewhere.

The paper is structured as follows. In Section 2, we give some preliminaries and explain assumptions (1.8) and (1.10). In Section 3, we prove the self-improving Lemma and as a Corollary we obtain implication (1.11). In Section 4 we face the weak problem and prove (1.12).

2. Minimal regularity for mappings of finite distortion

The topic of this section is to recall some basic facts concerning the minimal Orlicz-Sobolev regularity of mappings of finite distortion (see [23] and the monographs [3] and [21] for more details). We will be dealing with functions $\mathscr{A} : [1, \infty) \to [0, \infty)$ which are smooth, non-decreasing, onto, such that

$$\int_1^\infty \frac{\mathscr{A}(t)}{t^2}\,dt = \infty. \tag{2.1}$$

It was shown in [23] that the above assumption (together with other minor technical requirements) is critical for mappings of finite distortion K with $e^{\mathscr{A}(K)} \in L^1$ to be continuous, either constant or both discrete and open, and to satisfy Lusin's N-condition. The following inequality was verified in [23] (also see [3]), but we give a short proof below for the reader's convenience.

Lemma 2.1. *Let $\mathscr{A}$ be as above. Let P be defined by*

$$P(t) = \begin{cases} t & 0 \le t \le 1, \\ \dfrac{t}{\mathscr{A}^{-1}(2\log t)} & t > 1. \end{cases} \tag{2.2}$$

Then

$$P(xy) \le x + e^{(1/2)\mathscr{A}(y)}$$

whenever $x \ge 0$, $y \ge 1$.

Proof. First, note that $P(t) \le t$ for $t \ge 0$. Thus, if $xy \le e^{(1/2)\mathscr{A}(y)}$, then

$$P(xy) \le xy \le e^{(1/2)\mathscr{A}(y)}$$

and the desired inequality is obvious since $e^{\mathscr{A}(y)} \ge 1$. So we can restrict ourselves to the case $xy > e^{(1/2)\mathscr{A}(y)}$. But then $xy > 1$ and therefore

$$P(xy) = \frac{xy}{\mathscr{A}^{-1}(2\log(xy))} \le \frac{xy}{y} = x$$

because $\mathscr{A}$ is non-decreasing. Now the desired inequality easily follows. □

As a consequence, if $f : \Omega \to \mathbb{R}^n$ is a mapping of finite distortion, with distortion function K, and $p > 0$ is fixed, then

$$P(|Df(x)|^n) \le \frac{1}{p} J(x, f) + e^{\mathscr{A}(pK(x))}.$$

where P is as in (2.2). Therefore, if we further assume that $e^{\mathscr{A}(pK)} \in L^1$, we immediately obtain that $|Df|^n \in L^P_{loc}(\Omega; \mathbb{R})$. Since we are interested in comparing P

and $\mathscr{A}$, it is desirable to have precise estimates for $\mathscr{A}^{-1}$ in terms of $\mathscr{A}$. Such estimates easily follow, for instance, if we stay not too far from the borderline cases for (2.1). To make this more precise, we will represent $\mathscr{A}$ as

$$\mathscr{A}(t) = \frac{t}{L(t)}$$

where $L : [1, \infty) \to [1, \infty)$ is a smooth, non-decreasing function, growing to infinity more slowly than any power, that is,

$$\lim_{t \to \infty} L(t) = \infty \quad \text{and} \quad \lim_{t \to \infty} \frac{\log L(t)}{\log t} = 0. \tag{2.3}$$

We extend L and $\mathscr{A}$ for $0 \leq t \leq 1$ by letting $L(t) = 1$ and $\mathscr{A}(t) = 0$. In any case, we are only interested in the behavior of $\mathscr{A}$ at infinity. For technical reasons, we will restrict our attention to functions L such that

$$\frac{tL'(t)}{L(t)} \leq \frac{C_0}{\log(e + t)} \tag{2.4}$$

for some constant $C_0 \geq 0$ and all $t \geq 0$. This includes the examples in (1.9), that is,

$$\begin{aligned} L(t) &= \log(e + t) \\ L(t) &= \log(e + t) \log\log(e^e + t) \\ L(t) &= \log(e + t) \log\log(e^e + t) \log\log\log(e^{e^e} + t). \end{aligned}$$

Among other facts, (2.4) guarantees that L *does not see powers*, that is, $L(t^\alpha) \leq \alpha^{C_0} L(t)$ whenever $\alpha, t \geq 1$ (see Lemma 5.1 for more details). Note also that if L enjoys (2.3) and (2.4), then $\tilde{L}(t) = \log t L(\log t)$ also does.

Lemma 2.2. *If* $L : [1, \infty) \to [1, \infty)$ *is smooth, monotonically increasing, satisfying* (2.3) *and* (2.4), *then*

$$\lim_{t \to \infty} \frac{\mathscr{A}^{-1}(t)}{tL(t)} = 1.$$

Proof. We use Lemma 5.1. By (2.3), L grows not faster than any power. Thus, for each $\varepsilon > 0$ there is $s_\varepsilon > 0$ such that whenever $s > s_\varepsilon$

$$L(s^{1-\varepsilon}) \leq L(s/L(s)) \leq L(s).$$

Then, using (2.4), we see that

$$1 \leq \frac{L(s)}{L(s/L(s))} \leq \exp\Big(\log \frac{L(s)}{L(s^{1-\varepsilon})}\Big) \leq \exp\Big(\int_{s^{1-\varepsilon}}^{s} \frac{C_0\, dr}{r \log r}\Big) = \frac{1}{(1 - \varepsilon)^{C_0}}.$$

Hence

$$\lim_{s\to\infty}\frac{\mathscr{A}^{-1}(s)}{sL(s)}=\lim_{t\to\infty}\frac{L(t)}{L(t/L(t))}=1,$$

as claimed. □

One can therefore estimate the Orlicz function P at (2.2) as follows, for large values of t,

$$P(t)=\frac{t}{\mathscr{A}^{-1}(2\log t)}\simeq\frac{t}{\log(t)L(\log t)}. \tag{2.5}$$

In other words, both functions give rise to the same Orlicz space, with comparable norms. Some particular examples are:

$$\mathscr{A}(t)=t \quad\Rightarrow\quad P(t)\simeq\frac{t}{\log(e+t)},$$

$$\mathscr{A}(t)=\frac{t}{\log(e+t)} \quad\Rightarrow\quad P(t)\simeq\frac{t}{\log(e+t)\log\log(e^e+t)},$$

$$\mathscr{A}(t)=\frac{t}{\log(e+t)\log\log(e^e+t)} \quad\Rightarrow$$

$$P(t)\simeq\frac{t}{\log(e+t)\log\log(e^e+t)\log\log\log(e^{e^e}+t)}.$$

We remark here that in all the examples above P agrees with the Orlicz function $P_{\mathscr{A}}$ given in the following conjecture of Iwaniec and Martin [21, p. 267].

Conjecture 2.3. *Let $f:\Omega\to\mathbb{C}$ be a planar mapping of finite distortion K, such that $e^{\mathscr{A}(K)}\in L^1$. Then, $|Df|^2$ belongs locally to the Orlicz space $L^{P_{\mathscr{A}}}(\Omega)$, where*

$$P_{\mathscr{A}}(t)=\mathscr{A}(t)\Big(1+\int_1^t\frac{\mathscr{A}(s)}{s^2}\,ds\Big)^{-1}.$$

However, it is not true in general that P and $P_{\mathscr{A}}$ define the same Orlicz space, as shown by the examples

$$\mathscr{A}(t)=\frac{t}{(\log(e+t))^{1-\varepsilon}},$$

$$\mathscr{A}(t)=\frac{t}{\log(e+t)(\log\log(e^e+t))^{1-\varepsilon}}$$

$$\mathscr{A}(t)=\frac{t}{\log(e+t)\log\log(e^e+t)(\log\log\log(e^{e^e}+t))^{1-\varepsilon}},$$

where $\varepsilon > 0$. Because of this we restrict our attention to functions $\mathscr{A}$ such that the limit

$$\lim_{t\to\infty} \frac{P(t)}{P_{\mathscr{A}}(t)}$$

exists, and is positive and finite. That is, we require $\mathscr{A}$ to satisfy

$$\int_1^t \frac{\mathscr{A}(s)}{s^2}\,ds \simeq \frac{\mathscr{A}(t)\mathscr{A}^{-1}(\log t)}{t}. \tag{2.6}$$

For these $\mathscr{A}$, Lemma 2.1 holds with P replaced by $P_{\mathscr{A}}$, modulo some multiplicative constant.

3. Improved regularity for mappings of subexponential distortion

Let $\mathscr{A}$ be as in the previous section. That is, $\mathscr{A}(t) = 0$ for $0 \le t \le 1$ and $\mathscr{A} : [1,\infty) \to [0,\infty)$ is smooth, non-decreasing, and satisfies (2.4) and (2.6). In what follows, E will denote

$$E(t) = 1 + \int_1^t \frac{\mathscr{A}(s)}{s^2}\,ds.$$

Then (2.5) simply says that $P(t) \simeq P_0(t)$, where

$$P_0(t) = \frac{\mathscr{A}(t)}{E(t)}.$$

That is, if $f : \Omega \to \mathbb{R}^n$ is a mapping of finite distortion with distortion function K satisfying $e^{\mathscr{A}(pK)} \in L^1$ for some $p > 0$, then $|Df|^n \in L^{P_0}_{loc}(\Omega)$. Some particular examples are the following:

$$\mathscr{A}(t) = t \quad \Rightarrow \quad E(t) \simeq \log(e+t)$$

$$\mathscr{A}(t) = \frac{t}{\log(e+t)} \quad \Rightarrow \quad E(t) \simeq \log\log(e+t)$$

$$\mathscr{A}(t) = \frac{t}{\log(e+t)\log\log(e^e+t)} \quad \Rightarrow \quad E(t) \simeq \log\log\log(e+t).$$

The goal of this section is to show that this regularity improves as p grows. Such an improvement is controlled precisely by powers of E, and will be obtained in Corollary 3.3 as a consequence of the following lemma. Here we denote $\fint_B f = \frac{1}{|B|}\int_B f$.

LEMMA 3.1. *Let $d > 1$, $\sigma \in (0,1)$ and $p > 0$ be fixed, and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose that $f, g, h : \Omega \to \mathbb{R}$ are three non-negative functions, with*

$$f \in L^d_{loc}(\Omega), \quad e^{\mathscr{A}(pg^d)} \in L^1_{loc}(\Omega), \quad h \in L^d_{loc}(\Omega).$$

If there exists a constant $C_0 > 0$ such that the inequality

$$\left(\fint_B f^d \right)^{1/d} \le C_0 \fint_{2B} fg + C_0 \left(\fint_B h^d \right)^{1/d} \tag{3.1}$$

holds for all balls B with $2B \subset \Omega$, then there exist two constants $c_1 = c_1(n, d, C_0) > 0$ and $C_1 = C_1(n, d, C_0, \beta, p, \sigma) > 0$ such that

$$\fint_{\sigma B} f^d E\left(\frac{f}{\|f\|_{d,B}} \right)^\beta \le C_1 \|f\|^d_{d,B} \fint_B e^{\mathscr{A}(pg^d)} + C_1 \fint_B h^d E\left(\frac{h}{\|f\|_{d,B}} \right)^\beta \tag{3.2}$$

whenever $0 < \beta < c_1 p$, for all balls B such that $2B \subset \Omega$, where $\|f\|^d_{d,B} = \fint_B f^d$.

To prove this lemma, we modify the argument of [6, Lemma 1.2]. Among the needed results, we state the following standard one (cf. [6, p. 144]). For $\varphi \in L^1_{loc}(\mathbb{R}^n)$,

$$M\varphi(x) = \sup_{x \in B} \fint_B |f(y)|\, dy$$

is the Hardy-Littlewood maximal function of φ.

LEMMA 3.2. *For any $\varphi \in L^1(\mathbb{R}^n)$ and any $t > 0$,*

$$\frac{A(n)}{t} \int_{\{|\varphi| \ge t\}} |\varphi(x)|\, dx \le |\{M\varphi \ge t\}| \le \frac{B(n)}{t} \int_{\{|\varphi| \ge t/2\}} |\varphi(x)|\, dx$$

where $A(n)$, $B(n)$ are positive constants depending only on n.

PROOF OF LEMMA 3.1. Fix a ball $B_0 \subset \Omega$. It is not restrictive to assume that

$$\int_{B_0} f(x)^d\, dx = 1. \tag{3.3}$$

For $x \in \mathbb{R}^n$, denote by $d(x) = d(x, \mathbb{R}^n \backslash B_0)$ the Euclidean distance between x and $\mathbb{R}^n \backslash B_0$. We introduce the auxiliary functions

$$\begin{aligned} \tilde{f}(x) &= f(x)\, d(x)^{n/d}, \\ \tilde{h}(x) &= h(x)\, d(x)^{n/d}, \\ w(x) &= \chi_{B_0}(x), \end{aligned}$$

all of them supported on B_0. Standard arguments allow us to rewrite our starting inequality (3.1) as

$$\left(\fint_B \tilde{f}(x)^d\,dx\right)^{1/d} \le c_1(n,d,C_0)\fint_{2B}\tilde{f}(x)g(x)\,dx + c_2(n,d,C_0)\left(\fint_{2B}\tilde{h}(x)^d\,dx\right)^{1/d} + c_3(n,d)\left(\fint_{2B} w(x)\,dx\right)^{1/d}. \tag{3.4}$$

Since this holds for all balls B, we have a counterpart in terms of maximal functions,

$$M(\tilde{f}^d)(y)^{1/d} \le cM(\tilde{f}g)(y) + cM(\tilde{h}^d)(y)^{1/d} + cM(w)(y)^{1/d}$$

for some constant $c = c(d,n,C_0) \ge 1$. We can also rewrite this in terms of level sets,

$$|\{M(\tilde{f}^d) > \lambda^d\}| \le |\{cM(\tilde{f}g) > \lambda\}| + |\{cM(\tilde{h}^d) > \lambda^d\}| + |\{cM(w) > \lambda^d\}|$$

where the constant $c = c(n,d,C_0) \ge 1$ may have changed. Now, there exists $\lambda_1 = \lambda_1(n,d,C_0) > 0$ such that if $\lambda > \lambda_1 = \lambda_1(n,d,C_0)$ then the last term above vanishes. For the other terms, we use Lemma 3.2. We obtain for all $\lambda > \lambda_1$ the estimate

$$\int_{\{\tilde{f}>\lambda\}} \tilde{f}(x)^d\,dx \le c_4(n)\lambda^{d-1}\int_{\{c\tilde{f}g>\lambda\}}\tilde{f}(x)g(x)\,dx + c_4(n)\int_{\{c\tilde{h}>\lambda\}}\tilde{h}(x)^d\,dx, \tag{3.5}$$

where still $c = c(n,d,C_0) \ge 1$. We now introduce the auxiliary function

$$\Theta(\lambda) = \lambda E(\lambda)^{\beta-1}E'(\lambda) = \frac{1}{\lambda}\frac{\mathscr{A}(\lambda)}{E(\lambda)}E(\lambda)^\beta \tag{3.6}$$

and let $\Phi, \Psi : [0,\infty) \to \mathbb{R}$ be defined by

$$\Phi(\lambda) = \frac{1}{\lambda^{d-1}}\frac{d}{d\lambda}(\lambda^{d-1}\Theta(\lambda))$$

$$\Psi(\lambda) = \Theta(\lambda) + \frac{d-1}{\beta}E(t)^\beta.$$

Note that $\Psi'(\lambda) = \Phi(\lambda)$. Further, as E is non-decreasing on $(0,\infty)$, then also $\Theta(\lambda) \ge 0$ for $\lambda > 0$ so that

$$\Psi(\lambda) \ge \frac{d-1}{\beta}E(\lambda)^\beta. \tag{3.7}$$

On the other hand,

$$\Phi(\lambda) = \frac{\Theta(\lambda)}{\lambda}\left(d - 1 + \frac{\lambda\Theta'(\lambda)}{\Theta(\lambda)}\right)$$
$$= E(\lambda)^{\beta-1}E'(\lambda)\left(d + (\beta - 1)\frac{\lambda E'(\lambda)}{E(\lambda)} + \frac{\lambda E''(\lambda)}{E'(\lambda)}\right).$$

Thus by Lemma 5.1 (g) (note that $d > 1$), there exists $\lambda_2 = \lambda_2(d, \beta) > 0$ such that Φ is positive (hence both $\lambda \mapsto \lambda^{d-1}\Theta(\lambda)$ and Ψ are increasing) on the interval (λ_2, ∞).

Fix $\lambda_0 = \max(\lambda_1, \lambda_2)$, and let $j > \lambda_0$ be very large. We multiply both sides of (3.5) by $\Phi(\lambda)$, then integrate with respect to λ over (λ_0, j) and change the order of integration. We obtain

$$\int_{\{\tilde f>\lambda_0\}} \tilde f(x)^d \Big(\int_{\lambda_0}^{\min\{\tilde f(x), j\}} \Phi(\lambda)\, d\lambda\Big)\, dx$$
$$\leq c_4(n) \int_{\{c\tilde f g>\lambda_0\}} \tilde f(x) g(x) \Big(\int_{\lambda_0}^{\min\{c\tilde f(x)g(x), j\}} \lambda^{d-1}\Phi(\lambda)\, d\lambda\Big)\, dx$$
$$+ c_4(n) \int_{\{c\tilde h>\lambda_0\}} \tilde h(x)^d \Big(\int_{\lambda_0}^{\min\{c\tilde h(x), j\}} \Phi(\lambda)\, d\lambda\Big)\, dx.$$

If we denote $G(x) = \tilde f(x) g(x) \int_{\lambda_0}^{\min\{c\tilde f(x)g(x), j\}} \lambda^{d-1}\Phi(\lambda)\, d\lambda$, this leds us to

$$\int_{\{\tilde f>\lambda_0\}} \tilde f(x)^d \Psi(\tilde f_j(x))\, dx \tag{3.8}$$
$$\leq \int_{\{\tilde f>\lambda_0\}} \tilde f(x)^d \Psi(\lambda_0)\, dx + c_4(n) \int_{\{c\tilde f g>\lambda_0\}} G(x)\, dx$$
$$+ c_4(n) \int_{\{c\tilde h>\lambda_0\}} \tilde h(x)^d \Psi(\min\{c\tilde h(x), j\})\, dx$$

where $\tilde f_j(x) = \min\{\tilde f(x), j\}$. Now we proceed as follows. For the first term on the right hand side above, normalization (3.3) gives us that

$$\int_{\{\tilde f>\lambda_0\}} \tilde f(x)^d \Psi(\lambda_0)\, dx \leq C(n, d, C_0, \beta)|B_0|.$$

Concerning the second term at (3.8), we will break it into two terms,

$$\int_{\{c\tilde f g>\lambda_0\}} G(x)\, dx = \int_{\{c\tilde f g>\lambda_0, \tilde f \leq \lambda_0\}} G(x)\, dx + \int_{\{c\tilde f g>\lambda_0, \tilde f>\lambda_0\}} G(x)\, dx. \tag{3.9}$$

Note that there is no restriction in assuming $g \geq 1$, otherwise at the points x with $g(x) < 1$ we can replace g by $g+1$ and observe that still $\exp \mathscr{A}(p(g+1)^d) \in L^1$. Therefore

$$\min\{c\tilde{f}(x)g(x), j\} \leq cg(x)\tilde{f}_j(x),$$

where $\tilde{f}_j(x) = \min\{\tilde{f}(x), j\}$ and still $c = c(n,d,C_0)$, and hence

$$\begin{aligned}
G(x) &\leq \tilde{f}(x)g(x) \int_{\lambda_0}^{cg(x)\tilde{f}_j(x)} \lambda^{d-1}\Phi(\lambda)\,d\lambda \\
&\leq \tilde{f}(x)g(x)c^{d-1}\tilde{f}_j(x)^{d-1}g(x)^{d-1}\Theta(c\tilde{f}_j(x)g(x)) \\
&\leq c^{d-2}\tilde{f}(x)\tilde{f}_j(x)^{d-2}g(x)^{d-1}\mathscr{A}(c\tilde{f}_j(x)g(x))E(c\tilde{f}_j(x)g(x))^{\beta-1} \\
&= c^{d-2}aa_j^{d-2}b^{d-1}\mathscr{A}(ca_jb)E(ca_jb)^{\beta-1}
\end{aligned}$$

where $a = \tilde{f}(x)$, $a_j = \tilde{f}_j(x)$ and $b = g(x)$. Lemma 5.3 now gives us that

$$G(x) \leq \frac{C(n,d,C_0)}{p}\tilde{f}(x)^d E(\tilde{f}_j(x))^\beta + C(n,d,C_0,\beta,p)\exp(\mathscr{A}(pg(x)^d))$$

for almost every x. In particular, at points x where $\tilde{f}(x) \leq \lambda_0$ we obtain

$$G(x) \leq C(n,d,C_0,\beta,p)\exp(\mathscr{A}(pg(x)^d)),$$

because $t \mapsto t^d E(t)^\beta$ is increasing. Combining these estimates with (3.8) and (3.9), and using the lower bound for Ψ (3.7), we obtain

$$\begin{aligned}
\frac{d-1}{\beta}&\int_{\{\tilde{f}>\lambda_0\}} \tilde{f}(x)^d E(\tilde{f}_j(x))^\beta\,dx \\
&\leq \frac{C(n,d,C_0)}{p}\int_{\{\tilde{f}>\lambda_0\}} \tilde{f}(x)^d E(\tilde{f}_j(x))^\beta\,dx \\
&\quad + C(n,d,C_0,\beta,p)\int_{B_0} e^{\mathscr{A}(pg(x)^d)}\,dx \\
&\quad + C(n,d,C_0,\beta)\int_{B_0} \tilde{h}(x)^d E(\tilde{h}(x))^\beta\,dx.
\end{aligned}$$

Thus, by choosing $\beta < \frac{d-1}{2C(n,d,C_0)}p$, the first integral at the right hand side (which is clearly finite since $E(\tilde{f}_j)^\beta$ is bounded) is absorbed into the left hand side and after relabeling constants one obtains

$$\int_{\{\tilde{f}>\lambda_0\}} \tilde{f}(x)^d E(\tilde{f}_j(x))^\beta \, dx$$
$$\le C(n,d,C_0,\beta,p)\Big(\int_{B_0} e^{\mathscr{A}(pg(x)^d)}\,dx + \int_{B_0} \tilde{h}(x)^d E(\tilde{h}(x))^\beta \, dx\Big).$$

By letting $j \to \infty$,

$$\int_{\{\tilde{f}>\lambda_0\}} \tilde{f}(x)^d E(\tilde{f}(x))^\beta \, dx$$
$$\le C(n,d,C_0,\beta,p)\Big(\int_{B_0} e^{\mathscr{A}(pg(x)^d)}\,dx + \int_{B_0} \tilde{h}(x)^d E(\tilde{h}(x))^\beta \, dx\Big),$$

and therefore

$$\int_{B_0} \tilde{f}(x)^d E(\tilde{f}(x))^\beta \, dx$$
$$\le C(n,d,C_0,\beta,p)\Big(\int_{B_0} e^{\mathscr{A}(pg(x)^d)}\,dx + \int_{B_0} \tilde{h}(x)^d E(\tilde{h}(x))^\beta \, dx\Big).$$

In particular, we obtain that $\tilde{f}^d E(\tilde{f})^\beta \in L^1(B_0)$. Now notice that if $x \in \sigma B_0$ and $0 < \sigma < 1$ then $(1-\sigma)^n |B_0| \le c_n\, d(x)^n \le |B_0|$. Recalling the normalization (3.3), we get

$$⨍_{\sigma B_0} f^d E\Big(\frac{f}{\|f\|_{d,B_0}}\Big)^\beta dx$$
$$\le C(n,d,C_0,\beta,p,\sigma)\Big(\|f\|_{d,B_0} ⨍_{B_0} e^{\mathscr{A}(pg^d)}\,dx + \int_{B_0} h^d E\Big(\frac{h}{\|f\|_{d,B_0}}\Big)^\beta dx\Big)$$

where $\|f\|_{d,B_0}^d = ⨍_{B_0} f^d$. This gives us the desired inequality. □

Now we are ready to prove (1.11).

Corollary 3.3. *Let $\phi : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion, whose distortion function K satisfies*

$$e^{\mathscr{A}(pK)} \in L^1(\Omega).$$

Then $|D\phi|^n$ belongs to $L^{P_\beta}_{loc}(\Omega)$ for every $\beta < c(n)p$, and we have the estimate

$$⨍_{\sigma B_0} \frac{J(x,\phi)}{\|J(\cdot,\phi)\|_{1,B_0}} E\Big(\frac{J(x,\phi)}{\|J(\cdot,\phi)\|_{1,B_0}}\Big)^\beta dx \le C(n,\beta,p,\sigma) ⨍_{B_0} e^{\mathscr{A}(pK(x))}\,dx$$

for all balls B_0 with $2B_0 \subset \Omega$, and all $0 < \sigma < 1$.

PROOF. With the notation of Lemma 3.1, let $f = J(\cdot,\phi)^{1/d}$, $g = K^{1/d}$, $h = 0$ and $d = \frac{n+1}{n}$. Then all the integrability assumptions in Lemma 3.1 are satisfied, and concerning (3.1) there is nothing to say since it is equivalent to the inequality

$$\frac{1}{|B|}\int_B J(x,\phi)\,dx \leq C\Big(\frac{1}{|2B|}\int_{2B}|D\phi(x)|^{n^2/(n+1)}\Big)^{(n+1)/n}$$

which follows by standard arguments since ϕ is regular enough (see equation (2.3) in [9]). Then, by Lemma 3.1, we obtain the estimate

$$\fint_{\sigma B} J(x,\phi)E\Big(\Big(\frac{J(x,\phi)}{\|J(\cdot,\phi)\|_{1,B}}\Big)^{n/(n+1)}\Big)^{\beta}dx \leq C_1\|J(\cdot,\phi)\|_{1,B}\fint_B e^{\mathscr{A}(pK(x))}\,dx,$$

for every $\beta < c_1 p$, being c_1 and C_1 as in Lemma 3.1. By Lemma 5.1 (d), E does not see powers, so we can write this as

$$\fint_{\sigma B}\frac{J(x,\phi)}{\|J(\cdot,\phi)\|_{1,B}}E\Big(\frac{J(x,\phi)}{\|J(\cdot,\phi)\|_{1,B}}\Big)^{\beta}dx \leq C_1\fint_B e^{\mathscr{A}(pK(x))}\,dx.$$

Now the desired integrability for $|D\phi|^n$ comes from the inequality

$$P_\beta(xy) \leq \frac{C}{p}(xE(x)^\beta + e^{\mathscr{A}(py)})$$

which easily follows from Lemma 5.2. □

Concerning the sharpness of the above results, we have the following example. Let us write $\log^{(1)}(t) = \log t$, and for each $k = 2,3,\ldots$ write $\log^{(k)}(t) = \log^{(k-1)}(\log t)$.

EXAMPLE 3.4. *Given $k = 2,3,\ldots$, let us define for $x \in B_0$, $B_0 = B(0,r_0)$ and $r_0 > 0$ small enough*

$$\phi(x) = \frac{x}{|x|}\rho(|x|)$$

with $\rho(t) = \big(\log^{(k)}(e+\frac{1}{t})\big)^{-p/n}\big(\log^{(k+1)}(e+\frac{1}{t})\big)^{-1/n}$. It is not hard to see that

$$|D\phi(x)| = \frac{\rho(|x|)}{|x|} \quad \textit{and} \quad J(x,\phi) = \Big(\frac{\rho(|x|)}{|x|}\Big)^{n-1}\rho'(|x|)$$

so that the distortion function $K(x,\phi) = \frac{|D\phi(x)|^n}{J(x,\phi)}$ equals

$$\begin{aligned}K(x,\phi) &= \frac{\rho(|x|)}{|x|\rho'(|x|)}\\ &= \frac{n|x|}{p}\Big(e+\frac{1}{|x|}\Big)\log\Big(e+\frac{1}{|x|}\Big)\ldots\log^{(k)}\Big(e+\frac{1}{|x|}\Big)\frac{\log^{(k+1)}(e+\frac{1}{|x|})}{\log^{(k+1)}(e+\frac{1}{|x|})+\frac{1}{p}}.\end{aligned}$$

It can be easily checked that $\phi \in W^{1,1}_{loc}(B_0)$ *and* $J(\cdot,\phi) \in L^1_{loc}(B_0)$, *so that* ϕ *is a mapping of finite distortion. Further, we have* $e^{\mathscr{A}(pK)} \in L^1(B_0)$ *with*

$$\mathscr{A}(t) = \frac{t}{\log(e+t)\dots\log^{(k-1)}(e+t)}.$$

By Corollary 3.3, *the integrability of the differential* $|Df|^n$ *must be controlled by*

$$P_\beta(t) = \frac{t}{\log(e+t)\dots\log^{(k-1)}(e+t)}(\log^{(k)}(e+t))^{\beta-1},$$

and in fact

$$|D\phi|^n \in L^{P_\beta}_{loc}(B_0) \quad \Leftrightarrow \quad \beta < p.$$

This shows that in Corollary 3.3 *we cannot expect for a better upper bound than* $\beta < p$, *so that in particular* $c(n) \leq 1$. *Similarly, by letting* $f(x) = J(x,\phi)^{n/(n+1)}$, $g(x) = K(x,\phi)^{n/(n+1)}$, $h(x) = 0$ *and* $d = \frac{n+1}{n}$ *in Lemma* 3.1, *one sees that there is no room for improvement other than the precise value of the constant* c_1.

4. Weak mappings of finite distortion

Here we assume that $\mathscr{A}$, E are as in the previous sections. That is, $\mathscr{A} : [1,\infty) \to [0,\infty)$ is smooth, non-decreasing, satisfying conditions (2.1), (2.3), (2.4) and (2.6), and

$$E(t) = 1 + \int_1^t \frac{\mathscr{A}(s)}{s^2}\, ds.$$

Some properties of this function E are given in Lemma 5.1 below. In this section, we face the following question. Let $\Omega \subset \mathbb{R}^n$ be a domain. Suppose that $f \in W^{1,1}_{loc}(\Omega;\mathbb{R}^n)$ has differential $|Df|$ such that

$$|Df|^n \in L^{Q_\beta}_{loc}(\Omega) \quad \text{for some } \beta > 0,$$

where

$$Q_\beta(t) = \frac{\mathscr{A}(t)}{E(t)^{1+\beta}}.$$

We also assume our mapping f satisfies almost everywhere in Ω the distortion inequality

$$|Df(x)| \leq K(x)J(x,f)$$

with distortion function K such that $e^{\mathscr{A}(pK)} \in L^1$. We want to show that f is a *true* mapping of finite distortion, that is $J(\cdot,f) \in L^1_{loc}$, provided that $\beta > 0$ is small enough.

This question has already been treated for bounded K in the planar [1, 27] and spatial [21] cases. Also when $\mathscr{A}(t) = t$ a qualitatively sharp result was given in [9, Theorem 1.3].

Theorem 4.1. *Let $f \in W^{1,Q_\beta}_{loc}(\Omega;\mathbb{R}^n)$ be such that $e^{\mathscr{A}(pK)} \in L^1(\Omega)$. There exists a constant $c_n > 0$ such that if*

$$0 < \beta < c_n p,$$

then $J(\cdot, f)$ is locally integrable in Ω and therefore $f \in W^{1,Q_0}_{loc}(\Omega;\mathbb{R}^n)$. In particular, f is a mapping of finite distortion.

Proof. Let us fix a ball $B_0 = B(x_0, r)$, strictly included in Ω, and let $\varphi \in \mathscr{C}_0^\infty(B_0)$ be a positive function. Denote

$$g(x) = \begin{cases} |\varphi(x)Df(x)| + |f(x) \otimes \nabla\varphi(x)| & x \in B_0 \\ 0 & x \in \mathbb{R}^n \setminus B_0 \end{cases}$$

and $f = (f_1, \dots, f_n)$, and let $u = \varphi f_1$. For each $\lambda > 0$, let

$$F_\lambda = \{x \in B_0 : Mg(x) \le \lambda \text{ and } x \text{ is a Lebesgue point of } u\}.$$

Following the ideas of [10], one can show that there exists a constant $c = c(n)$ and a $c\lambda$-Lipschitz continuous function u_λ such that $u_\lambda = u$ on F_λ. Then the new function

$$f_\lambda = (u_\lambda, \varphi f_2, \dots, \varphi f_n)$$

belongs to the Sobolev space $W^{1,q}_{loc}(\Omega;\mathbb{R}^n)$ for all $q < n$ and has Lipschitz first component, so that one can integrate by parts

$$\int_{B_0} J(x, f_\lambda)\, dx = 0$$

and therefore

$$\int_{F_\lambda} J(x, \varphi f)\, dx \le -\int_{B_0 \setminus F_\lambda} J(x, f_\lambda)\, dx.$$

Arguing as in [9], this leds us to

$$\int_{\{g \le \lambda\}} \varphi(x)^n J(x, f)\, dx \le c(n) \int_{\{g \le 2\lambda\}} |f(x) \otimes \nabla\varphi(x)| g(x)^{n-1}\, dx + c(n)\lambda \int_{\{g > \lambda\}} g(x)^{n-1}\, dx. \tag{4.1}$$

We now introduce an auxiliary function Φ as

$$\frac{1}{\beta E(\lambda)^{\beta}} - \frac{\lambda E'(\lambda)}{E(\lambda)^{\beta+1}} = \int_{\lambda}^{\infty} \Phi(t)\,dt.$$

The above definition forces

$$\Phi(s) \le C\frac{E'(s)}{E(s)^{\beta+1}} \tag{4.2}$$

for $s \ge s_0$. We multiply both sides of (4.1) by $\Phi(\lambda)$, integrate over some interval (t,∞) and change the order of integration:

$$\begin{aligned} &\int_{B_0} \varphi(x)^n J(x,f)\Big(\int_{\max\{g(x),t\}}^{\infty} \Phi(\lambda)\,d\lambda\Big)\,dx \\ &\quad\le c(n)\int_{B_0} |f(x)\otimes\nabla\varphi(x)|g(x)^{n-1}\Big(\int_{\max\{(1/2)g(x),t\}}^{\infty} \Phi(\lambda)\,d\lambda\Big)\,dx \\ &\quad\quad + c(n)\int_{B_0\cap\{g>t\}} g(x)^{n-1}\Big(\int_{t}^{g(x)} \lambda\Phi(\lambda)\,d\lambda\Big)\,dx. \end{aligned} \tag{4.3}$$

Now we look for a lower bound for the left hand side at (4.3). By Lemma 5.1 (b), we can choose t_0 so that

$$0 \le \frac{tE'(t)}{E(t)} \le \frac{1}{2\beta}$$

for all $t > t_0$, and then for such a t we also have

$$\int_{B_0} \varphi(x)^n J(x,f)\frac{\max\{g(x),t\}E'(\max\{g(x),t\})}{E(\max\{g(x),t\})^{1+\beta}}\,dx \le \frac{1}{2}\int_{B_0} \frac{\varphi(x)^n J(x,f)}{\beta E(\max\{g(x),t\})^{\beta}}\,dx.$$

This fact, together with the definition of Φ, gives us a lower bound for the left hand side of (4.3),

$$\frac{1}{2}\int_{B_0} \frac{\varphi(x)^n J(x,f)}{\beta E(\max\{g(x),t\})^{\beta}}\,dx \le \int_{B_0} \varphi(x)^n J(x,f)\Big(\int_{\max\{g(x),t\}}^{\infty} \Phi(\lambda)\,d\lambda\Big)\,dx.$$

On the other hand, using the fact that $t \mapsto t\Phi(t)$ is positive and non-decreasing, and also (4.2), we get

$$\int_t^g \lambda\Phi(\lambda)\,d\lambda \le g^2\Phi(g) \le C\frac{g^2E'(g)}{E(g)^{\beta+1}}.$$

Thus (4.3) can be rewritten as

$$(4.4)\qquad \frac{1}{2\beta}\int_{B_0\cap\{g>t\}}\frac{\varphi(x)^n J(x,f)}{E(g(x))^\beta}\,dx+\frac{1}{2\beta}\int_{B_0\cap\{g\le t\}}\frac{\varphi(x)^n J(x,f)}{E(t)^\beta}\,dx$$
$$\le c(n)\int_{B_0}\frac{|f(x)\otimes\nabla\varphi(x)|g(x)^{n-1}}{\beta E\big(\max\{\frac12 g(x),t\}\big)^\beta}\,dx$$
$$+\,c(n)\int_{B_0\cap\{g>t\}}\frac{g(x)^{n+1}E'(g(x))}{E(g(x))^{\beta+1}}\,dx.$$

Note that the assumption $Df\in L^{P_\beta}_{loc}$ says that all the integrals above are finite. We now use the definition of g, the convexity of $t\mapsto t^{n-1}\mathscr{A}(t)$ for t large enough, the identity $|Df|^n=KJ$ and Lemma 5.2 with $\beta=0$ to get

$$\begin{aligned}
&g^{n+1}E'(g)\\
&\quad= g^{n-1}\mathscr{A}(g)\le c(n)(\varphi^{n-1}|Df|^{n-1}\mathscr{A}(\varphi|Df|)+|f\otimes\nabla\varphi|^{n-1}\mathscr{A}(|f\otimes\nabla\varphi|))\\
&\quad= c(n)\varphi^{n-1}J^{(n-1)/n}K^{(n-1)/n}\mathscr{A}(\varphi J^{1/n}K^{1/n})+c(n)|f\otimes\nabla\varphi|^{n-1}\mathscr{A}(|f\otimes\nabla\varphi|)\\
&\quad\le \frac{c(n)}{p}\varphi^n JE(\varphi J^{1/n})+c(n,p)e^{\mathscr{A}(pK)}+c(n)|f\otimes\nabla\varphi|^{n-1}\mathscr{A}(|f\otimes\nabla\varphi|).
\end{aligned}$$

Next, we divide both sides by $E(g)^{1+\beta}$. This gives us that

$$\begin{aligned}
\frac{g^{n+1}E'(g)}{E(g)^{\beta+1}}&\le\frac{c(n)}{p}\varphi^n J\frac{E(\varphi J^{1/n})}{E(g)^{\beta+1}}+c(n,p)\frac{e^{\mathscr{A}(pK)}}{E(g)^{\beta+1}}+c(n)\frac{|f\otimes\nabla\varphi|^{n-1}\mathscr{A}(|f\otimes\nabla\varphi|)}{E(g)^{\beta+1}}\\
&\le\frac{c(n)}{p}\frac{\varphi^n J}{E(g)^\beta}+c(n,p)\frac{e^{\mathscr{A}(pK)}}{E(g)^{\beta+1}}+c(n)\frac{|f\otimes\nabla\varphi|^{n-1}\mathscr{A}(|f\otimes\nabla\varphi|)}{E(g)^{\beta+1}}.
\end{aligned}$$

Summarizing, (4.4) becames

$$(4.5)\qquad \Big(\frac{1}{2\beta}-\frac{c(n)}{p}\Big)\int_{B_0\cap\{g>t\}}\frac{\varphi(x)^n J(x,f)}{E(g(x))^\beta}\,dx+\frac{1}{2\beta}\int_{B_0\cap\{g\le t\}}\frac{\varphi(x)^n J(x,f)}{E(t)^\beta}\,dx$$
$$\le c(n)\int_{B_0}\frac{|f(x)\otimes\nabla\varphi(x)|g(x)^{n-1}}{\beta E\big(\max\{\frac12 g(x),t\}\big)^\beta}\,dx$$
$$+\,c(n,p)\int_{B_0\cap\{g>t\}}\frac{e^{\mathscr{A}(pK(x,f))}}{E(g(x))^{\beta+1}}\,dx$$
$$+\,c(n)\int_{B_0\cap\{g>t\}}\frac{|f(x)\otimes\nabla\varphi(x)|^{n-1}\mathscr{A}(|f(x)\otimes\nabla\varphi(x)|)}{E(g(x))^{\beta+1}}\,dx.$$

Here is where we choose $\beta = \frac{p}{4c(n)}$. Then, after multiplication by $E(t)^\beta$, we obtain

$$\begin{aligned}\frac{1}{2\beta}\int_{B_0\cap\{g\le t\}} &\varphi(x)^n J(x,f)\,dx \\ &\le c(n)\int_{B_0} |f(x)\otimes\nabla\varphi(x)| g(x)^{n-1}\frac{E(t)^\beta}{\beta E\big(\max\{\frac{1}{2}g(x),t\}\big)^\beta}\,dx \\ &\quad + c(n,p)\int_{B_0\cap\{g>t\}} e^{\mathscr{A}(pK(x,f))}\frac{E(t)^\beta}{E(g(x))^{\beta+1}}\,dx \\ &\quad + c(n)\int_{B_0\cap\{g>t\}} |f(x)\otimes\nabla\varphi(x)|^{n-1}\mathscr{A}(|f(x)\otimes\nabla\varphi(x)|)\frac{E(t)^\beta}{E(g(x))^{\beta+1}}\,dx\end{aligned}$$

and letting $t\to\infty$ this finally gives

$$\frac{1}{2\beta}\int_{B_0}\varphi(x)^n J(x,f)\,dx \le c(n)\int_{B_0}|f(x)\otimes\nabla\varphi(x)|g(x)^{n-1}\,dx \tag{4.6}$$

because $|f\otimes\nabla\varphi|g^{n-1}$, $e^{\mathscr{A}(pK)}$ and $|f\otimes\nabla\varphi|\mathscr{A}(|f\otimes\nabla\varphi|)$ are all integrable on B_0. In particular, (4.6) says that $J(\cdot,f)\in L^1_{loc}$ and therefore f is a mapping of finite distortion. □

EXAMPLE 4.2. *Let ϕ be as in Example 3.4. We define*

$$f(x) = \frac{\phi(x)}{|\phi(x)|^2}$$

for $x\in B_0$. Easy computations show that $f\in W^{1,1}_{loc}(B_0)$. Indeed,

$$|Df(x)| = \frac{1}{|x|\rho(|x|)} \quad \text{and} \quad J(x,f) = \frac{|x|\rho'(|x|)}{\rho(|x|)}\frac{1}{|x|^n\rho(|x|)^n},$$

at almost every point $x\in B_0$, and clearly $J(\cdot,f)>0$ because ρ is strictly increasing. Thus f has a well defined distortion function K which actually agrees with $K(\cdot,\phi)$, that is

$$\begin{aligned}K(x,f) &= \frac{|Df(x)|^n}{J(x,f)}\frac{\rho(|x|)}{|x|\rho'(|x|)} \\ &= \frac{n|x|}{p}\Big(e+\frac{1}{|x|}\Big)\log\Big(e+\frac{1}{|x|}\Big)\dots\log^{(k)}\Big(e+\frac{1}{|x|}\Big)\frac{\log^{(k+1)}\big(e+\frac{1}{|x|}\big)}{\log^{(k+1)}\big(e+\frac{1}{|x|}\big)+\frac{1}{p}}.\end{aligned}$$

Therefore we have that $\exp(\mathscr{A}(pK))\in L^1$ for $\mathscr{A}$ as in Example 3.4, although the Jacobian determinant $J(\cdot,f)$ is not locally integrable, so f cannot be a mapping of finite distortion and $|Df|^n$ cannot belong to $L^{P_0}_\beta$. In fact, if

$$Q_\beta(t) = \frac{t}{\log(e+t)\dots\log^{(k-1)}(e+t)(\log^{(k)}(e+t))^{\beta+1}}$$

then Theorem 4.1 *forces* $|Df|^n$ *to belong to* $L^{Q_\beta}_{loc}(B_0)$ *only for* $\beta \geq c_n p$. *What actually happens is that*

$$|Df| \in L^{Q_\beta}_{loc} \quad \Leftrightarrow \quad \beta > p,$$

so that Theorem 4.1 *can only be improved by finding the precise value of* c_n.

5. Technicalities

As in the previous sections, $\mathscr{A} : [1,\infty) \to [0,\infty)$ is a smooth, onto, non-decreasing function, such that (2.1), (2.3), (2.4) and (2.6) hold. We extend it by 0 to $[0,1]$. We have represented $\mathscr{A}$ as

$$\mathscr{A}(t) = \frac{t}{L(t)}$$

where $L : [1,\infty) \to [1,\infty)$ is also smooth, non-decreasing, onto, and $L(t) = 1$ for $t \in [0,1]$. Recall as well that

$$E(t) = 1 + \int_1^t \frac{\mathscr{A}(s)}{s^2}\,ds = 1 + \int_1^t \frac{ds}{sL(s)}$$

is smooth, monotonically increasing to infinity, and E' is decreasing. We understand that $E(t) = 1$ for $t \in [0,1]$.

Proposition 5.1. *For* $\mathscr{A}$, *E and L as above, the following holds:*

(a) *There is a constant* $C > 0$ *such that* $E(t) \leq Ct$ *for all* $t \geq 1$.
(b) $\lim_{t\to\infty} \frac{tE'(t)}{E(t)} = 0$.
(c) *There exists* $C > 0$ *such that* $\frac{tE'(t)}{E(t)} \leq \frac{C}{\log t}$ *for all* $t \geq 0$.
(d) *L and E do not see powers, i.e. for each* $\alpha > 0$ *there is* $C = C(\alpha)$ *such that* $L(t^\alpha) \leq CL(t)$ *for all* $t \geq 1$, *and similarly for E.*
(e) *If* $t, s \geq t_0$ *then* $E(s+t) \leq E(s) + E(t)$.
(f) *If* $t \geq t_0$ *then* $E(ts) \leq C(E(t) + E(s))$.
(g) $\lim_{t\to 0} \frac{tE''(t)}{E'(t)} = -1$.

Proof. For (a), use just (2.1) and l'Hôpital's rule to see that

$$\lim_{t\to\infty} \frac{E(t)}{t} = \lim_{t\to\infty} \frac{1}{tL(t)} = 0,$$

and then the statement is clear. Claim (b) follows by the definition of E. Indeed, by the definition of E, (2.6) and Lemma 2.2 we have

$$\frac{tE'(t)}{E(t)} = \frac{\mathscr{A}(t)}{tE(t)} \simeq \frac{1}{\mathscr{A}^{-1}(2\log t)} \simeq \frac{1}{2\log tL(2\log t)} \leq \frac{C}{\log t}$$

from which (c) follows as well. Claim (d) is a consequence of (c) and (2.4). In fact, for L we proceed as follows,

$$0 < \log \frac{L(t^\alpha)}{L(t)} = \int_t^{t^\alpha} \frac{L'(s)}{L(s)} ds \leq C \int_t^{t^\alpha} \frac{ds}{s \log s} = C \log \alpha$$

whenever $\alpha > 1$. The same reasoning can be aplied to E. For (e), it is not restrictive to assume $1 \leq s \leq t$. Since E is smooth on $(1, \infty)$, we can use the mean value theorem, so that there exists $\xi \in (t, t+s)$ such that

$$\frac{E(t+s) - E(t)}{E(s)} = \frac{sE'(\xi)}{E(s)} \leq \frac{sE'(s)}{E(s)} = \frac{1}{E(s)L(s)} \leq 1$$

because E' is decreasing and $E(s), L(s) \geq 1$ for $s \geq 1$ (in particular, can take $t_0 = 1$). To show (f) we write

$$E(ts) \leq E((t+s)^2) \leq CE(t+s) \leq C(E(t) + E(s))$$

because E does not see powers. Finally, note that

$$1 + \frac{tE''(t)}{E'(t)} = \frac{t\mathscr{A}'(t)}{\mathscr{A}(t)} - 1 = -\frac{tL'(t)}{L(t)},$$

from which (g) can be obtained. □

Lemma 5.2. *Let $c, p > 0$ and $d > 1$ be fixed. For every $a \geq 0$, $b \geq 1$ and $\beta \geq 0$,*

$$a^{d-1} b^{d-1} \mathscr{A}(cab) E(cab)^{\beta-1} \leq \frac{C(d,c)}{p} a^d E(a)^\beta + C(d, \beta, p, c) e^{\mathscr{A}(pb^d)}.$$

Proof. For P defined as in Lemma 2.1, we have the following inequality

$$P(ab) \leq \frac{1}{p} a + e^{(1/2)\mathscr{A}(pb)},$$

for each $p > 0$. By relabelling variables, this can be written as

$$P(a^d b^d) \leq \frac{1}{p} a^d + e^{(1/2)\mathscr{A}(pb^d)}.$$

Due to (2.6), we have that $\frac{1}{C} \frac{\mathscr{A}(t)}{E(t)} \leq P(t) \leq C \frac{\mathscr{A}(t)}{E(t)}$. Thus the above inequality reads as

$$\frac{\mathscr{A}(a^d b^d)}{E(a^d b^d)} \leq C \Big(\frac{1}{p} a^d + e^{(1/2)\mathscr{A}(pb^d)} \Big)$$

or equivalently

$$a^{d-1}b^{d-1}\frac{ab}{L(a^d b^d)E(a^d b^d)} \le C\Big(\frac{1}{p}a^d + e^{(1/2)\mathscr{A}(pb^d)}\Big).$$

By Lemma 5.1 (d), neither L nor E do see powers, so that

$$a^{d-1}b^{d-1}\frac{\mathscr{A}(ab)}{E(ab)} \le C(d)\Big(\frac{1}{p}a^d + e^{(1/2)\mathscr{A}(pb^d)}\Big)$$

and we just paid the price that the constant at the right hand side now depends on d. Further, both $\mathscr{A}$ and E are doubling, so that

$$a^{d-1}b^{d-1}\frac{\mathscr{A}(cab)}{E(cab)} \le C(d,c)\Big(\frac{1}{p}a^d + e^{(1/2)\mathscr{A}(pb^d)}\Big) \tag{5.1}$$

again by suitably modifying the constant at the right hand side. This is precisely the desired inequality for $\beta = 0$. To get it as well for $\beta > 0$, we start by noting that

$$E(cab)^\beta \le 2E(a)^\beta + C(\beta,c)E(b)^\beta.$$

This follows from Lemma 5.1 (f) and the inequality $(x+y)^\beta \le 2x^\beta + C(\beta)y^\beta$. We then multiply the above inequality by (5.1),

$$\begin{aligned} &a^{d-1}b^{d-1}\frac{\mathscr{A}(cab)}{E(cab)}E(cab)^\beta \\ &\quad\le C(d,c)\Big(\frac{1}{p}a^d + e^{(1/2)\mathscr{A}(pb^d)}\Big)(2E(a)^\beta + C(\beta,c)E(b)^\beta) \end{aligned} \tag{5.2}$$

and then the desired inequality comes, provided that we show that

$$C(\beta,c)a^d E(b)^\beta \le a^d E(a)^\beta + C(d,\beta,p,c)e^{\mathscr{A}(pb^d)} \tag{5.3}$$

and

$$C(d,\beta,c), E(b)^\beta e^{(1/2)\mathscr{A}(pb^d)} \le C(d,\beta,p,c)e^{\mathscr{A}(pb^d)} \tag{5.4}$$

and

$$C(d,c)E(a)^\beta e^{(1/2)\mathscr{A}(pb^d)} \le \frac{C(d,c)}{p}a^d E(a)^\beta + C(d,\beta,p,c)e^{\mathscr{A}(pb^d)}. \tag{5.5}$$

For proving (5.3), we first see that if $C(\beta,c)E(b)^\beta \le E(a)^\beta$ then the inequality is obvious. Otherwise, we have $C(\beta,c)E(b) \ge E(a)$ and then

$$a^d E(b)^\beta \le E^{-1}(C(\beta,c)E(b))^d E(b)^\beta$$
$$\le E^{-1}(C(\beta,c,d)E(b))E(b)^\beta \le C(d,\beta,p,c)e^{\mathscr{A}(pb^d)}$$

which can be easily shown due to the slow growth properties of E. Inequality (5.4) is also an easy consequence of the slow growth of E. Finally, to prove (5.5), we can assume that $e^{(1/2)\mathscr{A}(pb^d)} \le \frac{1}{p}a^d$, since otherwise (5.5) is clear. Then one has that

$$a \le p^{1/d}\exp\Big(\frac{1}{2d}\mathscr{A}(pb^d)\Big).$$

Now, using Lemma 5.1 (a) and (d) we get

$$E(a) \le C(d,\beta)E(a^{d/\beta}) \le C(d,\beta)a^{d/\beta} \le C(d,\beta,p)\exp\Big(\frac{1}{2\beta}\mathscr{A}(pb^d)\Big)$$

which easily gives (5.5). □

LEMMA 5.3. *Let $c, p, \beta > 0$ and $d > 1$ be fixed. For every $a \ge 0$, $b \ge 1$, let $a_j = \min\{a, j\}$. Then*

$$aa_j^{d-2}b^{d-1}\mathscr{A}(ca_jb)E(ca_jb)^{\beta-1} \le \frac{C(d,c)}{p}a^d E(a_j)^\beta + C(d,\beta,p,c)e^{\mathscr{A}(pb^d)}.$$

PROOF. We use the above inequality. On one hand, there is no restriction in assuming that $a_j \ge 1$ for every j. Hence $a_j^{(d-1)/d} \le a_j$ and thus by Lemma 5.2

$$a_j^{d-2}b^{d-1}\mathscr{A}(ca_jb) = \frac{ca_j^{d-1}b^d}{L(ca_jb)} \le \frac{ca_j^{d-1}b^d}{L(ca_j^{(d-1)/d}b)} = (a_j^{(d-1)/d}b)^{d-1}\mathscr{A}(ca_j^{(d-1)/d}b)$$
$$\le \frac{C(d,c)}{p}a_j^{d-1}E(a_j) + \exp\Big(\mathscr{A}\Big(\frac{pb^d}{M}\Big)\Big)$$

where $M = M(d) > 1$ is a large constant, to be determined later. Hence

$$aa_j^{d-2}b^{d-1}\mathscr{A}(ca_jb)E(ca_jb)^{\beta-1}$$
$$\le a\Big(\frac{C(d,c)}{p}a_j^{d-1}E(a_j) + \exp\Big(\mathscr{A}\Big(\frac{pb^d}{M}\Big)\Big)\Big)E(ca_jb)^{\beta-1}$$
$$\le \frac{C(d,c)}{p}a\Big(a_j^{d-1} + \exp\Big(\mathscr{A}\Big(\frac{pb^d}{M}\Big)\Big)\Big)E(ca_jb)^\beta.$$

Now, since $(x+y)^\beta \le 2x^\beta + C(\beta)y^\beta$, we get

$$E(ca_jb)^\beta \le 2E(a_j)^\beta + C(\beta,c)E(cb)^\beta$$

and then a multiplication gives us

$$
\begin{aligned}
&aa_j^{d-2}b^{d-1}\mathscr{A}(ca_jb)E(ca_jb)^{\beta-1}\\
&\quad\le \frac{C(d,c)}{p}a\Big(a_j^{d-1}+\exp\Big(\mathscr{A}\Big(\frac{pb^d}{M}\Big)\Big)\Big)(2E(a_j)^\beta + C(\beta,c)E(cb)^\beta)\\
&\quad\le \frac{C(d,c)}{p}aa_j^{d-1}E(a_j)^\beta + \frac{C(d,c)}{p}aa_j^{d-1}C(\beta,c)E(cb)^\beta\\
&\qquad+\frac{C(d,c)}{p}a\exp\Big(\mathscr{A}\Big(\frac{pb^d}{M}\Big)\Big)E(a_j)^\beta + \frac{C(d,\beta,c)}{p}a\exp\Big(\mathscr{A}\Big(\frac{pb^d}{M}\Big)\Big)E(cb)^\beta
\end{aligned}
$$

and this gives us the desired inequality, provided that we check the estimates,

$$
aE(a_j)^\beta\exp\Big(\mathscr{A}\Big(\frac{pb^d}{M}\Big)\Big)\le a^dE(a_j)^\beta + C(d,\beta)\exp(\mathscr{A}(pb^d)) \tag{5.6}
$$

and

$$
aa_j^{d-1}C(\beta,c)E(cb)^\beta \le a^dE(a_j)^\beta + C(d,\beta,p,c)\exp(\mathscr{A}(pb^d)) \tag{5.7}
$$

and

$$
a\exp\Big(\mathscr{A}\Big(\frac{pb^d}{M}\Big)\Big)E(cb)^\beta \le a^dE(a_j)^\beta + C(d,\beta,p,c)\exp(\mathscr{A}(pb^d)). \tag{5.8}
$$

We check first (5.6). If $\exp\big(\mathscr{A}\big(\frac{pb^d}{M}\big)\big)\le a^{d-1}$, then we are done. If not, using that $E(t)\le|\log(t)|$,

$$
\begin{aligned}
aE(a_j)^\beta\exp(\mathscr{A}(pb^d/M)) &\le \left|\frac{1}{d-1}\mathscr{A}(pb^d/M)\right|^\beta \exp\Big(\frac{d}{d-1}\mathscr{A}(pb^d/M)\Big)\\
&\le C(d,\beta)\exp\Big(\frac{2d}{d-1}\mathscr{A}(pb^d/M)\Big)
\end{aligned}
$$

and we are reduced to find a constant $M>1$ large enough so that

$$
\exp\Big(\frac{2d}{d-1}\mathscr{A}(pb^d/M)\Big)\le \exp(\mathscr{A}(pb^d)).
$$

Equivalently, we must find M so that

$$
\frac{2d}{d-1}\mathscr{A}(x/M)\le\mathscr{A}(x),\quad x\ge p.
$$

But for this we only need M to be large enough, since

$$
\frac{\mathscr{A}(x/M)}{\mathscr{A}(x)} = \frac{L(x)}{ML(x/M)} = \frac{1}{M}\exp\int_{x/M}^{x}\frac{tL'(t)}{L(t)}\frac{dt}{t}\le\frac{(\log M)^{C_0}}{M}
$$

where C_0 is the constant in (2.4). Thus (5.6) follows. The inequality (5.8) can be similarly checked, and so we are reduced to prove (5.7). For this, there is nothing to say if $C(\beta,c)E(cb)^\beta \le E(a)^\beta$, since

$$aa_j^{d-1}E(a)^\beta \le a^d E(a_j)^\beta,$$

as $t \mapsto t^{d-1}E(t)^{-\beta}$ is increasing for large enough t. Otherwise, $E(a) \le C(\beta,c)E(cb)$ and then one just has to argue similarly as in the proof of (5.3). □

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References

[1] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math. 173 (1994), 37–60.

[2] K. Astala - T. Iwaniec - P. Koskela - G. Martin, *Mappings of BMO-bounded distortion*, Math. Ann. 317 (2000) 4, 703–726.

[3] K. Astala - T. Iwaniec - G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Math. Series vol. 48, Princeton University Press, 2009.

[4] K. Astala - J. Gill - S. Rohde - E. Saksman, *Optimal regularity for planar mappings of finite distortion*, http://arxiv.org/pdf/0801.4624.

[5] B. Boyarski, *Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients* (russian), Mat. Sb. N. S. 43(85) 1957, 451–503.

[6] M. Bildhauer - M. Fuchs - X. Zhong, *A lemma on the higher integrability of functions with applications to the regularity theory of two-dimensional generalized Newtonian fluids*, Manuscripta Math. 116 (2005), 135–156.

[7] G. David, *Solutions de l'equation de Beltrami avec* $\|\mu\|_\infty = 1$, Ann. Acad. Sci. Fenn. Ser. A I Math. 13 (1988), no. 1, 25–70.

[8] A. Elcrat - S. Meyers, *Some results on the regularity for solutions of non-linear elliptic systems and quasiregular functions*, Duke Math. J. 42 (1975), 121–136.

[9] D. Faraco - P. Koskela - X. Zhong, *Mappings of Finite Distortion: the degree of regularity*, Adv. Math. 190 (2005), no. 2, 300–318.

[10] D. Faraco - X. Zhong, *A short proof of the self-improving regularity of quasiregular mappings*, Proc. Amer. Math. Soc. 134 (2006), no. 1, 187–192.

[11] F. W. Gehring, *The* L^p *integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. 130 (1973), 265–277.

[12] F. Gianetti - L. Greco - A. Passarelli di Napoli, *Regularity of mappings of finite distortion*, preprint.

[13] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Ann. of Math Stud. 105, Princeton University Press, Princeton, NJ 1983.

[14] L. Greco, *Sharp integrability of nonnegative jacobians*, Rend. Mat. Appl. (7) 18 (1998), no. 3, 585–600.

[15] S. Hencl - P. Koskela, *Mappings of finite distortion: discreteness and openness for quasi-light mappings*, Ann. Inst. H. Poincar Anal. Non Linaire 22 (2005), no. 3, 331–342.

[16] T. Iwaniec, *p-harmonic tensors and quasiregular mappings*, Ann. of Math. (2) 136 (1992), no. 3, 589–624.

[17] T. Iwaniec, *The Gehring lemma*, Quasiconformal mappings and analysis (Ann Arbor, MI, 1995), 181–204, Springer, New York, 1998.

[18] T. Iwaniec - P. Koskela - G. Martin, *Mappings of BMO-distortion and Beltrami-type operators*, J. Anal. Math. 88 (2002), 337–381.

[19] T. Iwaniec - P. Koskela - G. Martin - C. Sbordone, *Mappings of Finite Distortion: $L^n \log^\alpha L$-integrability*, J. London Math. Soc. (2) 67 (1) (2003), 123–136.

[20] T. Iwaniec - P. Koskela - J. Onninen, *Mappings of finite distortion: monotonicity and continuity*, Invent. Math. 144 (2001), no. 3, 507–531.

[21] T. Iwaniec - G. Martin, *Geometric function theory and nonlinear analysis*, Oxford University Press, Oxford (2001).

[22] T. Iwaniec - G. Martin, *Quasiregular mappings in even dimensions*, Acta Math. 170 (1993), no. 1, 29–81.

[23] J. Kauhanen - P. Koskela - J. Maly - J. Onninen - X. Zhong, *Mappings of finite distortion: sharp Orlicz-conditions*, Rev. Mat. Iberoamericana 19 (2003), no. 3, 857–872.

[24] P. Koskela - K. Rajala, *Mappings of finite distortion: removable singularities*, Israel J. Math. 136 (2003), 269–283.

[25] Yu. G. Reshetnyak, *Space mappings with bounded distortion*, Transl. Math. Monographs, 73, Amer. Math. Soc., Providence, RI, (1989).

[26] J. Onninen - X. Zhong, *A note on mappings of finite distortion: the sharp modulus of continuity*, Michigan Math. J. 53 (2005), no. 2, 329–335.

[27] S. Petermichl - A. Volberg, *Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular*, Duke Math. J. 112 (2002), no. 2, 281–305.

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Albert Clop (corresponding author)
Departament de Matemàtiques
Facultat de Ciències, Campus de la U.A.B.
08193-Bellaterra (Barcelona)
Catalonia
albertcp@mat.uab.cat

Pekka Koskela
Department of Mathematics and Statistics
P.O. Box 35 (MaD)
FI-40014 University of Jyväskylä
Finland
pkoskela@maths.jyu.fi