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Solid Mechanics — *Deep foundations*, by PIERO VILLAGGIO, communicated on 11 June 2009.

Dedicated to the memory of Renato Caccioppoli

ABSTRACT. — The stress transmission between a rigid foundation and the ground below is traditionally formulated into mathematical terms as the elastic problem of finding the stress state in a half-plane loaded by a rigid indentor. But this model is not realistic since foundations are not built on the surface of the ground but below its level, at the bottom of an excavation. We here suggest a solution for a notched elastic half-plane loaded by a rigid punch applied at the throat of the cavity.

KEY WORDS: Soil mechanics, Plane Elasticity.

MATHEMATICS SUBJECT CLASSIFICATION 2000: 74M15.

1. INTRODUCTION

Foundations are rigid platforms designed for transmitting a vertical load on a soft substrate, like earth, distributing the pressure over a sufficiently large area in order to avoid the excessive stress concentrations at the interface. The typical model of a foundation, sketched in Fig. 1, is that of a rigid infinite beam, of rectangular cross-section of base $2L$ and height h, resting on an elastic half-space. Since the length of the beam is infinite, each cross-section of the system is a plane of symmetry and hence the stress state in the half-space can be determined by applying the methods of plane elasticity. In this case the elastic solution is that of a rigid beam with straight horizontal base indenting an elastic half-plane. This is a classic solution in plane elasticity. Its difficulty stems on the fact that the boundary conditions on the edge of the half-plane are mixed type. But still the problem was solved by different methods. Föppl [1], Sadowsky [6], Szabó [7, §22] expanded the solution in terms of a series of trigonometric functions, while Muskhelishvili [4] and Milne-Thomson [3] used the complex variable method.

However, modeling a foundation as a rigid plane punch is not realistic for its bottom is not resting on the edge of the half-plane, but at the bottom of a trench of a given deepth, say H, and given width $2L$ as shown in Fig. 2. But the elastic solution to an indentation problem like that drawn in Fig. 2 is not obtainable in an explicity form. We here study a different elastic problem formulated to illustrate the influence of the trench in the propagation of stresses inside the half-plane.

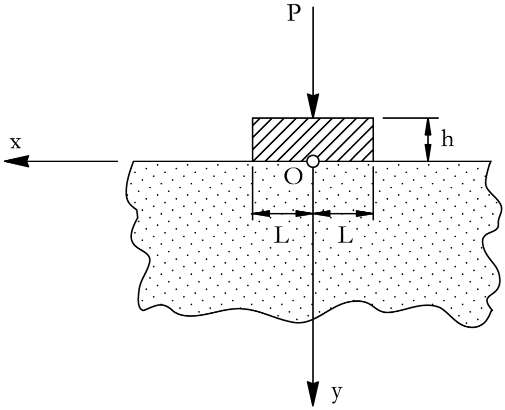

Figure 1. The scheme of a rigid foundation.

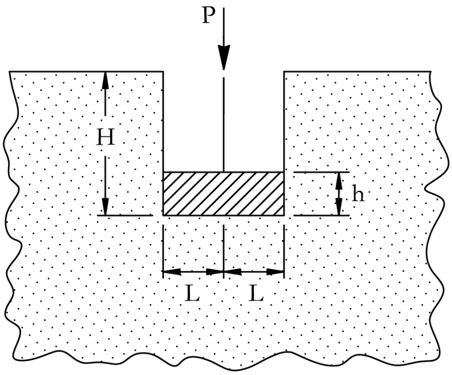

Figure 2. The trench.

2. Smooth contact

We consider, instead of the case illustrated in Fig. 2 a semi-infinite region with a rounded notch, as sketched in Fig. 3. Its profile can be analytically represented by taking a system of Cartesian x, y-coordinates with origin at 0 and using the representation in terms of complex variables (*cf.* Neuber [5, §4.11])

$$z = m(\zeta) = \zeta + i\eta_0 - \frac{1}{\zeta + i\eta_0}, \tag{1}$$

where $z = x + iy$ and $\zeta = \xi + i\eta$, and η_0 is a real constant ($\eta_0 > 1$). Equation (1) is a mapping associating points of the η-plane with points of the z-plane. Since $m(\zeta)$ is analytic, the mapping is conformal everywhere except at the points

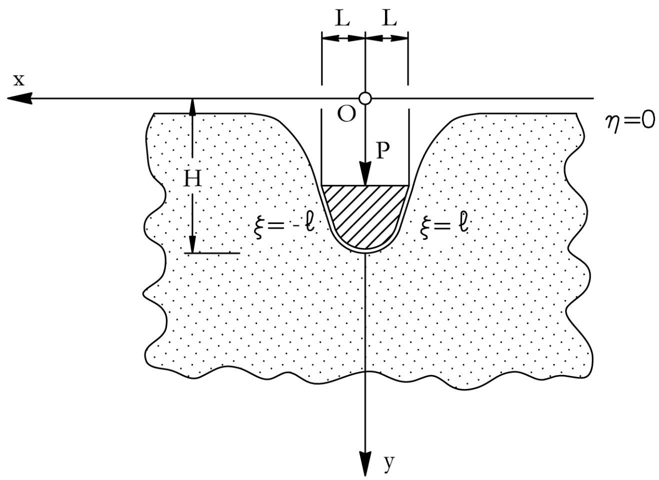

Figure 3. The notched half-plane.

$\zeta = i(\pm 1 - \eta_0)$ where $m'(\zeta) = 0$. Therefore the half-plane $\eta \geq 0$ of the ζ-plane is conformally mapped into the region of the z-plane situated below the curve

$$(2) \qquad x = \xi - \frac{\xi}{\xi^2 + \eta_0^2}, \quad y = \eta_0 + \frac{\eta_0}{\xi^2 + \eta_0^2}, \quad -\infty < \xi < \infty,$$

in the z-plane. In particular, the depth H is given by (see Fig. 3)

$$(3) \qquad H = y(0,0) = \eta_0 + \frac{1}{\eta_0},$$

and for the width $2L$ we may assume the expression

$$(4) \qquad 2L = x(l,0) - x(-l,0) = 2l\Big(1 - \frac{1}{l^2 + \eta_0^2}\Big),$$

where l is a suitably chosen value of ξ.

Let us now study the following elastic problem formulated in the ζ-plane. The half-plane $\eta \geq 0$ is subject to a vertical (parallel to the η-axis) rigid displacement $v = \delta$ applied in the interval $-l \leq \xi \leq l$ of its boundary $\eta = 0$, while the tangential stress $\tau_{\xi\eta}$ is zero. The complementary part of the boundary is unladed. This is the classical problem of smooth indentation of a half-plane. But in the z-plane, after use of tranformation (1), we can get the solution to the problem shown in Fig. 3.

The treatement of this mixed boundary value problem is classical (*cf.* Milne-Thomson [3, 4.23]). Let us introduce the complex function $W(\zeta)$ which determines the stress state in the half-plane $\eta \geq 0$. $W(\zeta)$ is holomorphic in the lower half-plane $\eta > 0$, denoted by L. In the region $\eta < 0$, denoted by R, where the material is absent, $W(\zeta)$ is defined by analytic continuation. Let us denote the values of $W(\zeta)$ as $\zeta \to t$, a point of the boundary, from L or R by $W^L(t)$, $W^R(t)$, respectively.

The boundary conditions are $\tau_{\xi\eta}(t,0) = 0$, which implies (*cf.* Milne-Thomson [3, 4.52])

$$\overline{W}(\zeta) + W(\zeta) \equiv 0, \tag{5}$$

and the mixed condition

$$v(t,0) = \delta \quad \text{for } |t| < l, \qquad \sigma_\eta(t,0) = 0 \quad \text{for } |t| > l. \tag{6}$$

This last equation, written in terms of $W(\zeta)$, reduces to a Riemann-Hilbert problem for the sectionally holomorphic function $W(\zeta)$. The problem can be further simplified by use of equation (5). Omitting the etails, we arrive at the following expression for $W(\zeta)$:

$$(m'(\zeta) + \overline{m}'(\zeta))W(\zeta) = \frac{C_0}{\sqrt{\zeta^2 - l^2}}, \tag{7}$$

where C_0 is a constant to be determined. C_0 must be imaginary in order to satisfy (5). A second condition to be met is

$$\lim_{\zeta\to\infty} \zeta W(\zeta) = -\frac{iP}{\pi}, \tag{8}$$

which indicates that iP is the vector resultant at the exernal force applied to the boundary. Therefore, since

$$\lim_{\zeta\to\infty} m'(\zeta) = \lim_{\zeta\to\infty} \overline{m}'(\zeta) = 1, \tag{9}$$

combination of (7) with (8) yields $C_0 = \frac{-2iP}{\pi}$ and hence $W(\zeta)$ has the explicit form

$$W(\zeta) = \frac{-2iP}{\pi(m'(\zeta) + \overline{m}'(\zeta))} \frac{1}{\sqrt{\zeta^2 - l^2}} = \frac{-2P}{\pi(m'(\zeta) + \overline{m}'(\zeta))} \frac{1}{\sqrt{l^2 - \zeta^2}}. \tag{10}$$

Once we have $W(\zeta)$, the stress state in the region is completely determined.

Here we limit ourselves to illustrate a particular feature of the solution. The pressure under the foundation is given by the formula (*cf.* Milne-Thomson [3, 4.30])

$$\begin{aligned} -2[p(t) - is(t)] &= W^L(t) - W^R(t) = (\text{after consideration of } (10)) \\ &= \frac{-4P}{\pi(m'(t) + \overline{m}'(t))} \frac{1}{\sqrt{l^2 - t^2}}. \end{aligned} \tag{11}$$

Since the right hand side of (11) is real, the tangential stress $s(t)$ is zero, as expected. The pressure $p(t)$ is instead

$$p(t) = \frac{2P}{\pi\left[1 + \frac{1}{(t+i\eta_0)^2} + 1 + \frac{1}{(t-i\eta_0)^2}\right]} \frac{1}{\sqrt{l^2 - t^2}}. \tag{12}$$

The pressure is, as expected, infinite at the edge points $t = \pm l$, while at the centre $t = 0$, has the value

$$p(0) = \frac{P}{\pi\left(1 - \frac{1}{\eta_0^2}\right)l}. \tag{13}$$

This result suggests an interesting comparison with the case of a flat punch where $\eta_0 \to \infty$ and the pressure $p(0) = \frac{P}{\pi l}$. In the case of a deep foundation, where $\eta_0 > 0$, $p(0)$, as given by (13), is higher than $\frac{P}{\pi l}$. This implies that, for deep foundations, the pressure concentration at the vertex of the notch tends to increase with the depth.

3. Complete adesion

The assumption of smooth contact examined above is sometimes not realistic because the bottom of a foundation may be rough enough to prevent the horizontal displacement of the material. In this case the boundary conditions of the problem must be changed. Instead of requiring the stress $\tau_{\xi\eta}$ to vanish in the interval $|t| < l$, we put $u(t, 0) = 0$, where u is the displacement in the x-direction.

The problem is known as that of "adherent" indentation, and its solution for the half-plane is classical. Its extension to a region with parabolic boundary was done by Paria [6] in an unduly ignored paper. We here repeat Paria's argument adapting it to the present case. The displacement is constant and so $u'(t) + iv'(t) = 0$ under the base. The solution of the corresponding Riemann-Hilbert problem is

$$m'(\zeta)W(\zeta) = P_0\chi(\zeta), \tag{14}$$

where P_0 is a constant and $\chi(\zeta)$ is the Plemelj function given by (*cf.* Milne-Thomson [3, 4.42])

$$\chi(\zeta) = \frac{(\zeta - l)^{\tau-1}}{(\zeta + l)^{\tau}}, \quad \tau = \frac{1}{2} + i\lambda, \quad \lambda = \frac{\ln\kappa}{\pi}, \tag{15}$$

where κ is an elastic constant ($1 < \kappa < 7$). The constant P_0 is again determined by a condition like (8), and hence its value is $P_0 = -\frac{iP}{\pi}$. Therefore the solution is

$$W(\zeta) = \frac{-iP}{m'(\zeta)}(\zeta - l)^{-(1/2)+i\lambda}(\zeta + l)^{-(1/2)-i\lambda}. \tag{16}$$

Thus the complete stress state in the region $\eta \geq 0$ is determined. In particular, pressure and shear under the foundations are given by

$$W^L(t) - W^R(t) = -2(p(t) - is(t)), \quad |t| < l. \tag{17}$$

Then, using (16) and recalling the property that

$$\chi^L(t) = -\frac{1}{\kappa}\chi^R(t), \quad |t| < l,$$

$$\chi^L(t) = \chi^R(t), \quad |t| > l,$$

equation (17) yields

$$p(t) - is(t) = \frac{(\kappa+1)P}{2\pi\kappa m'(t)} \frac{1}{\sqrt{l^2 - t^2}} e^{i\lambda} \ln\left(\frac{l+t}{l-t}\right). \tag{18}$$

At the vertex $t = 0$, we have $s(0) = 0$, and

$$p(0) = \frac{(\kappa+1)P}{2\pi\kappa l} \frac{1}{\left(1 - \frac{1}{\eta_0^2}\right)}. \tag{19}$$

This means that, in this case again, the pressure at the vertex of the cavity increases for deep cavities for which η_0 is close to 1. Moreover, since $\kappa > 1$, the pressure concentration is higher than that found in Section 3 in the same geometrical situation.

4. Aqueous foundations

Often the foundation rests on a medium that is not elastic, but a sort of wet sponge able to transmit a normal pressure $p(t)$ eventually accompanied by a tangential shear force $s(t)$, due to friction (Fig. 4). For definitess, we will assume that the magnitude of $p(t)$ is a constant p_0 along the whole contact interval $|t| < l$, and $s(t)$ has the expression $s(t) = fp_0\frac{t}{l}$, where f is the friction factor. As it is plausible, $s(0) = 0$ for symmetry, and is an odd function of t.

At this point the problem is purely statical. We must find the value of p_0 such that the vertical resultant of the tractions exerted along the arc having, in the x, y-plane, the parametric equation (2) with $-l \leq t \leq l$. Therefore the vertical resultant of the traction acting along the arc must balance the force P. The related equation is

$$\int_{-l}^{+l} p_0 n_y \, ds + \int_{-l}^{+l} fp_0 \frac{t}{l} t_y \, ds = P, \tag{20}$$

where n_y, t_y are the y-components of the unit normal vector and the unit tangential vector, respectively, and ds is the lenght of the arc element. But from (2), after replacement of ξ with t, we find

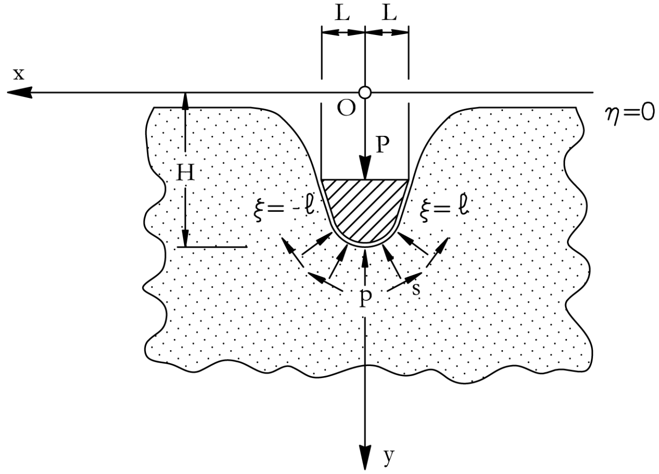

Figure 4. The floating foundation.

$$n_y\, ds = \frac{\partial x}{\partial t} dt, \quad t_y\, ds = \frac{\partial y}{\partial t} dt.$$

The first integral in (20) is immediate

$$\int_{-l}^{+l} p_0 \frac{\partial x}{\partial t} dt = p_0[x(l,0) - x(-l,0)] = p_0 2l\Big(1 - \frac{1}{l^2 + \eta_0^2}\Big).$$

The second is also explicitly evaluable (*cf.* Gradshteyn–Ryzhik [2, 2.175])

$$\int_{-l}^{+l} fp_0 \frac{t}{l} \frac{\partial y}{\partial t} dt = \int_{-l}^{+l} fp_0 \frac{t}{l}\Big(\frac{-2\eta_0 t}{(t^2 + \eta_0^2)^2}\Big) dt = 2fp_0 \frac{\eta_0}{l}\Big(\frac{l}{l^2 + \eta_0^2} - \frac{1}{\eta_0^2} \operatorname{arctg} \frac{l}{\eta_0}\Big).$$

Thus replacement of these values into (20) gives a relation for determining p_0 as a function of P. The expression of p_0 in terms of P is rather involved, but the inspection of two limiting cases may be useful to illustrate the result. For $\eta_0 \to \infty$, that is where the cavity is very flat, equation (20) yields $p_0 = \frac{P}{2l} = \frac{P}{2L}$, a constant pressure over a segment of lenght L. For $l \ll \eta_0$, the cavity is rather narrow and sharp, and neglection of l^2 with respect to η_o^2 leads to he following expression of (20)

(21) $$p_0 = \frac{P}{2l\big(1 - \frac{1}{\eta_0^2}\big)}.$$

Here again we find that in a deep foundation the normal pressure exerted by the terrain on the basis is higher than that arising in a superficial foundation of the same width.

5. Conclusion

Foundations are customarily regarded as plane rigid indentors able to transmit a vertical load on a substrate with a suitable distribution of the interfacial pressure. But, in practice, foundations are built not on the surface of the underlying medium but at the bottom of a trench excavated at a certain depth. The stress state under a deep foundation is more severe than that under a superficial foundation of the same width. However, deep foundations are necessary in order to rely upon a more consistent terrain, and also to prevent the toppling of the superposed structure. Inhabitants of lacustrian villages rested their huts on palafittes. Venetians of the 10^{th} century A.D. built stone palace and dams on wooden piles driven into the Lagoon.

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