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Calculus of Variations — *A sharp isoperimetric inequality in the plane involving Hausdorff distance*, by Angelo Alvino, Vincenzo Ferone, Carlo Nitsch.

Dedicated to the memory of Renato Caccioppoli

Abstract. — We show that among all the convex bounded domain in $\mathbb{R}^2$ having an assigned asymmetry index related to Hausdorff distance, there exists only one convex set (up to a similarity) which minimizes the isoperimetric deficit. We also show how to construct this set. The result can be read as a sharp improvement of the isoperimetric inequality for convex planar domain.

Key words: Isoperimetric inequality, Bonnesen-style inequality, Hausdorff distance, isoperimetric deficit.

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1. Introduction

The classical isoperimetric inequality in the plane states that, among all the subsets of $\mathbb{R}^2$ of prescribed finite measure, the disk has the smallest perimeter, namely

$$P(E) \geq (4\pi|E|)^{1/2}, \quad \text{with equality if and only if } E \text{ is a disk.}$$

Here $|E|$ and $P(E)$ denote, as usual, the measure and the perimeter of the set $E \subset \mathbb{R}^2$.

It is almost impossible to give exhaustive references concerning the isoperimetric inequality, therefore we refer the reader to some pioneering papers [2, 5, 16, 19], to the paper by De Giorgi [9] in the general framework of finite perimeter sets in $\mathbb{R}^n$, to the reviews [12, 20, 25] and to the books [7, 8].

In [3, 4] Bonnesen introduced some remarkable inequalities which imply the isoperimetric one (see also the reviews [7, 21]). For example, we recall that for bounded convex planar sets he proved that

$$P(E)^2 - 4\pi|E| \geq 4\pi d^2. \tag{1.1}$$

Here d is the thickness of the minimal annulus containing the boundary of E and we remark that the constant 4π and the exponent 2 on the right hand side are optimal. The chief tool in the proof was a symmetrization technique known as annular symmetrization. Later Bonnesen's work led to the study of a wider class of inequalities nowadays known as Bonnesen-style isoperimetric inequalities (see [7, 21]).

Following Osserman [7, 21] we say that a Bonnesen-style isoperimetric inequality can be written in the form

$$P(E)^2 - 4\pi|E| \geq F(E),$$

where the function F is nonnegative, vanishes only on the disks, and somehow measures how much E deviates from a disk. There are many different kinds of functions F satisfying these properties, and each one leads to a different refinement of the standard isoperimetric inequality.

Typical situations addressed in the literature are those where the function F depends on the set E through the so-called *Fraenkel asymmetry index* or through the Hausdorff distance from a ball. For the first case we quote the results contained in [15, 13, 10, 1]. In [15, 1] it is considered the case of convex planar sets and the best form of the inequality is given, while in [13, 10] the n-dimensional case is addressed.

As regards the second case, it is clear that inequality (1.1) can be written in terms of Hausdorff distance from a disk (see also [14])

$$P(E)^2 - 4\pi|E| \geq 16\pi d_H(E, C)^2,$$

where $d_H(A, B)$ denotes the Hausdorff distance between the sets $A, B \subset \mathbb{R}^2$, and C is the disk halfway between the inner and the outer circle of the annulus of minimal width that contains the boundary of E. A sharp estimate of this type can be found also in [11], for nearly spherical domains in $\mathbb{R}^n$.

In this paper we are interested in those functions F whose dependence on the set E is only through the *Hausdorff asymmetry index* $\delta(E)$ defined as the *translative Hausdorff distance* of E from a disk D_R having the same measure,

$$\delta(E) = \min_{x \in \mathbb{R}^2} d_H(E, D_R(x)),$$

where $D_R(x)$ is the disk centered at x, such that $|E| = |D_R(x)|$. We provide a sharp Bonnesen–style inequality for planar convex domains E involving just $P(E)^2 - 4\pi|E|$ and $\delta(E)$. Obviously the trivial relation $d \geq \delta$ already implies the following inequality

$$P(E)^2 - 4\pi|E| \geq 4\pi\delta(E)^2.$$

However such an inequality is not sharp. Actually there exists a maximal function G such that it holds

$$P(E)^2 - 4\pi|E| \geq G(\delta(E)). \tag{1.2}$$

The determination of the function G relies on the investigation of the shape of the *optimal* sets, i.e., those sets which minimize the left hand side of (1.2), for fixed $|E|$ and $\delta(E)$.

We show that, for any $0 \le \delta < +\infty$, it is possible to compute $G(\delta)$. In particular, for any fixed value of $|E|$, we work out the analytic expression of the set E with asymmetry index $\delta(E) = \delta$, which achieves the equality sign in (1.2). Moreover we prove that such a set is unique up to translations. Our result is based on a new symmetrization technique introduced in [1]. It is closely related to the circular symmetrization, a technique which is well suited to the bidimensional framework (see also [17, 18]). Using this tool we show how to reshape a given planar convex set keeping, step by step, its measure and its Hausdorff asymmetry index fixed and shortening the perimeter. The procedure eventually provides the family of *optimal* sets.

As a corollary to our result we provide the sharp inequality

$$P(E)^2 - 4\pi|E| \ge 16\delta(E)^2.$$

2. Main statement

In order to state our main result we define the class of *lenses* $\mathscr{Y}_\alpha$ as the family of convex set E satisfying the following properties:

- E has measure α;
- E is symmetric with respect to a straight line such that the part of it which stays on one side of the line coincides with a circular segment (the smallest part of a disk cut by a chord).

Such a class satisfies the following properties.

Proposition 2.1. *For any given positive α and any given positive δ there exists a unique set $Y_\delta \in \mathscr{Y}_\alpha$ such that*

$$\delta(Y_\delta) = \delta.$$

In particular, for such a set, δ will be the difference of the radii of the disk having the same measure, and of the smallest circumscribed disk. Moreover it holds

$$4\pi^2 = \lim_{\delta\to 0} \frac{P(Y_\delta)^2 - 4\pi\alpha}{\delta^2} > \frac{P(Y_\delta)^2 - 4\pi\alpha}{\delta^2} > \lim_{\delta\to+\infty} \frac{P(Y_\delta)^2 - 4\pi\alpha}{\delta^2} = 16.$$

We are now able to state our main result.

Theorem 2.1. *For every convex set $\Omega \in \mathbb{R}^2$, the set $Y_{\delta(\Omega)} \in \mathscr{Y}_{|\Omega|}$ satisfies the inequality*

$$P(\Omega) \ge P(Y_{\delta(\Omega)}),$$

equality holding if and only if $\Omega = Y_{\delta(\Omega)}$, up to translations.

As a consequence we have the following corollary.

Corollary 2.1. *Every convex set* $\Omega \in \mathbb{R}^2$ *satisfies the inequalities*

$$P(\Omega)^2 - 4\pi|\Omega| \geq 16\delta(\Omega)^2,$$
$$P(\Omega)^2 - 4\pi|\Omega| \geq \delta(\Omega)^2(4\pi^2 - H(\delta(\Omega)), \tag{2.1}$$

for some positive $H(\delta) = O(\delta)$.

We postpone the proof of Proposition 2.1 until the last section where we carry over a detailed study of the class $\mathscr{Y}_\alpha$.

Remark 2.1. *We observe that inequality* (2.1) *can be obtained in a different way. Namely, one can consider a one-parameter family of sets* E_δ, $\delta > 0$, *which smoothly converge to a disk as* $\delta \to 0$, *being* $\delta(E_\delta) = \delta$. *The computation of the second derivative of* $P(E_\delta)$ *with respect to* δ *gives:*

$$\lim_{\delta \to 0} \frac{P(E_\delta)^2 - 4\pi|E_\delta|}{\delta^2} \geq 4\pi^2.$$

3. Proof of Theorem 2.1

Let Ω be an open bounded and convex subset of $\mathbb{R}^2$, and D be a circle of radius $R = \left(\frac{|\Omega|}{\pi}\right)^{1/2}$ that achieves the minimal Hausdorff distance to Ω, i.e.:

$$\delta(\Omega) \equiv d_H(\Omega, D).$$

We refer to the last condition as the *optimality condition* for D with respect to Ω. From now on we shall use as the origin of the coordinate system in $\mathbb{R}^2$ the center O of D. We also denote by D_i and D_e the two disks $D_{R-\delta(\Omega)}(O)$ and $D_{R+\delta(\Omega)}(O)$. It is trivial to check that $D_i \subseteq \Omega \subseteq D_e$.

Since Ω is also starshaped with respect to O, we shall use $\rho(\theta)$ to denote a generic Lipschitz radial function which parametrizes the boundary of Ω with respect to the angular variable θ. Such a parametrization will by possibly chosen case by case. The optimality condition immediately implies

$$\max_\theta \{R - \rho(\theta), \rho(\theta) - R\} = \delta(\Omega), \tag{3.1}$$

roughly speaking there exists at least one point in which the boundary of Ω touches either the boundaries of D_i or D_e.

We also observe that, if D is the optimal disk, a line l passing through O cannot split the plane into two open halfplanes T_i and T_e, such that all the points where the boundary of Ω eventually touches the boundary of D_i belong to T_i, while all the points where the boundary of Ω eventually touches the boundary of D_e belong to T_e. Indeed, this would imply that there exists a slight translation of the set Ω in the direction normal to l and towards the halfplane T_i, such that still $D_i \subseteq \Omega \subseteq D_e$ but $\partial\Omega$ will have no intersection with both the boundaries of D_i

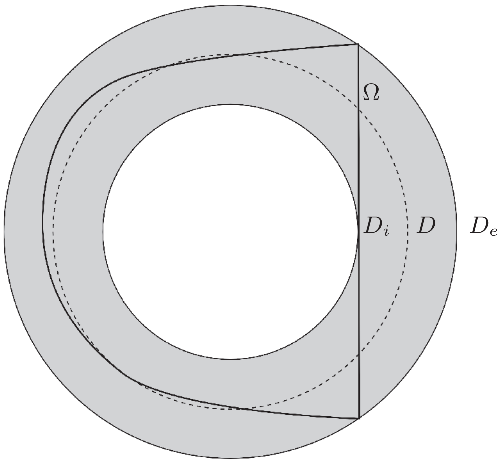

Figure 1. A convex set for which **Case 3** happens.

and D_e contradicting (3.1). It follows that at least one of the following four cases certainly happens.

Case 1 There exists a line l passing through O such that $l \cap D_e$ is included in Ω.
Case 2 There exists a line l passing through O such that $l \cap \Omega$ is included in D_i.
Case 3 There exist three points P_1 P_2 and P_3 of the boundary of Ω such that, P_1 and P_2 also belong to ∂D_e, P_3 lies inside the acute angle having vertex in O and bounded by the halflines passing through P_1 and P_2, and $\operatorname{dist}(P_3, \partial D_i) = \min_{x \in \partial\Omega} \operatorname{dist}(x, \partial D_i)$.
Case 4 There exist three points P_1 P_2 and P_3 of the boundary of Ω such that, P_1 and P_2 also belong to ∂D_i, P_3 lies inside the acute angle having vertex in O and bounded by the halflines passing through P_1 and P_2, and moreover $\operatorname{dist}(P_3, \partial D_e) = \min_{x \in \partial\Omega} \operatorname{dist}(x, \partial D_e)$.

In Figure 1 and Figure 2 we represent two examples of convex sets for which **Case 3** or **Case 4** happens.

We claim that any element of the family $\mathscr{Y}_\alpha$ has the property of being the unique set (up to translations) with the smallest possible perimeter among all the sets having same measure and same *Hausdorff asymmetry index*. In particular we shall see that for any given convex set Ω it holds $P(\Omega) \geq P(Y_{\delta(\Omega)})$, where $Y_{\delta(\Omega)}$ belongs to the family $\mathscr{Y}_{|\Omega|}$. We shall provide the proof of such an assertion for each one of the four aforementioned cases.

3.1. Case 1. By hypotheses there exists a line l passing through O intersecting ∂D_e in two points P_1 and P_2 which belong to $\partial\Omega$. We want to find a convex set having the same property, but also same δ and measure as Ω, and the least possible perimeter. We can restrict our attention to those sets which have two horthogonal axes of symmetry: the line l, and the line intersecting l in O. In fact,

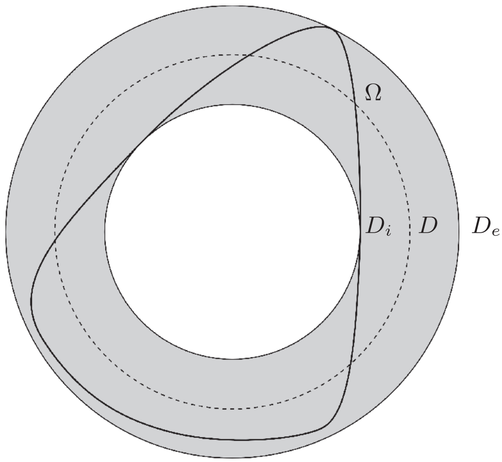

Figure 2. A convex set for which **Case 4** happens.

assuming that Ω does not posses such a symmetry, we denote by Ω^* the Steiner symmetric of Ω with respect to theese two axis, and from well-known properties of the Steiner symmetrizzation we know that Ω^* is convex, $|\Omega^*| = |\Omega|$, $\Omega^* \subseteq D_e$, $|\partial\Omega^*| < |\partial\Omega|$, and moreover P_1 and P_2 belong to $\partial\Omega^*$. Using well known isoperimetric properties of the circular arcs, it is easy to prove that the unique set with the least possible perimeter is the lens $Y_{\delta(\Omega)}$ belonging to the family $\mathscr{Y}_{|\Omega|}$. In view of Proposition 2.1 our claim is proved.

Remark 3.1. *It is important to observe that the proof of* **Case 1** *works in hypothesis weaker than convexity, for instance the starshapedness is enough.*

3.2. Case 2. By hypotheses there exist two parallel lines l_1 and l_2 tangent to D_i and such that the set Ω lies in the strip S between these two lines and contains D_i. We want to find a set having the same property, which is also included in D_e, which has the same measure as Ω, and such that it has the least possible perimeter. This set exists and it is unique [22] (possibly up to translations) and we shall denote it by E_r, where $r = \delta(\Omega)^2/|\Omega|$. We observe that there exists a feasible range of values of r, namely $[0, r_{\max}]$, such that E_r actually exists. Indeed, it is easy to prove that when δ is too big with respect to $|\Omega|$ then the set given by the intersection of the strip S and the disk D_e is too small to contain a set having measure $|\Omega|$.

Some important properties of the set E_r can be found in [22]. In particular E_r is either given by the convex hull of two disks both being translations of D_i, in case r is large enough, E_r is a convex set which includes the largest possible (in terms of measure) convex hull of two balls lying in the intersection between the strip S and the disk D_e. In the first case our claim follows as a consequence of the following result, whose proof is postponed until the last section.

PROPOSITION 3.1. *If Ω is the convex hull of two ball having the same radius then*

$$P(\Omega) \geq P(Y_{\delta(\Omega)})$$

whenever $|\Omega| = |Y_{\delta(\Omega)}|$.

In the second case, since the largest possible convex hull is certainly tangent to D_e, E_r will certainly contain a diameter of D_e and therefore E_r can be treated as in **Case 1** (see Remark 3.1).

REMARK 3.2. *It is important to observe that the proof of* **Case 2** *works even if Ω is not convex provided it is starshaped and lies between two parallel lines l_1 and l_2 both tangent to the inner disk D_i.*

3.3. CASE 3. By hypotheses there exists a parametrization of the radial function $\rho(\theta)$ such that $\rho(0) = \rho(\theta_1) = R + \delta(\Omega)$ for some $\theta_1 < \pi$ and moreover there exists $0 < \theta_2 < \theta_1$ such that $\rho(\theta_2) = \min_\theta \rho(\theta)$. We reshape the set Ω as follows: first of all we replace the restriction ρ_1 of the radial function ρ to the domain $[0, \theta_1]$ with its symmetric increasing rarrangement see [17], namely $\rho_{1\#}(\theta - \theta_1/2)$; the same is done with the complementary part ρ_2, restriction of ρ to the domain $[\theta_1, 2\pi]$, which we replace by $\rho_{2\#}(\theta - (\theta_1 + 2\pi)/2)$. The new radial function that we denote by $\hat{\rho}$ describes the boundary of a star shaped set having same measure as Ω, but in view of the properties of rearrangements [6, 17, 23, 24], also a shorter perimeter.

We can assume that $\rho(\theta_1/2 + \pi) = \min_{\theta \in [\theta_1, 2\pi]} \hat{\rho} > \min_{\theta \in [0, \theta_1]} \hat{\rho} = \rho(\theta_1/2)$ otherwise, if $\rho(\theta_1/2) = \rho(\theta_1/2 + \pi)$ we replace the restriction of $\hat{\rho}$ to the set $[\theta_1/2, \theta_1/2 + \pi]$ with its symmetric decreasing rearrangement, and the restriction of $\hat{\rho}$ to $[\theta_1/2 + \pi, \theta_1 + 2\pi]$ with its symmetric decreasing rearrangement [17]. The resulting function will describe the boundary of a starshaped set having a perimeter shorter then $P(\Omega)$ and containing a diameter of D_e, and the proof will continue as in **Case 1**.

Assuming that $\rho(\theta_1/2) < \rho(\theta_1/2 + \pi)$ we have $0 = |\{\theta \in [\theta_1, 2\pi] : \hat{\rho} < \min \hat{\rho} + \sigma\}| < |\{\theta \in [0, \theta_1] : \hat{\rho} < \min \hat{\rho} + \sigma\}|$ for some $\sigma > 0$ small enough. On the other hand we can also assume that $|\{\theta \in [\theta_1, 2\pi] : \hat{\rho} < \max \hat{\rho}\}| > |\{\theta \in [0, \theta_1] : \hat{\rho} < \max \hat{\rho}\}|$ otherwise the starshaped set described by $\hat{\rho}$ can be treated as in **Case 1**, see Remark 3.1. Therefore by continuity there exist $\min \rho < t < \max \rho$ such that

$$|\{\theta \in [\theta_1, 2\pi] : \hat{\rho} < t\}| \leq |\{\theta \in [0, \theta_1] : \hat{\rho} < t\}|$$

and

$$|\{\theta \in [\theta_1, 2\pi] : \hat{\rho} \leq t\}| \geq |\{\theta \in [0, \theta_1] : \hat{\rho} \leq t\}|.$$

As a consequence there exists $0 < \bar{\theta} < \theta_1/2$ such that

$$\hat{\rho}(\theta_1/2 - \bar{\theta}) = \hat{\rho}((\theta_1 - 2\pi)/2 + \bar{\theta}) = t$$

and by simmetry

$$\hat{\rho}(\theta_1/2 + \bar{\theta}) = \hat{\rho}((\theta_1 + 2\pi)/2 - \bar{\theta}) = t$$

We consider now $\hat{\rho}_1$ and $\hat{\rho}_2$ restriction of $\hat{\rho}$ to $[(\theta_1 - 2\pi)/2 + \bar{\theta}, \theta_1/2 - \bar{\theta}]$ and $[\theta_1/2 + \bar{\theta}, (\theta_1 + 2\pi)/2 - \bar{\theta}]$. We replace $\hat{\rho}_1$ with its symmetric decreasing rearrangement namely $\hat{\rho}_1^{\#}(\theta - (\theta_1 - \pi)/2)$, and $\hat{\rho}_2$ by $\hat{\rho}_2^{\#}(\theta - (\theta_1 + \pi)/2)$. We obtain in this way a radial function which describes the boundary of a starshaped set having shorter perimeter, which can be treated as in **Case 1**.

3.4. Case 4. By hypotheses there exists a parametrization of the radial function $\rho(\theta)$ such that $\rho(0) = \rho(\theta_1) = R - \delta(\Omega)$ for some $\theta_1 < \pi$ and moreover there exists $0 < \theta_2 < \theta_1$ such that $\rho(\theta_2) = \max_\theta \rho(\theta)$. We reshape the set Ω as follows: first of all we replace the restriction ρ_1 of the radial function ρ to the domain $[0, \theta_1]$ with its symmetric decreasing rarrangement, namely $\rho_1^{\#}(\theta - \theta_1/2)$; the same is done with the complementary part ρ_2, restriction of ρ to the domain $[\theta_1, 2\pi]$, which we replace by $\rho_2^{\#}(\theta - (\theta_1 + 2\pi)/2)$. The new radial function that we denote by $\hat{\rho}$ describes the boundary of a star shaped set having same measure as Ω, but in view of the properties of the symmetric decreasing rearrangements, also a shorter perimeter.

We can assume that $\rho(\theta_1/2 + \pi) = \max_{\theta \in [\theta_1, 2\pi]} \hat{\rho} < \max_{\theta \in [0, \theta_1]} \hat{\rho} = \rho(\theta_1/2)$ otherwise, if $\rho(\theta_1/2) = \rho(\theta_1/2 + \pi)$ we replace the restriction of $\hat{\rho}$ to the set $[\theta_1/2, \theta_1/2 + \pi]$ with its symmetric increasing rearrangement, and the restriction of $\hat{\rho}$ to $[\theta_1/2 + \pi, \theta_1 + 2\pi]$ with its symmetric increasing rearrangement. The resulting function $\tilde{\rho}$ will describe the boundary of a starshaped set $\tilde{\Omega}$, such that $P(\tilde{\Omega}) \leq P(\Omega)$, and touching the boundary of D_i in two points symmetric with respect to the origin O. In this case the proof can continue as in **Case 2**, see Remark 3.2, provided that the set lies between two parallel lines tangent to D_i. The last condition can be obtained arguing as in [1], indeed the convexity of the set Ω and the fact that $D_i \subseteq \Omega$ implies that ρ is an absolutely continuous function which almost everywhere satisfies (see [26])

$$|\rho'(\theta)| \leq \frac{\rho(\theta)}{R - \delta(\Omega)} \sqrt{\rho(\theta)^2 - (R - \delta(\Omega))^2} \quad \text{a.e. } \theta \in [0, 2\pi] \tag{3.2}$$

By well known properties of rearrangements (see [18, 23, 24]) the function $\tilde{\rho}$ is also an absolutely continuous function which satisfies (3.2) and since the same inequality holds with equality sign for the radial functions describing any line tangent to D_i the set $\tilde{\Omega}$ lies in a strip between two of such parallel lines.

Assuming that $\rho(\theta_1/2) > \rho(\theta_1/2 + \pi)$ we have $0 = |\{\theta \in [\theta_1, 2\pi] : \hat{\rho} > \min \hat{\rho} - \sigma\}| < |\{\theta \in [0, \theta_1] : \hat{\rho} > \min \hat{\rho} - \sigma\}|$ for some $\sigma > 0$ small enough. On the other hand we can also assume that $|\{\theta \in [\theta_1, 2\pi] : \hat{\rho} > \min \hat{\rho}\}| > |\{\theta \in [0, \theta_1] : \hat{\rho} > \min \hat{\rho}\}|$ otherwise the starshaped set described by $\hat{\rho}$ can be treated as in **Case 2**.

Therefore by continuity there exist $\min \rho < t < \max \rho$ such that

$$|\{\theta \in [\theta_1, 2\pi] : \hat{\rho} > t\}| \leq |\{\theta \in [0, \theta_1] : \hat{\rho} > t\}|$$

and

$$|\{\theta \in [\theta_1, 2\pi] : \hat{\rho} \geq t\}| \geq |\{\theta \in [0, \theta_1] : \hat{\rho} \geq t\}|.$$

As a consequence there exists $0 < \bar{\theta} < \theta_1/2$ such that

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We consider now $\hat{\rho}_1$ and $\hat{\rho}_2$ restriction of $\hat{\rho}$ to $[(\theta_1 - 2\pi)/2 + \bar{\theta}, \theta_1/2 - \bar{\theta}]$ and $[\theta_1/2 + \bar{\theta}, (\theta_1 + 2\pi)/2 - \bar{\theta}]$. We replace $\hat{\rho}_1$ with its symmetric increasing rearrangement namely $\hat{\rho}_{1\#}(\theta - (\theta_1 - \pi)/2)$, and $\hat{\rho}_2$ by $\hat{\rho}_{2\#}(\theta - (\theta_1 + \pi)/2)$. We obtain in this way a radial function which describes the boundary of a starshaped set which, arguing as before, lies in a strip bounded by two parallel lines tangent to D_i, and can be treated as in **Case 2**.

4. Two classes of convex sets

In this section we study the ratio

$$\frac{P(E)^2 - 4\pi|E|}{\delta(E)^2}, \tag{4.1}$$

when the set E belongs to two different classes, namely, we will consider the classes of "stadia" and of "lenses".

Class $\mathscr{X}_\alpha$ (stadia)

The class $\mathscr{X}_\alpha$ contains any convex set E satisfying the following properties:

- E has measure α;
- E is the union of a rectangle with two half disks having the diameter which coincides with two opposite sides.

As we will see, any element in the class $\mathscr{X}_\alpha$ can be identified by its Hausdorff asymmetry index δ. Indeed, if we denote by a and b (see Figure 3) the measures

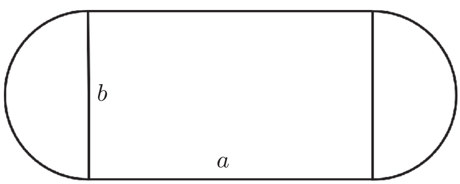

Figure 3. A set in $\mathscr{X}_\alpha$.

of the sides, being b the diameter of the two half disks joined to the rectangle, it is very easy to compute perimeter, measure and Hausdorff asymmetry index of $E \in \mathcal{X}_\alpha$:

$$P(E) = 2a + \pi b, \tag{4.2}$$

$$|E| = ab + \frac{\pi}{4} b^2, \tag{4.3}$$

$$\delta(E) = \max\left\{ \sqrt{\frac{\alpha}{\pi}} - \frac{b}{2}, \frac{a+b}{2} - \sqrt{\frac{\alpha}{\pi}} \right\}. \tag{4.4}$$

Actually, equality (4.3) states a constraint from which it is possible to write a in terms of b and α

$$a = \frac{\alpha}{b} - \frac{\pi}{4} b. \tag{4.5}$$

The above relation states that, for fixed α, a is a decreasing function of b for $0 < b \le 2\sqrt{\frac{\alpha}{\pi}}$. Using (4.4) and (4.5), for fixed α, $\delta(E)$ can be written as a decreasing function of b, for $0 < b \le 2\sqrt{\frac{\alpha}{\pi}}$,

$$\delta(E) = \begin{cases} \sqrt{\frac{\alpha}{\pi}} - \frac{b}{2} & \text{if } \frac{2\sqrt{\pi\alpha}}{8-\pi} \le b \le 2\sqrt{\frac{\alpha}{\pi}}, \\ \frac{\alpha}{2b} + \left(\frac{1}{2} - \frac{\pi}{8}\right) b - \sqrt{\frac{\alpha}{\pi}} & \text{if } 0 < b \le \frac{2\sqrt{\pi\alpha}}{8-\pi}. \end{cases} \tag{4.6}$$

This means that we can parametrize the sets in $\mathcal{X}_\alpha$ in terms of the Hausdorff asymmetry index, that is, we will denote by X_δ the set such that $X_\delta \in \mathcal{X}_\alpha$ and $\delta(X_\delta) = \delta$.

Using (4.2), (4.5) and (4.6), the ratio (4.1) for the set X_δ can be calculated in terms of δ:

$$\frac{P(X_\delta)^2 - 4\pi|X_\delta|}{\delta^2} = \begin{cases} \left(\pi \frac{2\sqrt{\pi\alpha} - \pi\delta}{\sqrt{\pi\alpha} - \pi\delta}\right)^2 & \text{if } 0 \le \delta \le 2\frac{4-\pi}{8-\pi}\sqrt{\frac{\alpha}{\pi}}, \\ \left(\frac{4\alpha - \pi g(\delta)^2}{2\delta g(\delta)}\right)^2 & \text{if } \delta > 2\frac{4-\pi}{8-\pi}\sqrt{\frac{\alpha}{\pi}}, \end{cases} \tag{4.7}$$

where, taking into account (4.6), $g(t)$ is the decreasing function, for $t > 2\frac{4-\pi}{8-\pi}\sqrt{\frac{\alpha}{\pi}}$,

$$g(t) = \frac{4(\sqrt{\pi}t + \sqrt{\alpha}) - 2\sqrt{4(\sqrt{\pi}t + \sqrt{\alpha})^2 - \alpha\pi(4-\pi)}}{\sqrt{\pi}(4-\pi)}. \tag{4.8}$$

Using (4.7) and (4.8), it is easy to prove that, for a fixed α, one has

$$\begin{aligned} &\frac{P(X_\delta)^2 - 4\pi\alpha}{\delta^2} \text{ is increasing w.r.t. } \delta \text{ if } 0 \le \delta \le 2\frac{4-\pi}{8-\pi}\sqrt{\frac{\alpha}{\pi}}, \\ &\frac{P(X_\delta)^2 - 4\pi\alpha}{\delta^2} \text{ is decreasing w.r.t. } \delta \text{ if } \delta > 2\frac{4-\pi}{8-\pi}\sqrt{\frac{\alpha}{\pi}}. \end{aligned} \tag{4.9}$$

Furthermore, it is possible to evaluate the behaviour of the ratio (4.7) when δ goes to zero and when δ diverges, that is, when the stadium tends to a disk or to a line. We have:

$$\begin{aligned} &\lim_{\delta\to 0} \frac{P(X_\delta)^2 - 4\pi\alpha}{\delta^2} = 4\pi^2, \\ &\lim_{\delta\to +\infty} \frac{P(X_\delta)^2 - 4\pi\alpha}{\delta^2} = 16. \end{aligned} \tag{4.10}$$

Class $\mathscr{Y}_\alpha$ (lenses)

The class $\mathscr{Y}_\alpha$ contains any convex set E satisfying the following properties:

- E has measure α;
- E is symmetric with respect to a straight line such that the part of it which stays on one side of the line coincides with a circular segment (the smallest part of a disk cut by a chord).

As we will see, any element in the class $\mathscr{Y}_\alpha$ can be identified by its Hausdorff asymmetry index δ. To show this fact we fix the reference axes (x, y) in such a way that x-axis coincides with the above mentioned line of symmetry and we describe the set by using two parameters $r > 0$ and $\theta \in \,]0, \pi/2]$ (see Figure 4) which are the radius of the disk and half of the angle subtended by the chord. We have:

$$E = \{(x, y) \in \mathbb{R}^2 : |x| \le r\sin\theta, |y| \le \sqrt{r^2 - x^2} - r\cos\theta\}.$$

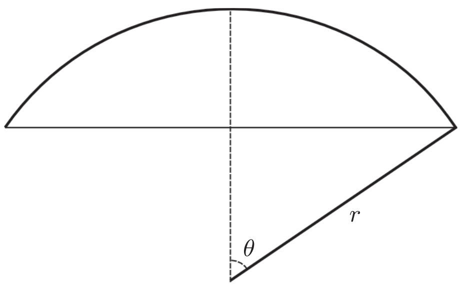

Figure 4. Half of a set in $\mathscr{Y}_\alpha$.

It is very easy to compute perimeter, measure and Hausdorff asymmetry index of E:

$$P(E) = 4r\theta, \tag{4.11}$$

$$|E| = 2r^2(\theta - \sin\theta\cos\theta), \tag{4.12}$$

$$\delta(E) = \max\{r\sin\theta - \sqrt{\alpha/\pi}, \sqrt{\alpha/\pi} - r(1-\cos\theta)\}, \tag{4.13}$$

Actually, equality (4.12) states a constraint from which it is possible to write r in terms of θ and α

$$r = \sqrt{\frac{\alpha}{2(\theta - \sin\theta\cos\theta)}}. \tag{4.14}$$

The above relation states that, for fixed α, r is a decreasing function of θ, for $0 < \theta \le \pi/2$. Using (4.13) and (4.14), for fixed α, $\delta(E)$ can be written as a decreasing function of θ, for $0 < \theta \le \pi/2$. Indeed, it is possible to prove that, if r is given by (4.14), then,

$$r\sin\theta - \sqrt{\alpha/\pi} \ge \sqrt{\alpha/\pi} - r(1-\cos\theta), \quad 0 < \theta \le \pi/2, \tag{4.15}$$

that is,

$$\sqrt{\frac{\alpha}{2(\theta - \sin\theta\cos\theta)}}(1 + \sin\theta - \cos\theta) \ge 2\sqrt{\frac{\alpha}{\pi}}, \quad 0 < \theta \le \pi/2.$$

If we square the above inequality, it is equivalent to

$$\pi(1 + \sin\theta - \cos\theta) - 4\theta + (4-\pi)\sin\theta\cos\theta \ge 0, \quad 0 < \theta \le \pi/2. \tag{4.16}$$

The above inequality can be proven computing the derivative of the function $h(\theta) = \pi(1 + \sin\theta - \cos\theta) - 4\theta + (4-\pi)\sin\theta\cos\theta$ which appears on the left hand side of (4.16),

$$\begin{aligned} h'(\theta) &= \pi(\cos\theta + \sin\theta) - 4 + (4-\pi)(\cos^2\theta - \sin^2\theta) \\ &= \frac{2\tan\frac{\theta}{2}}{\left(1 + \tan^2\frac{\theta}{2}\right)^2}\left(\pi + (3\pi - 16)\tan\frac{\theta}{2} + \pi\tan^2\frac{\theta}{2} - \pi\tan^3\frac{\theta}{2}\right). \end{aligned}$$

The observation that the polynomial $\pi + (3\pi - 16)t + \pi t^2 - \pi t^3$ has a negative derivative for $t \in [0,1]$ gives the desired result (4.16). This means that (4.15) holds true and, taking into account (4.13) and (4.14), we can finally write $\delta(E)$ as a decreasing function of θ:

$$\delta(E) = \sqrt{\frac{\alpha\sin^2\theta}{2(\theta - \sin\theta\cos\theta)}} - \sqrt{\frac{\alpha}{\pi}}, \quad 0 < \theta \le \pi/2. \tag{4.17}$$

The above relation states that we can determine θ as a function of $\delta(E)$, that is,

$$\theta = f(\delta(E)), \quad 0 \leq \delta(E) < +\infty, \tag{4.18}$$

where $f(t)$ is the inverse function of the one given in (4.17), which applies the interval $[0, +\infty[$ into $]0, \pi/2]$.

This means that we can parametrize the sets in $\mathscr{Y}_\alpha$ in terms of the Hausdorff asymmetry index, that is, we will denote by Y_δ the set such that $Y_\delta \in \mathscr{Y}_\alpha$ and $\delta(Y_\delta) = \delta$.

For the set Y_δ, using (4.11), (4.14) and (4.18), the ratio (4.1) can be calculated in terms of δ:

$$\frac{P(Y_\delta)^2 - 4\pi|Y_\delta|}{\delta^2} = 4\alpha \frac{2f(\delta)^2 - \pi(f(\delta) - \sin f(\delta)\cos f(\delta))}{\delta^2(f(\delta) - \sin f(\delta)\cos f(\delta))}, \tag{4.19}$$

where $f(t)$ is the decreasing function defined in (4.18).

It is possible to prove that, for a fixed α,

$$\frac{P(Y_\delta)^2 - 4\pi\alpha}{\delta^2} \text{ is decreasing w.r.t. } \delta \text{ in } [0, +\infty[\,. \tag{4.20}$$

In order to prove the above statement we first put $\varphi = 2f(\delta)$ in (4.19) obtaining

$$\frac{P(Y_\delta)^2 - 4\pi\alpha}{\delta^2} = 8\pi\Phi(\varphi), \tag{4.21}$$

where

$$\Phi(\varphi) = \frac{\varphi^2 - \pi(\varphi - \sin\varphi)}{(\sqrt{\pi(1 - \cos\varphi)} - \sqrt{2(\varphi - \sin\varphi)})^2}, \quad \varphi \in\,]0, \pi].$$

Then we show that $\Phi(\varphi)$ is an increasing function of φ. Indeed, we have

$$\Phi'(\varphi) = \frac{\Phi_1(\varphi)\Phi_2(\varphi)}{\Phi_3(\varphi)\sqrt{\varphi - \sin\varphi}}, \tag{4.22}$$

where

$$\begin{aligned}
\Phi_1(\varphi) &= 2\sqrt{1 - \cos\varphi} - \varphi\sqrt{1 + \cos\varphi}, \\
\Phi_2(\varphi) &= \varphi\sqrt{2(1 + \cos\varphi)} - (\pi - \varphi)\sqrt{\pi(\varphi - \sin\varphi)}, \\
\Phi_3(\varphi) &= (\sqrt{\pi(1 - \cos\varphi)} - \sqrt{2(\varphi - \sin\varphi)})^3.
\end{aligned}$$

It is immediate to show that

$$\Phi_1(\varphi) \geq 0 \quad \text{and} \quad \Phi_3(\varphi) \geq 0, \quad \varphi \in\,]0, \pi]. \tag{4.23}$$

As regards $\Phi_2(\varphi)$, we observe that

$$\Phi_2(\varphi) \geq \varphi\sqrt{2(1+u(\varphi))} - (\pi - \varphi)\sqrt{\pi(\varphi - v(\varphi))}, \quad \varphi \in \,]0,\pi], \tag{4.24}$$

where

$$u(\varphi) = \begin{cases} 1 - \dfrac{\varphi^2}{2} & \text{if } 0 \leq \varphi \leq \dfrac{2}{5}\pi, \\ \dfrac{10}{\pi}\left(\dfrac{\pi}{2} - \varphi\right)\cos\left(\dfrac{2}{5}\pi\right) & \text{if } \dfrac{2}{5}\pi < \varphi \leq \dfrac{\pi}{2}, \\ \dfrac{\pi}{2} - \varphi & \text{if } \dfrac{\pi}{2} < \varphi \leq \dfrac{3}{5}\pi, \\ -1 + \dfrac{(\pi-\varphi)^2}{6} - \dfrac{(\pi-\varphi)^4}{24} & \text{if } \dfrac{3}{5}\pi < \varphi \leq \pi, \end{cases} \tag{4.25}$$

$$v(\varphi) = \begin{cases} \varphi - \dfrac{\varphi^3}{6} & \text{if } 0 \leq \varphi \leq \dfrac{2}{5}\pi, \\ 1 + \dfrac{10}{\pi}\left(\varphi - \dfrac{\pi}{2}\right)\left(1 - \sin\left(\dfrac{2}{5}\pi\right)\right) & \text{if } \dfrac{2}{5}\pi < \varphi \leq \dfrac{\pi}{2}, \\ 1 + \dfrac{10}{\pi}\left(\dfrac{\pi}{2} - \varphi\right)\left(1 - \sin\left(\dfrac{2}{5}\pi\right)\right) & \text{if } \dfrac{\pi}{2} < \varphi \leq \dfrac{3}{5}\pi, \\ \pi - \varphi - \dfrac{(\pi-\varphi)^3}{6} & \text{if } \dfrac{3}{5}\pi < \varphi \leq \pi. \end{cases} \tag{4.26}$$

Using (4.25) and (4.26) in (4.24) we have $\Phi_2(\varphi) \geq 0$ in $]0,\pi]$. This inequality, together with (4.17), (4.18), (4.21), (4.22) and (4.23), gives (4.20).

Furthermore, it is possible to evaluate the behaviour of the ratio (4.19) when δ goes to zero and when δ diverges, that is, when the lens tends to a disk or to a line. We have:

$$\lim_{\delta\to 0} \frac{P(X_\delta)^2 - 4\pi\alpha}{\delta^2} = \lim_{\varphi\to\pi} 8\pi\Phi(\varphi) = 4\pi^2, \tag{4.27}$$

$$\lim_{\delta\to +\infty} \frac{P(X_\delta)^2 - 4\pi\alpha}{\delta^2} = \lim_{\varphi\to 0} 8\pi\Phi(\varphi) = 16. \tag{4.28}$$

We conclude this section observing that properties (4.20), (4.28) proven above immediately imply Proposition 2.1. Moreover, a simple argument allows us to prove Proposition 3.1.

Proof of Proposition 3.1. We claim that for each $0 \leq \delta < +\infty$, if $X_\delta \in \mathscr{X}_\alpha$ and $Y_\delta \in \mathscr{Y}_\alpha$, we have:

$$P(X_\delta) \geq P(Y_\delta). \tag{4.29}$$

In view of (4.9), (4.10), (4.20) and (4.27) the assertion follows for

$$0 \le \delta \le 2\frac{4-\pi}{8-\pi}\sqrt{\frac{\alpha}{\pi}}.$$

In the case

$$\delta > 2\frac{4-\pi}{8-\pi}\sqrt{\frac{\alpha}{\pi}},$$

inequality (4.29) is a consequence of the fact that circular arcs minimize perimeter. □

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