Rend. Lincei Mat. Appl. 20 (2009), 379–386

Calculus of Variations — *On a Sobolev-type inequality*, by ANGELO ALVINO.

Dedicated to the memory of Renato Caccioppoli

ABSTRACT. — A new proof of the classical Sobolev inequality in $\mathbb{R}^n$ with the best constant is given. The result follows from an intermediate inequality which connects in a sharp way the L^p norm of the gradient of a function u to L^{p^*} and L^{p^*}-weak norms of u, where $p \in]1, n[$ and $p^* = \frac{np}{n-p}$ is the Sobolev exponent.

KEY WORDS: Sobolev inequality, Isoperimetric inequalities, one-dimensional Calculus of Variations.

MATHEMATICAL SUBJECT CLASSIFICATION: 49K20, 26D10, 39B62.

1. INTRODUCTION

The celebrated Sobolev inequality states that

$$S(n,p)\|u\|_{L^{p^*}} \le \| \, |\nabla u| \, \|_{L^p}, \tag{1}$$

where u is a sufficiently smooth function, defined in R^n, ∇u is the gradient of u, $p \in]1, n[$, $p^* = \frac{np}{n-p}$.

The optimal value of $S(n,p)$ in (1) is

$$\pi 2^{1/n} n^{1/p} (n-p)^{(p-1)/p} (p-1)^{1/n-(p-1)/p} p^{-1/n} \left[\frac{\Gamma\left(\frac{n}{p}\right) \Gamma\left(n - \frac{n}{p}\right)}{\Gamma(n)\Gamma\left(\frac{n}{2}\right)} \right]^{1/n}. \tag{2}$$

This means that (2) is the infimum of the functional

$$F(u) = \frac{\| \, |\nabla u| \, \|_{L^p}}{\|u\|_{L^{p^*}}};$$

it is actually attained (see [1], [8] and, also, [2]) when

$$u(x) = \frac{h}{[1 + k|x|^{p/(p-1)}]^{(n-p)/p}}, \tag{3}$$

where h, k are positive constants.

The proof proceeds in two steps. The first one consists of a symmetrization procedure: u is replaced by its rearrangement $u^\#$ which is a spherically symmetric function and decreases with respect to $|x|$. Moreover u, $u^{\#}$ have the same distribution function, hence they have the same L^{p^*} norm. On the other side, the L^p norm of the gradient decreases as a consequence of the following Pólya Principle

$$\int_{\mathbb{R}^n} |\nabla u^{\#}|^p \, dx \leq \int_{\mathbb{R}^n} |\nabla u|^p \, dx. \tag{4}$$

In conclusion $F(u) \geq F(u^{\#})$; so only radial functions compete in reaching the best constant in (1).

We stress the central role of (4) and recall that it follows from a combined use of the Hölder inequality and the classical isoperimetric inequality

$$P(E) \geq n^{(n-1)/n} \omega_n^{1/n} \min\{|E|, |\mathbb{R}^n \backslash E|\}^{(n-1)/n};$$

here $|E|$ is the Lebesgue measure of a Caccioppoli set E, $P(E)$ is the perimeter of E in the sense of De Giorgi [5],

$$\omega_n = \frac{n\pi^{n/2}}{\Gamma\left(1 + \frac{n}{2}\right)}$$

is the measure of the unitary $(n-1)$-dimensional sphere.

The problem thus becomes a classical question of one-dimensional Calculus of Variation with constraints. It can be dealt with turning it into a Lagrange Problem whose extremals are available. These form a Mayer field; introducing the Weierstrass excess function leads to the result.

As for the second step our proof appeals to simpler tools for free functionals of the Calculus of Variations. A more general Sobolev-type inequality, involving the norm of u in a Marcinkiewicz space, is established. The classical Sobolev inequality (1), with the optimal value (2) of the constant, easily follows.

2. Main result

Let $a > 0$ and consider the following one-parameter family of extremals (3)

$$u_\varepsilon(x) = u_\varepsilon(|x|) = \frac{\varepsilon^{(n-p)/p}}{[1 + (a\varepsilon|x|)^{p/(p-1)}]^{(n-p)/p}}. \tag{5}$$

These functions have the same L^{p^*} norm

$$\|u_\varepsilon\|_{L^{p^*}}^{p^*} = a^{-n} 2\pi^{n/2} \left(\frac{p-1}{p}\right) \frac{\Gamma\left(\frac{n}{p}\right)\Gamma\left(n - \frac{n}{p}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(n)}.$$

Moreover they all solve the nonlinear partial differential equation

$$-\Delta_p u_\varepsilon = n\left(\frac{n-p}{p-1}\right)^{p-1} a^p u_\varepsilon^{p^*-1},$$

which is the Euler-Lagrange equation of the functional

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p \, dx - \frac{1}{p}\frac{(n-p)^p}{(p-1)^{p-1}} a^p \int_{\mathbb{R}^n} |u|^{p^*} \, dx,$$

or

$$J(u) = \frac{\omega_n}{p}\int_0^\infty |u'|^p r^{n-1} \, dr - \frac{\omega_n}{p}\frac{(n-p)^p}{(p-1)^{p-1}} a^p \int_0^\infty |u|^{p^*} r^{n-1} \, dr \tag{6}$$

if u is a radial function.

The curve

$$y = \frac{(p-1)^{(n-p)(p-1)/p^2}}{p^{(n-p)/p}} (ar)^{-(n-p)/p} = \gamma_a(r), \quad r > 0, \tag{7}$$

envelopes the graphs $y = u_\varepsilon(r)$; these cover the region of the first quadrant which lies below the curve (7) and will be called T.

If v is a non negative, sufficiently smooth, compactly supported, radial function let

$$\|v\|_{p^*,\infty} = \sup_{r>0}[r^{n/p^*} v(r)]$$

be its norm in the Marcinkiewicz space of the functions weakly L^{p^*}. If we choose

$$a = \frac{(p-1)^{(p-1)/p}}{p} \frac{1}{\|v\|_{p^*,\infty}^{p/(n-p)}}, \tag{8}$$

the minimum value such that $v(r) \leq \gamma_a(r)$, for all r positive, the envelope (7) becomes

$$y = \|v\|_{p^*,\infty} r^{-(n-p)/p} = \gamma(r).$$

Each graph $y = u_\varepsilon(r)$ touches the envelope at a point which splits it into two curves $C_1(\varepsilon)$, $C_2(\varepsilon)$. These two families of curves are the trajectories of two different fields of extremals of the functional (6), and both defined in the same set T. We denote by $(1, q_1(r, y))$ the former and by $(1, q_2(r, y))$ the latter. As usual, $q_1(r, y)$ is the slope of the extremal of the first family passing through (r, y); $q_2(r, y)$ has an analogous meaning. The envelope also touches the graph of v at least in a point $P = (\alpha, \gamma(\alpha))$ which splits it into two arcs Γ_1, Γ_2. Moreover, we simply denote by C_1, C_2, respectively, the arcs of the families $C_1(\varepsilon)$, $C_2(\varepsilon)$ passing through P.

In Figure 1 (2) the graphs of the envelope $y = \gamma(r)$, Γ_1 (Γ_2), C_1 (C_2) are sketched, together with some further arcs of extremals.

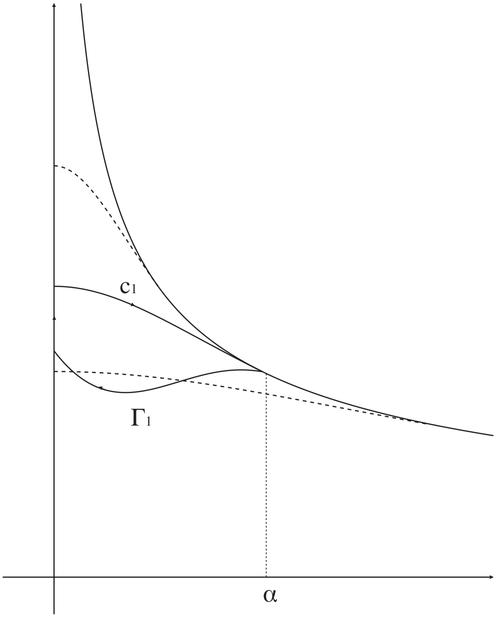

Figure 1

Setting

$$f(r, v, v') = \frac{\omega_n}{p} r^{n-1} \left[|v'|^p - \frac{(n-p)^p}{(p-1)^{p-1}} a^p |v|^{p^*} \right]$$

gives

$$J(v) = \int_0^\alpha f(r, v, v')\, dr + \int_\alpha^\infty f(r, v, v')\, dr = J_1(v) + J_2(v).$$

We begin by estimating $J_1(v)$ from below; to this aim we refer to the first field of extremals.

Since f is convex with respect to the last variable, we get

$$\mathscr{E}(r, v, v', q_1) = f(r, v, v') - f(r, v, q_1) - (v' - q_1) f_{v'}(r, v, q_1) \geq 0,$$

where $\mathscr{E}$ is the well-known Weierstrass excess function. Therefore

(9) $$J_1(v) \geq \int_0^\alpha [f(r, v, q_1) + (v' - q_1) f_{v'}(r, v, q_1)]\, dr.$$

Now we use classical arguments of one-dimensional Calculus of Variations (see, for example, [6], [7]).

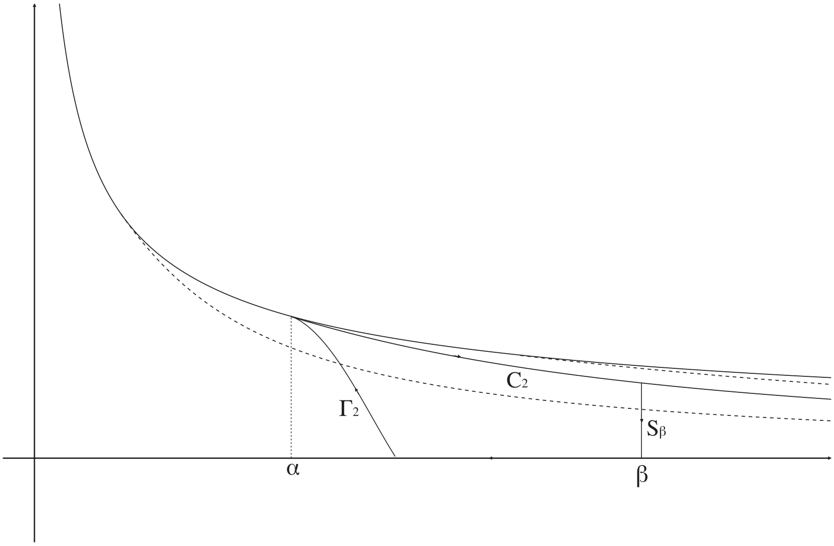

Figure 2

Since the 1-form

$$[f(r,v,q_1) - q_1 f_{v'}(r,v,q_1)]\,dr + f_{v'}(r,v,q_1)\,dv \tag{10}$$

is exact, the integral on the right-hand side of (9) equals the line integral of (10) along a segment of the vertical axis, which is null, plus the integral line along the curve C_1 (see Figure 1). The latter is

$$J_1(u_\varepsilon) = \int_0^\alpha f(r, u_\varepsilon, u_\varepsilon')\,dr.$$

Thus, we have

$$J_1(v) \geq J_1(u_\varepsilon). \tag{11}$$

A similar procedure applies to $J_2(v)$. We integrate the exact 1-form

$$[f(r,v,q_2) - q_2 f_{v'}(r,v,q_2)]\,dr + f_{v'}(r,v,q_2)\,dv \tag{12}$$

along the closed path delineated in Figure 2.

A simple asymptotic argument allows us to claim that the line integral of (12) along the vertical segment S_β is infinitesimal when β goes to infinity. Therefore

$$J_2(v) = \int_\alpha^\infty f(r,v,v')\,dr \geq J_2(u_\varepsilon) = \int_\alpha^\infty f(r,u_\varepsilon,u_\varepsilon')\,dr. \tag{13}$$

Collecting (11) and (13) gives $J(v) \geq J(u_\varepsilon)$. Hence, computing $J(u_\varepsilon)$ leads to

$$\int_{\mathbb{R}^n} |\nabla v|^p \, dx \geq a^p \frac{(n-p)^p}{(p-1)^{p-1}} \|v\|_{L^{p^*}}^{p^*} + a^{p-n} 2\pi^{n/2} \frac{(n-p)^{p-1}}{(p-1)^{p-2}} \frac{\Gamma\left(\frac{n}{p}\right)\Gamma\left(n-\frac{n}{p}\right)}{\Gamma(n)\Gamma\left(\frac{n}{2}\right)}.$$

If we recall the value (8) of a, by density arguments, we have the following result.

THEOREM 2.1. *If v belongs to the Sobolev space $W^{1,p}(\mathbb{R}^n)$ and $p \in]1, n[$, then*

$$\|v\|_{p^*,\infty}^{p^2/(n-p)} \|\nabla v\|_p^p \geq A(n,p) \|v\|_{p^*}^{p^*} + B(n,p) \|v\|_{p^*,\infty}^{p^*}, \tag{14}$$

where

$$A(n,p) = \frac{(n-p)^p}{p^p}$$

and

$$B(n,p) = 2\pi^{n/2} \frac{(n-p)^{p-1}}{(p-1)^{n-1-n/p}} p^{n-p} \frac{\Gamma\left(\frac{n}{p}\right)\Gamma\left(n-\frac{n}{p}\right)}{\Gamma(n)\Gamma\left(\frac{n}{2}\right)}.$$

REMARK 2.1. *Handling with a sole extremal field leads to trivial outcomes. Namely it is not possible to assemble the graphs of v and of an extremal, and make a closed path along which calculate the integral of an exact* 1*-form as above. This becomes possible if one thinks of the extremal fields as a unique field defined on a surface, a sort of cylinder, squashed onto T. In some sense we deal with a sheet with two pages: when an extremal touches the envelope it passes from one page to another. Therefore, the extremals can be viewed as closed paths which describe a complete ring. The same happens to the graph of v when it touches the envelope. In some sense the graphs of v and of each extremal are in the same homotopy class.*

REMARK 2.2. *Recently the problem of the optimality of the Sobolev constant has been tackled by different tools* (*see* [4]). *Instead of a symmetrization procedure and the Pólya inequality* (4), *mass transport methods and a subtle result by Brenier* [3] *are used. Both methods have deep, but different, geometric flavours.*

3. THE SOBOLEV INEQUALITY

Inequality (14) can be viewed as a generalization of the Sobolev inequality. Namely (1) can be deduced from (14) dividing by $\|v\|_{p^*,\infty}^{p^2/(n-p)}$ and minimizing the right-hand side with respect to $\|v\|_{p^*,\infty}$.

We can also argue in a different way.

For instance, if $p = 2$ and $n = 3$, (14) becomes

$$\|\nabla v\|_2^2 \geq \frac{1}{4} \frac{\|v\|_6^6}{\|v\|_{6,\infty}^4} + \pi^2 \|v\|_{6,\infty}^2.$$

By Young inequality we get

$$\|\nabla v\|_2^2 \geq 3\left(\frac{\pi^2-\sigma^2}{4}\right)^{2/3}\|v\|_6^2 + \sigma^2\|v\|_{6,\infty}^2 \tag{15}$$

for any $\sigma \in [0, \pi]$. If $\sigma = 0$ we obtain the Sobolev inequality, whereas, if $\sigma = \pi$, we have

$$\|\nabla v\|_{L^2} \geq \pi\|v\|_{6,\infty}. \tag{16}$$

However the value of the constant in (16) is not sharp, as the following result shows.

Theorem 3.1. *Let $u \in W^{1,2}(\mathbb{R}^n)$. Then*

$$(n-2)\omega_n\|u\|_{2n/(n-2),\infty}^2 \leq \|\nabla u\|_{L^2}^2. \tag{17}$$

It is obviously sufficient to deal with spherically decreasing and spherically symmetric functions. For the sake of simplicity we assume

$$\sup_{r>0}(r^{(n-2)/2}u(r)) = r_0^{(n-2)/2}u(r_0) = 1 \tag{18}$$

for a suitable $r_0 > 0$. Among all functions satisfying (18) the one with the lowest energy is

$$w(r) = \begin{cases} r_0^{-(n-2)/2} & \text{if } r \leq r_0 \\ r_0^{(n-2)/2}r^{2-n} & \text{if } r > r_0 \end{cases}$$

The energy of w is $(n-2)\omega_n$, then we get (17). Moreover the constant is sharp.

Remark 3.1. *As for* (15), *if $S < 3(\pi^2/4)^{2/3}$, one could ask for the best constant $C(S)$ such that*

$$\|\nabla v\|_2^2 \geq S\|v\|_6^2 + C(S)\|v\|_{6,\infty}^2.$$

Analogous question can be set when we remove any restriction on p and n.

References

[1] T. Aubin: *Problèmes isopérimetriques et espaces de Sobolev*, J. Differential Geometry 11 (1976) n. 4, 573–598.

[2] G. A. Bliss: *An integral inequality*, J. London Math. Soc. 5 (1930), 40–46.

[3] Y. Brenier: *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. 44 (1991) n. 4, 375–417.

[4] D. Cordero-Erausquin - B. Nazaret - C. Villani: *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, Adv. Math. 182 (2004) n. 2, 307–332.

[5] E. De Giorgi: *Su una teoria generale della misura* $(r-1)$ *dimensionale in uno spazio a r dimensioni*, Ann. Mat. Pura Appl. (4) 36 (1954), 191–213.
[6] H. Sagan: Introduction to the Calculus of Variations, Dover Pubbl., New York (1992).
[7] G. Talenti: Calcolo delle Variazioni, Pitagora Ed. (1977).
[8] G. Talenti: *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) 110 (1976), 353–372.

Received 14 May 2009,
and in revised form 15 July 2009.

Dip. di Matematica e Appl. ''R. Caccioppoli''
Università degli Studi di Napoli ''Federico II''
Complesso Univ. Monte S. Angelo
via Cintia, 80126 Napoli (Italy)
angelo.alvino@unina.it